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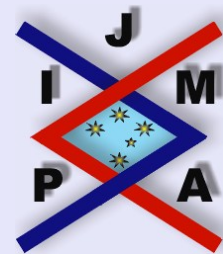
A RELATION TO HARDY-HILBERT'S INTEGRAL INEQUALITY AND MULHOLLAND'S INEQUALITY

BICHENG YANG

Department of Mathematics
Guangdong Institute of Education
Guangzhou, Guangdong 510303
P. R. China.

EMail: bcyang@pub.guangzhou.gd.cn

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Abstract

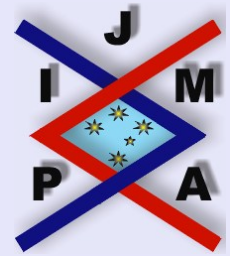
This paper deals with a relation between Hardy-Hilbert's integral inequality and Mulholland's integral inequality with a best constant factor, by using the Beta function and introducing a parameter λ . As applications, the reverse, the equivalent form and some particular results are considered.

2000 Mathematics Subject Classification: 26D15.

Key words: Hardy-Hilbert's integral inequality; Mulholland's integral inequality; β function; Hölder's inequality.

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1. Introduction

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f, g \geq 0$ satisfy $0 < \int_0^\infty f^p(x)dx < \infty$ and $0 < \int_0^\infty g^q(x)dx < \infty$, then one has two equivalent inequalities as (see [1]):

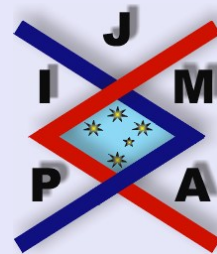
$$(1.1) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\{ \int_0^\infty f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty g^q(x)dx \right\}^{\frac{1}{q}};$$

$$(1.2) \quad \int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y} dx \right)^p dy < \left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \right]^p \int_0^\infty f^p(x)dx,$$

where the constant factors $\frac{\pi}{\sin(\pi/p)}$ and $\left[\frac{\pi}{\sin(\pi/p)} \right]^p$ are all the best possible. Inequality (1.1) is called Hardy- Hilbert’s integral inequality, which is important in analysis and its applications (cf. Mitrinovic et al. [2]).

If $0 < \int_1^\infty \frac{1}{x} F^p(x)dx < \infty$ and $0 < \int_1^\infty \frac{1}{y} G^q(y)dy < \infty$, then the Mulholland’s integral inequality is as follows (see [1, 3]):

$$(1.3) \quad \int_1^\infty \int_1^\infty \frac{F(x)F(y)}{xy \ln xy} dx dy < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\{ \int_1^\infty \frac{F^p(x)}{x} dx \right\}^{\frac{1}{p}} \left\{ \int_1^\infty \frac{G^q(y)}{y} dy \right\}^{\frac{1}{q}},$$



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where the constant factor $\frac{\pi}{\sin(\frac{\pi}{p})}$ is the best possible. Setting $f(x) = F(x)/x$, and $g(y) = G(y)/y$ in (1.3), by simplification, one has (see [12])

$$(1.4) \quad \int_1^\infty \int_1^\infty \frac{f(x)g(y)}{\ln xy} dx dy < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\{ \int_1^\infty x^{p-1} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_1^\infty x^{q-1} g^q(x) dx \right\}^{\frac{1}{q}}.$$

We still call (1.4) Mulholland's integral inequality.

In 1998, Yang [11] first introduced an independent parameter λ and the β function for given an extension of (1.1) (for $p = q = 2$). Recently, by introducing a parameter λ , Yang [8] and Yang et al. [10] gave some extensions of (1.1) and (1.2) as: If $\lambda > 2 - \min\{p, q\}$, $f, g \geq 0$ satisfy $0 < \int_0^\infty x^{1-\lambda} f^p(x) dx < \infty$ and $0 < \int_0^\infty x^{1-\lambda} g^q(x) dx < \infty$, then one has two equivalent inequalities as:

$$(1.5) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dx dy < k_\lambda(p) \left\{ \int_0^\infty x^{1-\lambda} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty x^{1-\lambda} g^q(x) dx \right\}^{\frac{1}{q}}$$

and

$$(1.6) \quad \int_0^\infty y^{(p-1)(\lambda-1)} \left[\int_0^\infty \frac{f(x)}{(x+y)^\lambda} dx \right]^p dy < [k_\lambda(p)]^p \int_0^\infty x^{1-\lambda} f^p(x) dx,$$

where the constant factors $k_\lambda(p)$ and $[k_\lambda(p)]^p$ ($k_\lambda(p) = B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)$, $B(u, v)$ is the β function) are all the best possible. By introducing a parameter



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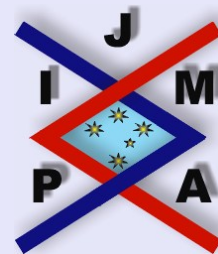
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α , Kuang [5] gave an extension of (1.1), and Yang [9] gave an improvement of [5] as: If $\alpha > 0$, $f, g \geq 0$ satisfy $0 < \int_0^\infty x^{(p-1)(1-\alpha)} f^p(x) dx < \infty$ and $0 < \int_0^\infty x^{(q-1)(1-\alpha)} g^q(x) dx < \infty$, then

$$(1.7) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\alpha + y^\alpha} dx dy < \frac{\pi}{\alpha \sin\left(\frac{\pi}{p}\right)} \left\{ \int_0^\infty x^{(p-1)(1-\alpha)} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty x^{(q-1)(1-\alpha)} g^q(x) dx \right\}^{\frac{1}{q}},$$

where the constant $\frac{\pi}{\alpha \sin(\pi/p)}$ is the best possible. Recently, Sulaiman [6] gave some new forms of (1.1) and Hong [14] gave an extension of Hardy-Hilbert's inequality by introducing two parameters λ and α . Yang et al. [13] provided an extensive account of the above results.

The main objective of this paper is to build a relation to (1.1) and (1.4) with a best constant factor, by introducing the β function and a parameter λ , related to the double integral $\int_a^b \int_a^b \frac{f(x)g(y)}{(u(x)+u(y))^\lambda} dx dy$ ($\lambda > 0$). As applications, the reversion, the equivalent form and some particular results are considered.



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2. Some Lemmas

First, we need the formula of the β function as (cf. Wang et al. [7]):

$$(2.1) \quad B(u, v) := \int_0^\infty \frac{1}{(1+t)^{u+v}} t^{u-1} dt = B(v, u) \quad (u, v > 0).$$

Lemma 2.1 (cf. [4]). *If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\omega(\sigma) > 0$, $f, g \geq 0$, $f \in L_\omega^p(E)$ and $g \in L_\omega^q(E)$, then one has the Hölder's inequality with weight as:*

$$(2.2) \quad \int_E \omega(\sigma) f(\sigma) g(\sigma) d\sigma \leq \left\{ \int_E \omega(\sigma) f^p(\sigma) d\sigma \right\}^{\frac{1}{p}} \left\{ \int_E \omega(\sigma) g^q(\sigma) d\sigma \right\}^{\frac{1}{q}};$$

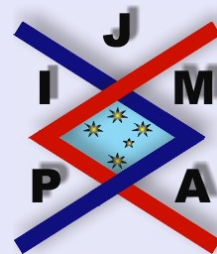
if $p < 1$ ($p \neq 0$), with the above assumption, one has the reverse of (2.2), where the equality (in the above two cases) holds if and only if there exists non-negative real numbers c_1 and c_2 , such that they are not all zero and $c_1 f^p(\sigma) = c_2 g^q(\sigma)$, a. e. in E .

Lemma 2.2. *If $p \neq 0, 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\phi_r = \phi_r(\lambda) > 0$ ($r = p, q$), $\phi_p + \phi_q = \lambda$, and $u(t)$ is a differentiable strict increasing function in (a, b) ($-\infty \leq a < b \leq \infty$) such that $u(a+) = 0$ and $u(b-) = \infty$, for $r = p, q$, define $\omega_r(x)$ as*

$$(2.3) \quad \omega_r(x) := (u(x))^{\lambda - \phi_r} \int_a^b \frac{(u(y))^{\phi_r - 1} u'(y)}{(u(x) + u(y))^\lambda} dy \quad (x \in (a, b)).$$

Then for $x \in (a, b)$, each $\omega_r(x)$ is constant, that is

$$(2.4) \quad \omega_r(x) = B(\phi_p, \phi_q) \quad (r = p, q).$$



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Proof. For fixed x , setting $v = \frac{u(y)}{u(x)}$ in (2.3), one has

$$\begin{aligned}\omega_r(x) &= (u(x))^{\lambda-\phi_r} \int_a^b \frac{(u(y))^{\phi_r-1} u'(y)}{(u(x))^\lambda (1+u(y)/u(x))^\lambda} dy \\ &= (u(x))^{\lambda-\phi_r} \int_0^\infty \frac{(vu(x))^{\phi_r-1}}{(u(x))^\lambda (1+v)^\lambda} u(x) dv \\ &= \int_0^\infty \frac{v^{\phi_r-1}}{(1+v)^\lambda} dv \quad (r = p, q).\end{aligned}$$

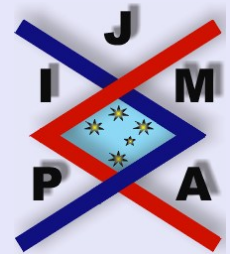
By (2.1), one has (2.4). The lemma is proved. \square

Lemma 2.3. *If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\phi_r > 0$ ($r = p, q$), satisfy $\phi_p + \phi_q = \lambda$, and $u(t)$ is a differentiable strict increasing function in (a, b) ($-\infty \leq a < b \leq \infty$) satisfying $u(a+) = 0$ and $u(b-) = \infty$, then for $c = u^{-1}(1)$ and $0 < \varepsilon < q\phi_p$,*

$$\begin{aligned}(2.5) \quad I &:= \int_c^b \int_c^b \frac{(u(x))^{\phi_q - \frac{\varepsilon}{p} - 1} u'(x)}{(u(x) + u(y))^\lambda} (u(y))^{\phi_p - \frac{\varepsilon}{q} - 1} u'(y) dx dy \\ &> \frac{1}{\varepsilon} B\left(\phi_p - \frac{\varepsilon}{q}, \phi_q + \frac{\varepsilon}{q}\right) - O(1); \end{aligned}$$

if $0 < p < 1$ (or $p < 0$), with the above assumption and $0 < \varepsilon < -q\phi_q$ (or $0 < \varepsilon < q\phi_p$), then

$$(2.6) \quad I < \frac{1}{\varepsilon} B\left(\phi_p - \frac{\varepsilon}{q}, \phi_q + \frac{\varepsilon}{q}\right).$$



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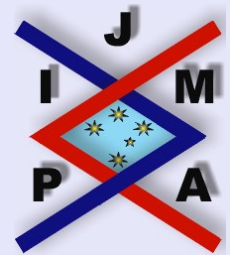
Proof. For fixed x , setting $v = \frac{u(y)}{u(x)}$ in I , one has

$$\begin{aligned}
 I &:= \int_c^b (u(x))^{\phi_q - \frac{\varepsilon}{p} - 1} u'(x) \left[\int_c^b \frac{(u(y))^{\phi_p - \frac{\varepsilon}{q} - 1}}{(u(x) + u(y))^\lambda} u'(y) dy \right] dx \\
 &= \int_c^b (u(x))^{-1 - \varepsilon} u'(x) \int_{\frac{1}{u(x)}}^\infty \frac{1}{(1+v)^\lambda} v^{\phi_p - \frac{\varepsilon}{q} - 1} dv dx \\
 &= \int_c^b \frac{u'(x) dx}{(u(x))^{1+\varepsilon}} \int_0^\infty \frac{v^{\phi_p - \frac{\varepsilon}{q} - 1}}{(1+v)^\lambda} dv \\
 (2.7) \quad &\quad - \int_c^b \frac{u'(x)}{(u(x))^{1+\varepsilon}} \int_0^{\frac{1}{u(x)}} \frac{v^{\phi_p - \frac{\varepsilon}{q} - 1}}{(1+v)^\lambda} dv dx \\
 &> \frac{1}{\varepsilon} \int_0^\infty \frac{v^{\phi_p - \frac{\varepsilon}{q} - 1}}{(1+v)^\lambda} dv - \int_c^b \frac{u'(x)}{(u(x))} \left[\int_0^{\frac{1}{u(x)}} v^{\phi_p - \frac{\varepsilon}{q} - 1} dv \right] dx \\
 &= \frac{1}{\varepsilon} \int_0^\infty \frac{v^{\phi_p - \frac{\varepsilon}{q} - 1}}{(1+v)^\lambda} dv - \left(\phi_p - \frac{\varepsilon}{q} \right)^{-2}.
 \end{aligned}$$

By (2.1), inequality (2.5) is valid. If $0 < p < 1$ (or $p < 0$), by (2.7), one has

$$I < \int_c^b \frac{u'(x)}{(u(x))^{1+\varepsilon}} dx \int_0^\infty \frac{1}{(1+v)^\lambda} v^{\phi_p - \frac{\varepsilon}{q} - 1} dv,$$

and then by (2.1), inequality (2.6) follows. The lemma is proved. \square



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3. Main Results

Theorem 3.1. *If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\phi_r > 0$ ($r = p, q$), $\phi_p + \phi_q = \lambda$, $u(t)$ is a differentiable strict increasing function in (a, b) ($-\infty \leq a < b \leq \infty$), such that $u(a+) = 0$ and $u(b-) = \infty$, and $f, g \geq 0$ satisfy $0 < \int_a^b \frac{(u(x))^{p(1-\phi_q)-1}}{(u'(x))^{p-1}} f^p(x) dx < \infty$ and $0 < \int_a^b \frac{(u(x))^{q(1-\phi_p)-1}}{(u'(x))^{q-1}} g^q(x) dx < \infty$, then*

$$(3.1) \quad \int_a^b \int_a^b \frac{f(x)g(y)}{(u(x) + u(y))^\lambda} dx dy$$

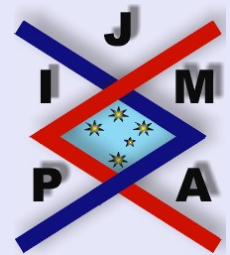
$$< B(\phi_p, \phi_q) \left\{ \int_a^b \frac{(u(x))^{p(1-\phi_q)-1}}{(u'(x))^{p-1}} f^p(x) dx \right\}^{\frac{1}{p}}$$

$$\times \left\{ \int_a^b \frac{(u(x))^{q(1-\phi_p)-1}}{(u'(x))^{q-1}} g^q(x) dx \right\}^{\frac{1}{q}},$$

where the constant factor $B(\phi_p, \phi_q)$ is the best possible. If $p < 1$ ($p \neq 0$), $\{\lambda; \phi_r > 0$ ($r = p, q$), $\phi_p + \phi_q = \lambda\} \neq \phi$, with the above assumption, one has the reverse of (3.1), and the constant is still the best possible.

Proof. By (2.2), one has

$$J := \int_a^b \int_a^b \frac{f(x)g(y)}{(u(x) + u(y))^\lambda} dx dy$$



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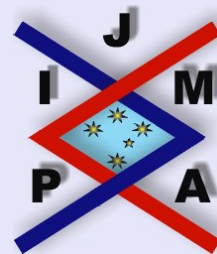


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$$\begin{aligned}
 &= \int_a^b \int_a^b \frac{1}{(u(x) + u(y))^\lambda} \left[\frac{(u(x))^{(1-\phi_q)/q} (u'(y))^{1/p}}{(u(y))^{(1-\phi_p)/p} (u'(x))^{1/q}} f(x) \right] \\
 &\quad \times \left[\frac{(u(y))^{(1-\phi_p)/p} (u'(x))^{1/q}}{(u(x))^{(1-\phi_q)/q} (u'(y))^{1/p}} g(y) \right] dx dy \\
 &\leq \left\{ \int_a^b \left[\int_a^b \frac{(u(y))^{\phi_p-1} u'(y)}{(u(x) + u(y))^\lambda} dy \right] \frac{(u(x))^{(p-1)(1-\phi_q)}}{(u'(x))^{p-1}} f^p(x) dx \right\}^{\frac{1}{p}} \\
 (3.2) \quad &\quad \times \left\{ \int_a^b \left[\int_a^b \frac{(u(x))^{\phi_q-1} u'(x)}{(u(x) + u(y))^\lambda} dx \right] \frac{(u(y))^{(q-1)(1-\phi_p)}}{(u'(y))^{q-1}} g^q(y) dy \right\}^{\frac{1}{q}}.
 \end{aligned}$$

If (3.2) takes the form of equality, then by (2.2), there exist non-negative numbers c_1 and c_2 , such that they are not all zero and

$$c_1 \frac{u'(y)(u(x))^{(p-1)(1-\phi_q)}}{(u(y))^{1-\phi_p}(u'(x))^{p-1}} f^p(x) = c_2 \frac{u'(x)(u(y))^{(q-1)(1-\phi_p)}}{(u(x))^{1-\phi_q}(u'(y))^{q-1}} g^q(y),$$

a.e. in $(a, b) \times (a, b)$.

It follows that

$$c_1 \frac{(u(x))^{p(1-\phi_q)}}{(u'(x))^p} f^p(x) = c_2 \frac{(u(y))^{q(1-\phi_p)}}{(u'(y))^q} g^q(y) = c_3, \text{ a.e. in } (a, b) \times (a, b),$$

where c_3 is a constant. Without loss of generality, suppose $c_1 \neq 0$. One has

$$\frac{(u(x))^{p(1-\phi_q)-1}}{(u'(x))^{p-1}} f^p(x) = \frac{c_3 u'(x)}{c_1 u(x)}, \text{ a.e. in } (a, b),$$

which contradicts $0 < \int_a^b \frac{(u(x))^{p(1-\phi_q)-1}}{(u'(x))^{p-1}} f^p(x) dx < \infty$. Then by (2.3), one has

$$(3.3) \quad J < \left\{ \int_a^b \omega_p(x) \frac{(u(x))^{p(1-\phi_q)-1}}{(u'(x))^{p-1}} f^p(x) dx \right\}^{\frac{1}{p}} \\ \times \left\{ \int_0^\infty \omega_q(x) \frac{(u(x))^{q(1-\phi_p)-1}}{(u'(x))^{q-1}} g^q(x) dx \right\}^{\frac{1}{q}},$$

and in view of (2.4), it follows that (3.1) is valid.

For $0 < \varepsilon < q\phi_p$, setting $f_\varepsilon(x) = g_\varepsilon(x) = 0$, $x \in (a, c)$ ($c = u^{-1}(1)$);

$$f_\varepsilon(x) = (u(x))^{\phi_q - \frac{\varepsilon}{p} - 1} u'(x), \quad g_\varepsilon(x) = (u(x))^{\phi_p - \frac{\varepsilon}{q} - 1} u'(x),$$

$x \in [c, b)$, we find

$$(3.4) \quad \left\{ \int_a^b \frac{(u(x))^{p(1-\phi_q)-1}}{(u'(x))^{p-1}} f_\varepsilon^p(x) dx \right\}^{\frac{1}{p}} \\ \times \left\{ \int_a^b \frac{(u(x))^{q(1-\phi_p)-1}}{(u'(x))^{q-1}} g_\varepsilon^q(x) dx \right\}^{\frac{1}{q}} = \frac{1}{\varepsilon}.$$

If the constant factor $B(\phi_p, \phi_q)$ in (3.1) is not the best possible, then, there exists a positive constant $k < B(\phi_p, \phi_q)$, such that (3.1) is still valid if one replaces



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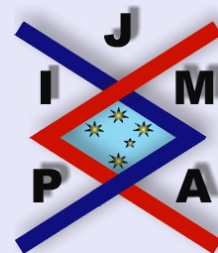
$B(\phi_p, \phi_q)$ by k . In particular, by (2.6) and (3.4), one has

$$\begin{aligned} & B\left(\phi_p - \frac{\varepsilon}{q}, \phi_q + \frac{\varepsilon}{q}\right) - \varepsilon O(1) \\ & < \varepsilon \int_a^b \int_a^b \frac{f_\varepsilon(x)g_\varepsilon(y)}{(u(x) + u(y))^\lambda} dx dy \\ & < \varepsilon k \left\{ \int_a^b \frac{(u(x))^{p(1-\phi_q)-1}}{(u'(x))^{p-1}} f_\varepsilon^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_a^b \frac{(u(x))^{q(1-\phi_p)-1}}{(u'(x))^{q-1}} g_\varepsilon^q(x) dx \right\}^{\frac{1}{q}} = k, \end{aligned}$$

and then $B(\phi_p, \phi_q) \leq k$ ($\varepsilon \rightarrow 0^+$). This contradicts the fact that $k < B(\phi_p, \phi_q)$. Hence the constant factor $B(\phi_p, \phi_q)$ in (3.1) is the best possible.

For $0 < p < 1$ (or $p < 0$), by the reverse of (2.2) and using the same procedures, one can obtain the reverse of (3.1). For $0 < \varepsilon < -q\phi_q$ (or $0 < \varepsilon < q\phi_p$), setting $f_\varepsilon(x)$ and $g_\varepsilon(x)$ as the above, we still have (3.4). If the constant factor $B(\phi_p, \phi_q)$ in the reverse of (3.1) is not the best possible, then, there exists a positive constant $K > B(\phi_p, \phi_q)$, such that the reverse of (3.1) is still valid if one replaces $B(\phi_p, \phi_q)$ by K . In particular, by (2.7) and (3.4), one has

$$\begin{aligned} & B\left(\phi_p - \frac{\varepsilon}{q}, \phi_q + \frac{\varepsilon}{q}\right) \\ & > \varepsilon \int_a^b \int_a^b \frac{f_\varepsilon(x)g_\varepsilon(y)}{(u(x) + u(y))^\lambda} dx dy \\ & > \varepsilon K \left\{ \int_a^b \frac{(u(x))^{p(1-\phi_q)-1}}{(u'(x))^{p-1}} f_\varepsilon^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_a^b \frac{(u(x))^{q(1-\phi_p)-1}}{(u'(x))^{q-1}} g_\varepsilon^q(x) dx \right\}^{\frac{1}{q}} = K, \end{aligned}$$



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and then $B(\phi_p, \phi_q) \geq K$ ($\varepsilon \rightarrow 0^+$). This contradiction concludes that the constant in the reverse of (3.1) is the best possible. The theorem is proved. \square

Theorem 3.2. *Let the assumptions of Theorem 3.1 hold.*

(i) *If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, one obtains the equivalent inequality of (3.1) as follows*

$$(3.5) \quad \int_a^b \frac{u'(y)}{(u(y))^{1-p\phi_p}} \left[\int_a^b \frac{f(x)}{(u(x) + u(y))^\lambda} dx \right]^p dy < [B(\phi_p, \phi_q)]^p \int_a^b \frac{(u(x))^{p(1-\phi_q)-1}}{(u'(x))^{p-1}} f^p(x) dx;$$

(ii) *If $0 < p < 1$, one obtains the reverse of (3.5) equivalent to the reverse of (3.1);*

(iii) *If $p < 0$, one obtains inequality (3.5) equivalent to the reverse of (3.1),*

where the constants in the above inequalities are all the best possible.

Proof. Set

$$g(y) := \frac{u'(y)}{(u(y))^{1-p\phi_p}} \left[\int_a^b \frac{f(x)}{(u(x) + u(y))^\lambda} dx \right]^{p-1},$$



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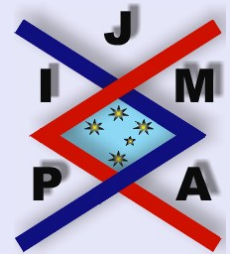
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and use (3.1) to obtain

$$\begin{aligned}
 0 &< \int_a^b \frac{(u(y))^{q(1-\phi_p)-1}}{(u'(y))^{q-1}} g^q(y) dy \\
 &= \int_a^b \frac{u'(y)}{(u(y))^{1-p\phi_p}} \left[\int_a^b \frac{f(x)}{(u(x)+u(y))^\lambda} dx \right]^p dy \\
 &= \int_a^b \int_a^b \frac{f(x)g(y)}{(u(x)+u(y))^\lambda} dx dy \leq B(\phi_p, \phi_q) \\
 &\quad \times \left\{ \int_a^b \frac{(u(x))^{p(1-\phi_q)-1}}{(u'(x))^{p-1}} f^p(x) dx \right\}^{\frac{1}{p}} \\
 (3.6) \quad &\quad \times \left\{ \int_a^b \frac{(u(y))^{q(1-\phi_p)-1}}{(u'(y))^{q-1}} g^q(y) dy \right\}^{\frac{1}{q}} ;
 \end{aligned}$$

$$\begin{aligned}
 0 &< \left\{ \int_a^b \frac{(u(y))^{q(1-\phi_p)-1}}{(u'(y))^{q-1}} g^q(y) dy \right\}^{1-\frac{1}{q}} \\
 &= \left\{ \int_a^b \frac{u'(y)}{(u(y))^{1-p\phi_p}} \left[\int_a^b \frac{f(x)}{(u(x)+u(y))^\lambda} dx \right]^p dy \right\}^{\frac{1}{p}} \\
 (3.7) \quad &\leq B(\phi_p, \phi_q) \left\{ \int_a^b \frac{(u(x))^{p(1-\phi_q)-1}}{(u'(x))^{p-1}} f^p(x) dx \right\}^{\frac{1}{p}} < \infty.
 \end{aligned}$$

It follows that (3.6) takes the form of strict inequality by using (3.1); so does



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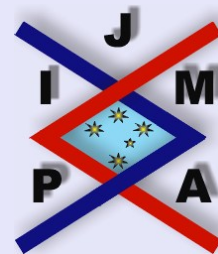
(3.7). Hence one can get (3.5). On the other hand, if (3.5) is valid, by (2.2),

$$\begin{aligned}
 & \int_a^b \int_a^b \frac{f(x)g(y)}{(u(x) + u(y))^\lambda} dx dy \\
 &= \int_a^b \left[\frac{(u'(y))^{\frac{1}{p}}}{(u(y))^{\frac{1}{p} - \phi_p}} \int_a^b \frac{f(x)}{(u(x) + u(y))^\lambda} dx \right] \left[\frac{(u(y))^{\frac{1}{p} - \phi_p}}{(u'(y))^{\frac{1}{p}}} g(y) \right] dy \\
 &\leq \left\{ \int_a^b \frac{u'(y)}{(u(y))^{1 - p\phi_p}} \left[\int_a^b \frac{f(x)}{(u(x) + u(y))^\lambda} dx \right]^p dy \right\}^{\frac{1}{p}} \\
 (3.8) \quad & \times \left\{ \int_a^b \frac{(u(y))^{q(1 - \phi_p) - 1}}{(u'(y))^{q-1}} g^q(y) dy \right\}^{\frac{1}{q}}.
 \end{aligned}$$

Hence by (3.5), (3.1) yields. It follows that (3.1) and (3.5) are equivalent.

If the constant factor in (3.5) is not the best possible, one can get a contradiction that the constant factor in (3.1) is not the best possible by using (3.8). Hence the constant factor in (3.5) is still the best possible.

If $0 < p < 1$ (or $p < 0$), one can get the reverses of (3.6), (3.7) and (3.8), and thus concludes the equivalence. By (3.6), for $0 < p < 1$, one can obtain the reverse of (3.5); for $p < 0$, one can get (3.5). If the constant factor in the reverse of (3.5) (or simply (3.5)) is not the best possible, then one can get a contradiction that the constant factor in the reverse of (3.1) is not the best possible by using the reverse of (3.8). Thus the theorem is proved. \square



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4. Some Particular Results

We point out that the constant factors in the following particular results of Theorems 3.1 – 3.2 are all the best possible.

4.1. The first reversible form

Corollary 4.1. *Let the assumptions of Theorems 3.1 – 3.2 hold. For*

$$\phi_r = \left(1 - \frac{1}{r}\right) (\lambda - 2) + 1 \quad (r = p, q),$$

$$0 < \int_a^b \frac{(u(x))^{1-\lambda}}{(u'(x))^{p-1}} f^p(x) dx < \infty$$

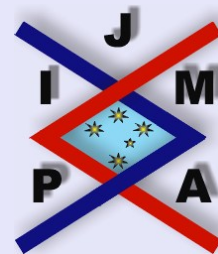
and

$$0 < \int_a^b \frac{(u(x))^{1-\lambda}}{(u'(x))^{q-1}} g^q(x) dx < \infty,$$

setting $k_\lambda(p) = B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)$,

(i) *If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 2 - \min\{p, q\}$, then we have the following two equivalent inequalities:*

$$(4.1) \quad \int_a^b \int_a^b \frac{f(x)g(y)}{(u(x) + u(y))^\lambda} dx dy$$



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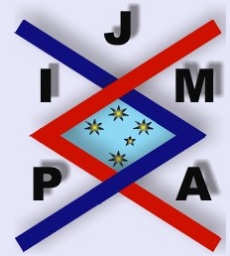


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$$< k_{\lambda}(p) \left\{ \int_a^b \frac{(u(x))^{1-\lambda}}{(u'(x))^{p-1}} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_a^b \frac{(u(x))^{1-\lambda}}{(u'(x))^{q-1}} g^q(x) dx \right\}^{\frac{1}{q}}$$

and

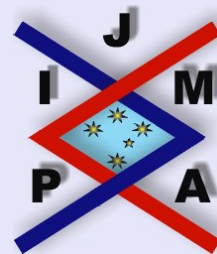
$$(4.2) \quad \int_a^b \frac{u'(y)}{(u(y))^{(p-1)(1-\lambda)}} \left[\int_a^b \frac{f(x)}{(u(x) + u(y))^{\lambda}} dx \right]^p dy < [k_{\lambda}(p)]^p \int_a^b \frac{(u(x))^{1-\lambda}}{(u'(x))^{p-1}} f^p(x) dx.$$

(ii) If $0 < p < 1$ and $2 - p < \lambda < 2 - q$, one obtains two equivalent reverses of (4.1) and (4.2),

(iii) If $p < 0$ and $2 - q < \lambda < 2 - p$, we have the reverse of (4.1) and the inequality (4.2), which are equivalent. In particular, by (4.1),

(a) setting $u(x) = x^{\alpha}$ ($\alpha > 0, x \in (0, \infty)$), one has

$$(4.3) \quad \int_0^{\infty} \int_0^{\infty} \frac{f(x)g(y)}{(x^{\alpha} + y^{\alpha})^{\lambda}} dx dy < \frac{1}{\alpha} k_{\lambda}(p) \left\{ \int_0^{\infty} x^{p-1+\alpha(2-\lambda-p)} f^p(x) dx \right\}^{\frac{1}{p}} \times \left\{ \int_0^{\infty} x^{q-1+\alpha(2-\lambda-q)} g^q(x) dx \right\}^{\frac{1}{q}};$$



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(b) setting $u(x) = \ln x$, $x \in (1, \infty)$, one has

$$(4.4) \quad \int_1^\infty \int_1^\infty \frac{f(x)g(y)}{(\ln xy)^\lambda} dx dy$$

$$< k_\lambda(p) \left\{ \int_1^\infty x^{p-1} (\ln x)^{1-\lambda} f^p(x) dx \right\}^{\frac{1}{p}}$$

$$\times \left\{ \int_1^\infty x^{q-1} (\ln x)^{1-\lambda} g^q(x) dx \right\}^{\frac{1}{q}};$$

(c) setting $u(x) = e^x$, $x \in (-\infty, \infty)$, one has

$$(4.5) \quad \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{f(x)g(y)}{(e^x + e^y)^\lambda} dx dy$$

$$< k_\lambda(p) \left\{ \int_{-\infty}^\infty e^{(2-p-\lambda)x} f^p(x) dx \right\}^{\frac{1}{p}}$$

$$\times \left\{ \int_{-\infty}^\infty e^{(2-q-\lambda)x} g^q(x) dx \right\}^{\frac{1}{q}};$$

(d) setting $u(x) = \tan x$, $x \in (0, \frac{\pi}{2})$, one has

$$(4.6) \quad \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{f(x)g(y)}{(\tan x + \tan y)^\lambda} dx dy$$

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$$< k_\lambda(p) \left\{ \int_0^{\frac{\pi}{2}} \frac{\tan^{1-\lambda} x}{\sec^{2(p-1)} x} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^{\frac{\pi}{2}} \frac{\tan^{1-\lambda} x}{\sec^{2(q-1)} x} g^q(x) dx \right\}^{\frac{1}{q}} ;$$

(e) setting $u(x) = \sec x - 1$, $x \in (0, \frac{\pi}{2})$, one has

$$(4.7) \quad \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{f(x)g(y)}{(\sec x + \sec y - 2)^\lambda} dx dy$$

$$< k_\lambda(p) \left\{ \int_0^{\frac{\pi}{2}} \frac{(\sec x - 1)^{1-\lambda}}{(\sec x \tan x)^{p-1}} f^p(x) dx \right\}^{\frac{1}{p}}$$

$$\times \left\{ \int_0^{\frac{\pi}{2}} \frac{(\sec x - 1)^{1-\lambda}}{(\sec x \tan x)^{q-1}} g^q(x) dx \right\}^{\frac{1}{q}} .$$

4.2. The second reversible form

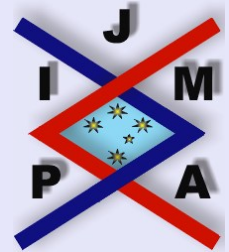
Corollary 4.2. *Let the assumptions of Theorems 3.1 – 3.2 hold. For*

$$\phi_r = \frac{\lambda - 1}{2} + \frac{1}{r} \quad (r = p, q),$$

$$0 < \int_a^b \frac{(u(x))^{p\frac{1-\lambda}{2}}}{(u'(x))^{p-1}} f^p(x) dx < \infty$$

and

$$0 < \int_a^b \frac{(u(x))^{q\frac{1-\lambda}{2}}}{(u'(x))^{q-1}} g^q(x) dx < \infty,$$



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setting $\tilde{k}_\lambda(p) = B\left(\frac{p\lambda-p+2}{2p}, \frac{q\lambda-q+2}{2q}\right),$

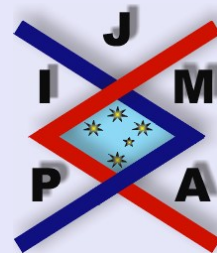
(i) If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \lambda > 1 - 2 \min\{\frac{1}{p}, \frac{1}{q}\},$ then one can get two equivalent inequalities as follows:

$$(4.8) \int_a^b \int_a^b \frac{f(x)g(y)}{(u(x) + u(y))^\lambda} dx dy < \tilde{k}_\lambda(p) \left\{ \int_a^b \frac{(u(x))^{p\frac{1-\lambda}{2}}}{(u'(x))^{p-1}} f^p(x) dx \right\}^{\frac{1}{p}} \times \left\{ \int_a^b \frac{(u(x))^{q\frac{1-\lambda}{2}}}{(u'(x))^{q-1}} g^q(x) dx \right\}^{\frac{1}{q}} ;$$

$$(4.9) \int_a^b \frac{u'(y)}{(u(y))^{p\frac{1-\lambda}{2}}} \left[\int_a^b \frac{f(x)}{(u(x) + u(y))^\lambda} dx \right]^p dy < [\tilde{k}_\lambda(p)]^p \int_a^b \frac{(u(x))^{p\frac{1-\lambda}{2}}}{(u'(x))^{p-1}} f^p(x) dx,$$

(ii) If $0 < p < 1, 1 - \frac{2}{p} < \lambda < 1 - \frac{2}{q},$ one can get two equivalent reversions of (4.8) and (4.9),

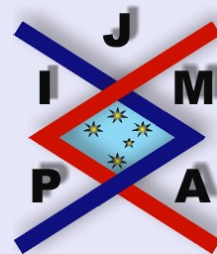
(iii) If $p < 0, 1 - \frac{2}{q} < \lambda < 1 - \frac{2}{p},$ one can get the reversion of (4.8) and inequality (4.9), which are equivalent. In particular, by (4.8),



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(a) setting $u(x) = x^\alpha$ ($\alpha > 0, x \in (0, \infty)$), one has

$$(4.10) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x^\alpha + y^\alpha)^\lambda} dx dy$$

$$< \frac{1}{\alpha} \tilde{k}_\lambda(p) \left\{ \int_0^\infty x^{p-1+\alpha(1-p\frac{1+\lambda}{2})} f^p(x) dx \right\}^{\frac{1}{p}}$$

$$\times \left\{ \int_0^\infty x^{q-1+\alpha(1-q\frac{1+\lambda}{2})} g^q(x) dx \right\}^{\frac{1}{q}};$$

(b) setting $u(x) = \ln x$, $x \in (1, \infty)$, one has

$$(4.11) \quad \int_1^\infty \int_1^\infty \frac{f(x)g(y)}{(\ln xy)^\lambda} dx dy$$

$$< \tilde{k}_\lambda(p) \left\{ \int_1^\infty x^{p-1} (\ln x)^{p\frac{1-\lambda}{2}} f^p(x) dx \right\}^{\frac{1}{p}}$$

$$\times \left\{ \int_1^\infty x^{q-1} (\ln x)^{q\frac{1-\lambda}{2}} g^q(x) dx \right\}^{\frac{1}{q}};$$

(c) setting $u(x) = e^x$, $x \in (-\infty, \infty)$, one has

$$(4.12) \quad \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{f(x)g(y)}{(e^x + e^y)^\lambda} dx dy$$

$$\langle \tilde{k}_\lambda(p) \left\{ \int_{-\infty}^{\infty} e^{(1-p\frac{1+\lambda}{2})x} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_{-\infty}^{\infty} e^{(1-q\frac{1+\lambda}{2})x} g^q(x) dx \right\}^{\frac{1}{q}} \rangle ;$$

(d) setting $u(x) = \tan x$, $x \in (0, \frac{\pi}{2})$, one has

$$(4.13) \quad \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{f(x)g(y)}{(\tan x + \tan y)^\lambda} dx dy$$

$$\langle \tilde{k}_\lambda(p) \left\{ \int_0^{\frac{\pi}{2}} \frac{\tan^{p\frac{1-\lambda}{2}} x}{\sec^{2(p-1)} x} f^p(x) dx \right\}^{\frac{1}{p}} \right.$$

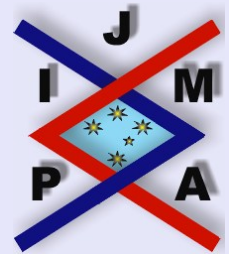
$$\left. \times \left\{ \int_0^{\frac{\pi}{2}} \frac{\tan^{q\frac{1-\lambda}{2}} x}{\sec^{2(q-1)} x} g^q(x) dx \right\}^{\frac{1}{q}} \right. ;$$

(e) setting $u(x) = \sec x - 1$, $x \in (0, \frac{\pi}{2})$, one has

$$(4.14) \quad \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{f(x)g(y)}{(\sec x + \sec y - 2)^\lambda} dx dy$$

$$\langle \tilde{k}_\lambda(p) \left\{ \int_0^{\frac{\pi}{2}} \frac{(\sec x - 1)^{p\frac{1-\lambda}{2}}}{(\sec x \tan x)^{p-1}} f^p(x) dx \right\}^{\frac{1}{p}} \right.$$

$$\left. \times \left\{ \int_0^{\frac{\pi}{2}} \frac{(\sec x - 1)^{q\frac{1-\lambda}{2}}}{(\sec x \tan x)^{q-1}} g^q(x) dx \right\}^{\frac{1}{q}} \right. .$$



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4.3. The form which does not have a reverse

Corollary 4.3. *Let the assumptions of Theorems 3.1 – 3.2 hold. For*

$$\phi_r = \frac{\lambda}{r}(r = p, q), \quad \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1, \lambda > 0,$$

$$0 < \int_a^b \frac{(u(x))^{(p-1)(1-\lambda)}}{(u'(x))^{p-1}} f^p(x) dx < \infty$$

and

$$0 < \int_a^b \frac{(u(x))^{(q-1)(1-\lambda)}}{(u'(x))^{q-1}} g^q(x) dx < \infty,$$

then one can get two equivalent inequalities as:

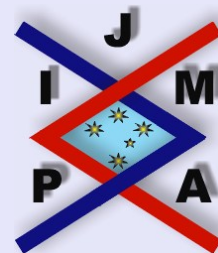
$$(4.15) \quad \int_a^b \int_a^b \frac{f(x)g(y)}{(u(x) + u(y))^\lambda} dx dy$$

$$< B \left(\frac{\lambda}{p}, \frac{\lambda}{q} \right) \left\{ \int_a^b \frac{(u(x))^{(p-1)(1-\lambda)}}{(u'(x))^{p-1}} f^p(x) dx \right\}^{\frac{1}{p}}$$

$$\times \left\{ \int_a^b \frac{(u(x))^{(q-1)(1-\lambda)}}{(u'(x))^{q-1}} g^q(x) dx \right\}^{\frac{1}{q}};$$

$$(4.16) \quad \int_a^b \frac{u'(y)}{(u(y))^{1-\lambda}} \left[\int_a^b \frac{f(x)}{(u(x) + u(y))^\lambda} dx \right]^p dy$$

$$< \left[B \left(\frac{\lambda}{p}, \frac{\lambda}{q} \right) \right]^p \int_a^b \frac{(u(x))^{(p-1)(1-\lambda)}}{(u'(x))^{p-1}} f^p(x) dx.$$



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In particular, by (4.15),

(a) setting $u(x) = x^\alpha (\alpha > 0; x \in (0, \infty))$, one has

$$(4.17) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x^\alpha + y^\alpha)^\lambda} dx dy$$

$$< \frac{1}{\alpha} B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) \left\{ \int_0^\infty x^{(p-1)(1-\alpha\lambda)} f^p(x) dx \right\}^{\frac{1}{p}}$$

$$\times \left\{ \int_0^\infty x^{(q-1)(1-\alpha\lambda)} g^q(x) dx \right\}^{\frac{1}{q}};$$

(b) setting $u(x) = \ln x, x \in (1, \infty)$, one has

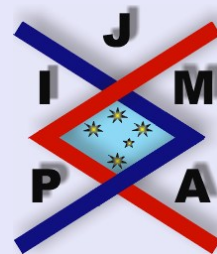
$$(4.18) \quad \int_1^\infty \int_1^\infty \frac{f(x)g(y)}{(\ln xy)^\lambda} dx dy$$

$$< B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) \left\{ \int_1^\infty x^{p-1} (\ln x)^{(p-1)(1-\lambda)} f^p(x) dx \right\}^{\frac{1}{p}}$$

$$\times \left\{ \int_1^\infty x^{q-1} (\ln x)^{(q-1)(1-\lambda)} g^q(x) dx \right\}^{\frac{1}{q}};$$

(c) setting $u(x) = e^x, x \in (-\infty, \infty)$, one has

$$(4.19) \quad \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{f(x)g(y)}{(e^x + e^y)^\lambda} dx dy$$



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$$< B \left(\frac{\lambda}{p}, \frac{\lambda}{q} \right) \left\{ \int_{-\infty}^{\infty} e^{(1-p)\lambda x} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_{-\infty}^{\infty} e^{(1-q)\lambda x} g^q(x) dx \right\}^{\frac{1}{q}};$$

(d) setting $u(x) = \tan x, x \in (0, \frac{\pi}{2})$, one has

$$(4.20) \quad \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{f(x)g(y)}{(\tan x + \tan y)^\lambda} dx dy$$

$$< B \left(\frac{\lambda}{p}, \frac{\lambda}{q} \right) \left\{ \int_0^{\frac{\pi}{2}} \frac{\tan^{(p-1)(1-\lambda)} x}{\sec^{2(p-1)} x} f^p(x) dx \right\}^{\frac{1}{p}}$$

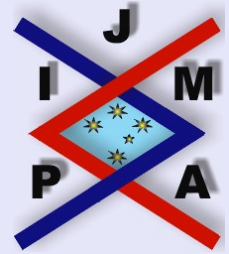
$$\times \left\{ \int_0^{\frac{\pi}{2}} \frac{\tan^{(q-1)(1-\lambda)} x}{\sec^{2(q-1)} x} g^q(x) dx \right\}^{\frac{1}{q}};$$

(e) setting $u(x) = \sec x - 1, x \in (0, \frac{\pi}{2})$, one has

$$(4.21) \quad \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{f(x)g(y)}{(\sec x + \sec y - 2)^\lambda} dx dy$$

$$< B \left(\frac{\lambda}{p}, \frac{\lambda}{q} \right) \left\{ \int_0^{\frac{\pi}{2}} \frac{(\sec x - 1)^{(p-1)(1-\lambda)}}{(\sec x \tan x)^{p-1}} f^p(x) dx \right\}^{\frac{1}{p}}$$

$$\times \left\{ \int_0^{\frac{\pi}{2}} \frac{(\sec x - 1)^{(q-1)(1-\lambda)}}{(\sec x \tan x)^{q-1}} g^q(x) dx \right\}^{\frac{1}{q}}.$$



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Remark 1. For $\alpha = 1$, (4.3) reduces to (1.5). For $\lambda = 1$, inequalities (4.3), (4.10) and (4.17) reduce to (1.7), and inequalities (4.4), (4.11) and (4.18) reduce to (1.4). It follows that inequality (3.5) is a relation between (1.4) and (1.7) (or (1.1)) with a parameter λ . Still for $\lambda = 1$, (4.5), (4.12) and (4.19) reduce to

$$(4.22) \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(x)g(y)}{e^x + e^y} dx dy$$

$$< \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\{ \int_{-\infty}^{\infty} e^{(1-p)x} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_{-\infty}^{\infty} e^{(1-q)x} g^q(x) dx \right\}^{\frac{1}{q}},$$

(4.6), (4.13) and (4.20) reduce to

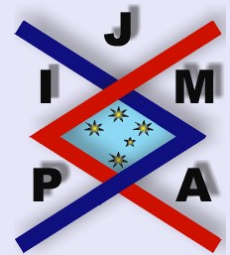
$$(4.23) \quad \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{f(x)g(y)}{\tan x + \tan y} dx dy$$

$$< \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\{ \int_0^{\frac{\pi}{2}} \cos^{2(p-1)} x f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^{\frac{\pi}{2}} \cos^{2(q-1)} x g^q(x) dx \right\}^{\frac{1}{q}},$$

and (4.7), (4.14) and (4.21) reduce to

$$(4.24) \quad \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{f(x)g(y)}{\sec x + \sec y - 2} dx dy$$

$$< \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\{ \int_0^{\frac{\pi}{2}} \left(\frac{\cos^2 x}{\sin x}\right)^{p-1} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^{\frac{\pi}{2}} \left(\frac{\cos^2 x}{\sin x}\right)^{q-1} g^q(x) dx \right\}^{\frac{1}{q}}.$$



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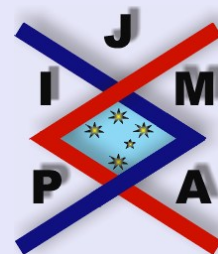
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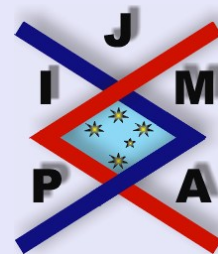
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