Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 7, Issue 3, Article 94, 2006

COEFFICIENTS OF INVERSE FUNCTIONS IN A NESTED CLASS OF STARLIKE FUNCTIONS OF POSITIVE ORDER

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Received 28 July, 2005; accepted 10 March, 2006
Communicated by H. Silverman

ABSTRACT. In the present paper we find the estimates on the n^{th} coefficients in the Maclaurin's series expansion of the inverse of functions in the class $\mathcal{S}_\delta(\alpha)$, $(0 \leq \delta < \infty, 0 \leq \alpha < 1)$, consisting of analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in the open unit disc and satisfying $\sum_{n=2}^{\infty} n^\delta \left(\frac{n-\alpha}{1-\alpha}\right)|a_n| \leq 1$. For each n these estimates are sharp when α is close to $zero$ or one and δ is close to $zero$. Further for the second, third and fourth coefficients the estimates are sharp for every admissible values of α and δ.

Key words and phrases: Univalent functions, Starlike functions of order α, Convex functions of order α, Inverse functions, Coefficient estimates.

2000 *Mathematics Subject Classification.* 30C45.

1. INTRODUCTION

Let $\mathcal{U}$ denote the *open* unit disc in the complex plane

$$\mathcal{U} := \{z \in \mathbb{C} : |z| < 1\}.$$

Let $\mathcal{S}$ be the class of *normalized analytic univalent* functions in $\mathcal{U}$ i.e. f is in $\mathcal{S}$ if f is one to one in $\mathcal{U}$, analytic and

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n; \qquad (z \in \mathcal{U}). \tag{1.1}$$

ISSN (electronic): 1443-5756

The present investigation is partially supported by National Board For Higher Mathematics, Department of Atomic Energy, Government of India. Sanction No. 48/2/2003-R&D-II/844.

227-05

The function $f \in \mathcal{S}$ is said to be in $\mathcal{S}^*(\alpha)$ $(0 \le \alpha < 1)$, the class of univalent *starlike functions of order* α, if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \qquad (z \in \mathcal{U})$$

and f is said to be in the class $\mathcal{CV}(\alpha)$ of univalent convex functions of order α if $zf' \in \mathcal{S}^*(\alpha)$. The linear mapping $f \to zf'$ is popularly known as the *Alexander transformation*. A well known sufficient condition, for the function f of the form (1.1) to be in the class $\mathcal{S}$, is

$$\sum_{n=2}^{\infty} n|a_n| \le 1 \qquad \text{(see e.g. [17, p. 212]).} \tag{1.2}$$

In fact, (1.2) is sufficient for f to be in the smaller class $\mathcal{S}^*(0) \equiv \mathcal{S}^*$ (see e.g [4]). An analogous sufficient condition for $\mathcal{S}^*(\alpha)$ $(0 \le \alpha < 1)$ is

$$\sum_{n=2}^{\infty} \left(\frac{n-\alpha}{1-\alpha}\right)|a_n| \le 1 \qquad \text{(see [15]).} \tag{1.3}$$

The Alexander transformation gives that

$$\sum_{n=2}^{\infty} n\left(\frac{n-\alpha}{1-\alpha}\right)|a_n| \le 1 \tag{1.4}$$

is a sufficient condition for f to be in $\mathcal{CV}(\alpha)$. We recall the following:

Definition 1.1 ([8, 12])**.** The function f given by the series (1.1) is said to be in the class $\mathcal{S}_\delta(\alpha)$ $(0 \le \alpha < 1, -\infty < \delta < \infty)$ if

$$\sum_{n=2}^{\infty} n^\delta\left(\frac{n-\alpha}{1-\alpha}\right)|a_n| \le 1 \tag{1.5}$$

is satisfied.

For each fixed n the function n^δ is increasing with respect to δ. Thus it follows that if $\delta_1 < \delta_2$, then $\mathcal{S}_{\delta_2}(\alpha) \subset \mathcal{S}_{\delta_1}(\alpha)$. Consequently, by (1.3), the functions in $\mathcal{S}_\delta(\alpha)$ are univalent starlike of order α if $\delta \ge 0$ and further if $\delta \ge 1$, then by (1.4), $\mathcal{S}_\delta(\alpha)$ contains only univalent convex functions of order α. Also we know (see e.g. [12, p. 224]) that if $\delta < 0$ then the class $\mathcal{S}_\delta(\alpha)$ contains non-univalent functions as well. Basic properties of the class $\mathcal{S}_\delta(\alpha)$ have been studied in [8, 11, 12, 13]. We also note that if $f \in \mathcal{S}_\delta(\alpha)$ then

$$|a_n| \le \frac{(1-\alpha)}{n^\delta(n-\alpha)}; \qquad (n = 2, 3, \dots)$$

and equality holds for each n only for functions of the form

$$f_n(z) = z + \frac{(1-\alpha)}{n^\delta(n-\alpha)}e^{i\theta}z^n, \qquad (\theta \in \mathbb{R}).$$

We shall use this estimate in our investigation.

The inverse f^{-1} of every function $f \in \mathcal{S}$, defined by $f^{-1}(f(z)) = z$, is analytic in $|w| < r(f), (r(f) \ge \frac{1}{4})$ and has Maclaurin's series expansion

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} b_n w^n \qquad \Big(|w| < r(f)\Big). \tag{1.6}$$

The De-Branges theorem [2], previously known as the Bieberbach conjecture; states that if the function f in $\mathcal{S}$ is given by the power series (1.1) then $|a_n| \le n$ $(n = 2, 3, \dots)$ with equality for each n only for the rotations of the Koebe function $\frac{z}{(1-z)^2}$. Early in 1923 Löwner [10] invented the famous parametric method to prove the Bieberbach conjecture for the third coefficient (i.e. $|a_3| \leq 3,\ f \in \mathcal{S}$). Using this method he also found sharp bounds on all the coefficients for the inverse functions in $\mathcal{S}$ (or $\mathcal{S}^*$). Thus, if $f \in \mathcal{S}$ (or $\mathcal{S}^*$) and f^{-1} is given by (1.6) then

$$|b_n| \leq \frac{1}{n+1}\binom{2n}{n};\ (n = 2, 3, \dots) \text{ (cf [10]; also see [5, p. 222])}$$

with equality for every n for the inverse of the Koebe function $k(z) = z/(1+z)^2$. Although the coefficient estimate problem for inverse functions in the whole class $\mathcal{S}$ was completely solved in early part of the last century; for certain subclasses of $\mathcal{S}$ only partial results are available in literature. For example, if $f \in \mathcal{S}^*(\alpha)$, $(0 \leq \alpha < 1)$ then the sharp estimates

$$|b_2| \leq 2(1-\alpha)$$

and

$$|b_3| \leq \begin{cases} (1-\alpha)(5-6\alpha); & 0 \leq \alpha \leq \frac{2}{3} \\ 1-\alpha; & \frac{2}{3} \leq \alpha < 1 \end{cases} \qquad \text{(cf. [7])}$$

hold. Further, if $f \in \mathcal{CV}$ then $|b_n| \leq 1\ (n = 2, 3, \dots, 8)$ (cf. [1, 9]), while $|b_{10}| > 1$ [6]. However the problem of finding sharp bounds for b_n for $f \in \mathcal{S}^*(\alpha)\ (n \geq 4)$ and for $f \in \mathcal{CV}\ (n \geq 9)$ still remains open.

The object of the present paper is to study the coefficient estimate problem for the inverse of functions in the class $\mathcal{S}_\delta(\alpha)$; $(\delta \geq 0, 0 \leq \alpha < 1)$. We find sharp bounds for $|b_2|$, $|b_3|$ and $|b_4|$ for $f \in S_\delta(\alpha)$ $(0 \leq \alpha < 1$ and $\delta \geq 0)$. We further show that for every positive integer $n \geq 2$ there exist positive real numbers ε_n, δ_n and t_n such that for every $f \in \mathcal{S}_\delta(\alpha)$ the following sharp estimates hold:

$$|b_n| \leq \begin{cases} \frac{2}{n2^{(n-1)\delta}}\binom{2n-3}{n-2}\left(\frac{1-\alpha}{2-\alpha}\right)^{n-1}; & (0 \leq \alpha \leq \varepsilon_n, 0 \leq \delta \leq \delta_n) \\ \frac{1-\alpha}{n^\delta(n-\alpha)}; & (1-t_n \leq \alpha < 1, \delta > 0). \end{cases} \tag{1.7}$$

For each $n = 2, 3, \dots,$ there are two different extremal functions; in contrast to only one extremal function for every n for the whole class $\mathcal{S}$ $(or\ \mathcal{S}^*(0))$. We also obtain crude estimates for $|b_n|$ $(n = 2, 3, 4, \dots;\ 0 \leq \alpha < 1, \delta > 0;\ f \in \mathcal{S}_\delta(\alpha))$. Our investigation includes some results of Silverman [16] for the case $\delta = 0$ and provides new information for $\delta > 0$.

2. NOTATIONS AND PRELIMINARY RESULTS

Let the function s given by the power series

$$s(z) = 1 + d_1 z + d_2 z^2 + \cdots \tag{2.1}$$

be analytic in a neighbourhood of the origin. For a real number p define the function h by

$$h(z) = (s(z))^p = (1 + d_1 z + d_2 z^2 + \cdots)^p = 1 + \sum_{k=1}^{\infty} C_k^{(p)} z^k. \tag{2.2}$$

Thus $C_k^{(p)}$ denotes the k^{th} coefficient in the Maclaurin's series expansion of the p^{th} power of the function $s(z)$. We need the following:

Lemma 2.1 ([14]). *Let the coefficients $C_k^{(p)}$ be defined as in (2.2), then*

$$C_{k+1}^{(p)} = \sum_{j=0}^{k}\left[p - \frac{(p+1)j}{k+1}\right] d_{k+1-j} C_j^{(p)}; \qquad (k = 0, 1, \dots;\ C_0^{(p)} = 1). \tag{2.3}$$

Lemma 2.2 ([16])**.** *If k and n are positive integers with $k \leq n-2$, then*

$$A_j = \binom{n+j-1}{j}\left(\frac{n(k+1-j)+j}{2^j(k+2-j)}\right)$$

is a strictly increasing function of j, $j = 1, 2, \dots, k$.

Lemma 2.3. *Let k and n be positive integers with $k \leq n-2$. Write*

$$A_j(\alpha,\delta) = \frac{(1-\alpha)}{2^{j\delta}}\binom{n+j-1}{j}\frac{(n(k+1-j)+j)}{(k+2-j)^\delta(k+2-j-\alpha)}\left(\frac{1-\alpha}{2-\alpha}\right)^j, \quad (0 \leq \alpha < 1,\ \delta > 0).$$

Then for each n there exist positive real numbers ε_n and δ_n such that $A_j(\alpha,\delta)$ is strictly increasing in j for $0 \leq \alpha < \varepsilon_n$, $0 \leq \delta < \delta_n$ and $j = 1, 2, \dots, k$.

Proof. Write

$$\begin{aligned} h_j(\alpha,\delta) &= A_{j+1}(\alpha,\delta) - A_j(\alpha,\delta) \\ &= \frac{(1-\alpha)^{j+1}}{2^{j\delta}(2-\alpha)^j}\binom{n+j-1}{j}\left[\frac{(n+j)(n(k-j)+(j+1))(1-\alpha)}{2^\delta(j+1)(k+1-j)^\delta(k+1-j-\alpha)(2-\alpha)}\right. \\ &\qquad \left. - \frac{(n(k+1-j)+j)}{(k+2-j)^\delta(k+2-j-\alpha)}\right]. \end{aligned}$$

We observe that for each fixed j $(j = 1, 2, \dots, k-1)$ $h_j(\alpha,\delta)$ is a continuous function of (α,δ). Also $\lim_{(\alpha,\delta)\to(0,0)} h_j(\alpha,\delta) = h_j(0,0) = A_{j+1}(0,0) - A_j(0,0) > 0$ by Lemma 2.2. Thus there exists an open circular disc $B(0, r_j)$ with center at $(0,0)$ and radius $r_j > 0$ such that $h_j(\alpha,\delta) > 0$ for $(\alpha,\delta) \in B(0,r_j)$ for each $j = 1, 2, \dots, k-1$. Consequently, $h_j(\alpha,\delta) > 0$ for all j $(j = 1, 2, \dots, k-1)$ and $(\alpha,\delta) \in B(0,r)$, where $r = \min_{1\leq j\leq k-1} r_j$. If we choose $\varepsilon_n = \delta_n = \frac{\sqrt{2}}{2}r$, then $A_j(\alpha,\delta)$ is strictly increasing in j for $0 \leq \alpha < \varepsilon_n$, $0 \leq \delta < \delta_n$ and $j = 1, 2, \dots, k$. The proof of Lemma 2.3 is complete. □

3. MAIN RESULTS

We have the following:

Theorem 3.1. *Let the function f, given by the series* (1.1) *be in $\mathcal{S}_\delta(\alpha)$ $(0 \leq \alpha < 1,\ 0 \leq \delta < \infty)$. Write*

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} b_n w^n, \qquad (|w| < r_0(f)) \tag{3.1}$$

for some $r_0(f) \geq \frac{1}{4}$. Then

(a)

$$|b_2| \leq \frac{(1-\alpha)}{2^\delta(2-\alpha)}; \qquad (0 \leq \alpha < 1,\ 0 \leq \delta < \infty). \tag{3.2}$$

Set

$$\delta_0 = \frac{\log 3 - \log 2}{\log 4 - \log 3} \qquad \textit{and} \qquad \delta_1 = \frac{\log 5}{\log 2} - 1. \tag{3.3}$$

(b) (i) *If* $0 \le \delta \le \delta_0$, *then*

$$|b_3| \le \begin{cases} \frac{2(1-\alpha)^2}{2^{2\delta}(2-\alpha)^2}; & (0 \le \alpha \le \alpha_0), \\ \frac{(1-\alpha)}{3^\delta(3-\alpha)}; & (\alpha_0 \le \alpha < 1), \end{cases} \tag{3.4}$$

where α_0 *is the only root, in the interval* $0 \le \alpha < 1$, *of the equation*

$$(2 \cdot 3^\delta - 2^{2\delta})\alpha^2 - 4(2 \cdot 3^\delta - 2^{2\delta})\alpha + (6 \cdot 3^\delta - 4 \cdot 2^{2\delta}) = 0. \tag{3.5}$$

(ii) *Further, if* $\delta > \delta_0$, *then*

$$|b_3| \le \frac{(1-\alpha)}{3^\delta(3-\alpha)}; \qquad (0 \le \alpha < 1). \tag{3.6}$$

(c) (i) *If* $0 \le \delta \le \delta_1$, *then*

$$|b_4| \le \begin{cases} \frac{5(1-\alpha)^3}{2^{3\delta}(2-\alpha)^3}; & (0 \le \alpha < \alpha_1), \\ \frac{(1-\alpha)}{4^\delta(4-\alpha)}; & (\alpha_1 \le \alpha < 1), \end{cases} \tag{3.7}$$

where α_1 *is the only root in the interval* $0 \le \alpha < 1$, *of the equation*

$$(2^{3\delta} - 5 \cdot 4^\delta)\alpha^3 - 6(2^{3\delta} - 5 \cdot 4^\delta)\alpha^2 - 3(15 \cdot 4^\delta - 4 \cdot 2^{3\delta})\alpha + 4(5 \cdot 4^\delta - 2 \cdot 2^{3\delta}) = 0. \tag{3.8}$$

(ii) *If* $\delta > \delta_1$, *then*

$$|b_4| \le \frac{(1-\alpha)}{4^\delta(4-\alpha)}; \qquad (0 \le \alpha < 1). \tag{3.9}$$

All the estimates are sharp.

Proof. We know from [7] that

$$b_n = \frac{1}{2\pi i n} \int_{|z|=r} \left[\frac{1}{f(z)}\right]^n dz.$$

For fixed n write

$$h(z) = \left[\frac{z}{f(z)}\right]^n = \frac{1}{\left(1 + \sum_{k=2}^{\infty} a_k z^{k-1}\right)^n} = 1 + \sum_{k=1}^{\infty} C_k^{(-n)} z^k.$$

Thus

$$nb_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{h(z)}{z^n} dz = \frac{h^{(n-1)}(0)}{(n-1)!} = C_{n-1}^{(-n)}.$$

Therefore a function, which maximizes $\left|C_{n-1}^{(-n)}\right|$ will also maximize $|b_n|$. Now write $w(z) = -\sum_{k=2}^{\infty} a_k z^{k-1}$ and $h(z) = (1 + w(z) + w^2(z) + \cdots)^n$, $(z \in \mathcal{U})$. It follows that all the coefficients in the expansion of $h(z)$ shall be nonnegative if $f(z)$ is of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \qquad (a_k \ge 0\,; k = 2, 3, \dots). \tag{3.10}$$

Consequently, $\max_{f \in \mathcal{S}_\delta(\alpha)} \left|C_{n-1}^{(-n)}\right|$ must occur for a function in $\mathcal{S}_\delta(\alpha)$ with the representation (3.10).

(a) Now

$$\left(\frac{z}{f(z)}\right)^2 = \left(1 - \sum_{k=2}^{\infty} a_k z^{k-1}\right)^{-2} = 1 + 2a_2 z + \cdots .$$

Therefore

$$C_1^{(-2)} = 2a_2 = \frac{2(1-\alpha)}{2^\delta(2-\alpha)}\lambda_2; \qquad (0 \le \lambda_2 \le 1, 0 \le \alpha < 1, 0 \le \delta < 1)$$

and the maximum $C_1^{(-2)}$ is obtained by replacing $\lambda_2 = 1$. Equivalently

$$|b_2| = \frac{C_1^{(-2)}}{2} \le \frac{1-\alpha}{2^\delta(2-\alpha)}; \qquad (0 \le \alpha < 1,\ 0 \le \delta < \infty).$$

We get (3.2). To show that equality holds in (3.2), consider the function $f_2(z)$ defined by

(3.11) $$f_2(z) = z - \frac{(1-\alpha)}{2^\delta(2-\alpha)} z^2; \qquad (z \in \mathcal{U},\ 0 \le \alpha < 1,\ 0 \le \delta < \infty).$$

For this function

$$\left(\frac{z}{f_2(z)}\right)^2 = 1 + \frac{2(1-\alpha)}{2^\delta(2-\alpha)} z + \cdots = 1 + C_1^{(-2)} z + \cdots$$

and

$$|b_2| = \frac{C_1^{(-2)}}{2} = \frac{(1-\alpha)}{2^\delta(2-\alpha)}.$$

The proof of (a) is complete.

To find sharp estimates for $|b_3|$, we consider

$$h(z) = \left(\frac{z}{f(z)}\right)^3 = (1 - a_2 z - a_3 z^2 - \cdots)^{-3} = 1 + \sum_{k=1}^{\infty} C_k^{(-3)} z^k.$$

By direct calculation or by taking $p = -3, d_k = -a_{k+1}$ in Lemma 2.1,we get,

(3.12) $$C_1^{(-3)} = 3a_2 \quad \text{and} \quad C_2^{(-3)} = 3a_3 + 2a_2 C_1^{(-3)} = 3a_3 + 6a_2^2.$$

Substituting $a_2 = \frac{(1-\alpha)\lambda_2}{2^\delta(2-\alpha)}$ and $a_3 = \frac{(1-\alpha)\lambda_3}{3^\delta(3-\alpha)}$, $(0 \le \lambda_2, \lambda_3 \le 1, \lambda_2 + \lambda_3 \le 1)$ in the equation (3.12) we obtain

$$C_2^{(-3)} = \frac{3(1-\alpha)}{3^\delta(3-\alpha)}\lambda_3 + \frac{6(1-\alpha)^2}{2^{2\delta}(2-\alpha)^2}\lambda_2^2.$$

Equivalently

(3.13) $$\frac{C_2^{(-3)}}{3} = (1-\alpha)\left\{\frac{\lambda_3}{3^\delta(3-\alpha)} + \frac{2(1-\alpha)\lambda_2^2}{2^{2\delta}(2-\alpha)^2}\right\}.$$

In order to maximize the right hand side of (3.13), write

$$G(\lambda_2, \lambda_3) = \frac{\lambda_3}{3^\delta(3-\alpha)} + \frac{2(1-\alpha)\lambda_2^2}{2^{2\delta}(2-\alpha)^2}; \qquad (0 \le \lambda_2 \le 1, 0 \le \lambda_3 \le 1, \lambda_2 + \lambda_3 \le 1).$$

The function $G(\lambda_2, \lambda_3)$ does not have a maximum in the interior of the square $\{(\lambda_2, \lambda_3) : 0 < \lambda_2 < 1, 0 < \lambda_3 < 1\}$, since $G_{\lambda_2} \ne 0, G_{\lambda_3} \ne 0$. Also if $\lambda_3 = 1$ then $\lambda_2 = 0$ and if $\lambda_2 = 1$ then $\lambda_3 = 0$. Therefore

$$\max_{\lambda_3=1} G(\lambda_2, \lambda_3) = \frac{1}{3^\delta(3-\alpha)} \quad \text{and} \quad \max_{\lambda_2=1} G(\lambda_2, \lambda_3) = \frac{2(1-\alpha)}{2^{2\delta}(2-\alpha)^2}.$$

Also

$$\max_{\lambda_3=0} G(\lambda_2,\lambda_3) = \frac{2(1-\alpha)}{2^{2\delta}(2-\alpha)^2} \quad \text{and} \quad \max_{\lambda_2=0} G(\lambda_2,\lambda_3) = \frac{1}{3^{\delta}(3-\alpha)}.$$

We get

$$\max_{\substack{0\le\lambda_2\le 1\\ 0\le\lambda_3\le 1}} G(\lambda_2,\lambda_3) = \max\left\{\frac{1}{3^{\delta}(3-\alpha)},\ \frac{2(1-\alpha)}{2^{2\delta}(2-\alpha)^2}\right\}.$$

Thus

$$\frac{C_3^{(-2)}}{3} \le (1-\alpha)\max\left\{\frac{1}{3^{\delta}(3-\alpha)},\ \frac{2(1-\alpha)}{2^{2\delta}(2-\alpha)^2}\right\}.$$

We now find the maximum of the above two terms. Note that the sign of the expression

$$\frac{1}{3^{\delta}(3-\alpha)} - \frac{2(1-\alpha)}{2^{2\delta}(2-\alpha)^2} = \frac{-F(\alpha)}{2^{2\delta}3^{\delta}(3-\alpha)(2-\alpha)^2}$$

depends on the sign of the quadratic polynomial $F(\alpha) = a(\delta)\alpha^2 - 4a(\delta)\alpha + c(\delta)$, where $a(\delta) = 3^{\delta}\cdot 2 - 2^{2\delta}$ and $c(\delta) = 2(3^{\delta+1} - 2^{2\delta+1})$. Observe that

$$a(\delta)\begin{cases} \ge 0 & \text{if } \delta \le \delta_0^* \\ < 0 & \text{if } \delta > \delta_0^* \end{cases}; \qquad \left(\delta_0^* = \frac{\log 2}{\log 4 - \log 3}\right)$$

$$c(\delta)\begin{cases} \ge 0 & \text{if } \delta \le \delta_0 \\ < 0 & \text{if } \delta > \delta_0 \end{cases}; \qquad \left(\delta_0 = \frac{\log 3 - \log 2}{\log 4 - \log 3}\right)$$

and $\delta_0 \le \delta_0^*$.

(b) (i) *The case* $0 \le \delta \le \delta_0$: Suppose $0 \le \delta \le \delta_0$ then $F(0) = c(\delta) \ge 0$, $F(1) = -2^{2\delta} < 0$ and since $a(\delta) \ge 0,\ F(\alpha)$ is positive for large values of α. Therefore $F(\alpha) \ge 0$ if $0 \le \alpha \le \alpha_0$ and $F(\alpha) < 0$ if $\alpha_0 < \alpha < 1$ where α_0 is the unique root of equation $F(\alpha) = 0$ in the interval $0 \le \alpha < 1$. Or equivalently $-F(\alpha) \le 0$ for $0 < \alpha \le \alpha_0$ and $-F(\alpha) > 0$ for $\alpha_0 < \alpha < 1$. Consequently,

$$|b_3| = \frac{C_2^{(-3)}}{3} \le \begin{cases} \frac{2(1-\alpha)^2}{2^{2\delta}(2-\alpha)^2}; & (0 \le \alpha \le \alpha_0); \\ \frac{(1-\alpha)}{3^{\delta}(3-\alpha)}; & (\alpha_0 \le \alpha < 1). \end{cases}$$

We get the estimate (3.4).

(ii) *The case* $\delta_0 < \delta$: We show below that if $\delta_0 < \delta \le \delta_0^*$ or $\delta_0^* < \delta$ then $F(\alpha) < 0$. Suppose $\delta_0 < \delta \le \delta_0^*$, then $a(\delta) \ge 0$. Consequently, $F(\alpha) > 0$ for large positive and negative values of α. Also $F(0) = c(\delta) < 0$ and $F(1) = -2^{2\delta} < 0$. Therefore $F(\alpha) < 0$ for every α in the real interval $0 \le \alpha < 1$. Similarly, if $\delta_0^* < \delta$, then $a(\delta) < 0$. Thus $F'(\alpha) = 2a(\delta)(\alpha - 2) > 0;\ (0 \le \alpha < 1)$. Or equivalently $F(\alpha)$ is an increasing function in $0 \le \alpha < 1$. Also $F(1) = -2^{2\delta} < 0$. Therefore $F(\alpha) < 0$ in $0 \le \alpha < 1$.

Since $-F(\alpha) > 0$ we have

$$|b_3| = \frac{C_2^{(-3)}}{3} \le \frac{(1-\alpha)}{3^{\delta}(3-\alpha)} \qquad (0 \le \alpha < 1;\ \delta > \delta_0).$$

This is precisely the estimate (3.6). We note that for the function $f_2(z)$ defined by (3.11)

$$\left(\frac{z}{f_2(z)}\right)^3 = \left(1-\frac{(1-\alpha)}{2^\delta(2-\alpha)}z\right)^{-3} = 1+\frac{3(1-\alpha)}{2^\delta(2-\alpha)}z+\frac{6(1-\alpha)^2}{2^{2\delta}(2-\alpha)^2}z^2+\cdots .$$

Therefore

$$|b_3| = \frac{C_2^{(-3)}}{3} = \frac{2(1-\alpha)^2}{2^{2\delta}(2-\alpha)^2}.$$

We get sharpness in (3.4) with $0 \le \alpha < \alpha_0$. Similarly for the function $f_3(z)$ defined by

$$f_3(z) = z-\frac{(1-\alpha)}{3^\delta(3-\alpha)}z^3; \qquad (z\in\mathcal{U}, 0\le\alpha<1, 0\le\delta<\infty), \tag{3.14}$$

we have

$$\left(\frac{z}{f_3(z)}\right)^3 = \left(1-\frac{(1-\alpha)}{3^\delta(3-\alpha)}z^2\right)^{-3} = 1+\frac{3(1-\alpha)}{3^\delta(3-\alpha)}z^2+\cdots$$

$$|b_3| = \frac{C_2^{(-3)}}{3} = \frac{(1-\alpha)}{3^\delta(3-\alpha)}.$$

This establishes the sharpness of (3.4) with $\alpha_0 \le \alpha < 1$ and (3.6). The proof of (b) is complete.

In order to find sharp estimates for $|b_4|$, we consider the function

$$h(z) = \left(\frac{z}{f(z)}\right)^4 = \left(1-\sum_{k=2}^{\infty}a_k z^{k-1}\right)^{-4} = 1+\sum_{k=1}^{\infty}C_k^{(-4)}z^k.$$

Taking $p=-4$ and $d_k = -a_{k+1}$ in Lemma 2.1, we get

$$C_1^{(-4)} = 4a_2; \qquad C_2^{(-4)} = 4a_3+10a_2^2; \qquad C_3^{(-4)} = 4a_4+20a_2a_3+20a_2^3.$$

Taking $a_2 = \frac{(1-\alpha)}{2^\delta(2-\alpha)}\lambda_2$, $a_3 = \frac{(1-\alpha)}{3^\delta(3-\alpha)}\lambda_3$ and $a_4 = \frac{(1-\alpha)}{4^\delta(4-\alpha)}\lambda_4$, where $0 \le \lambda_2,\lambda_3,\lambda_4 \le 1$ and $\lambda_2+\lambda_3+\lambda_4 \le 1$ we get

$$\begin{aligned}|b_4| &= \frac{C_3^{(-4)}}{4}\\ &= (1-\alpha)\left\{\frac{\lambda_4}{4^\delta(4-\alpha)}+\frac{5(1-\alpha)\lambda_2\lambda_3}{2^\delta3^\delta(2-\alpha)(3-\alpha)}+\frac{5(1-\alpha)^2\lambda_2^3}{2^{3\delta}(2-\alpha)^3}\right\}\\ &= (1-\alpha)L(\lambda_2,\lambda_3,\lambda_4) \qquad \text{(say)}.\end{aligned}$$

Since $L_{\lambda_2} \ne 0, L_{\lambda_3} \ne 0$ and $L_{\lambda_4} \ne 0$, the function L cannot have a local maximum in the interior of cube $0<\lambda_2<1, 0<\lambda_3<1, 0<\lambda_4<1$. Therefore the constraint $\lambda_2+\lambda_3+\lambda_4 \le 1$ becomes $\lambda_2+\lambda_3+\lambda_4 = 1$. Hence putting $\lambda_4 = 1-\lambda_2-\lambda_3$ we get

$$\begin{aligned}|b_4| &= \frac{C_3^{(-4)}}{4}\\ &= (1-\alpha)\left\{\frac{1-\lambda_2-\lambda_3}{4^\delta(4-\alpha)}+\frac{5(1-\alpha)\lambda_2\lambda_3}{2^\delta3^\delta(2-\alpha)(3-\alpha)}+\frac{5(1-\alpha)^2\lambda_2^3}{2^{3\delta}(2-\alpha)^3}\right\}\\ &= (1-\alpha)H(\lambda_2,\lambda_3) \qquad \text{(say)}.\end{aligned}$$

Thus we need to maximize $H(\lambda_2, \lambda_3)$ in the closed square $0 \le \lambda_2 \le 1$, $0 \le \lambda_3 \le 1$. Since

$$H_{\lambda_2\lambda_2} \cdot H_{\lambda_3\lambda_3} - (H_{\lambda_2\lambda_3})^2 = -\left(\frac{5(1-\alpha)}{2^\delta 3^\delta (2-\alpha)(3-\alpha)}\right)^2 < 0$$

the function H cannot have a local maximum in the interior of the square $0 \le \lambda_2 \le 1$, $0 \le \lambda_3 \le 1$. Further, if $\lambda_2 = 1$ then $\lambda_3 = 0$ and if $\lambda_3 = 1$ then $\lambda_2 = 0$. Therefore

$$\max_{\lambda_2=1} H(\lambda_2, \lambda_3) = H(1,0) = \frac{5(1-\alpha)^2}{2^{3\delta}(2-\alpha)^3},$$

$$\max_{\lambda_3=1} H(\lambda_2, \lambda_3) = H(0,1) = 0,$$

$$\begin{aligned}\max_{0<\lambda_2<1} H(\lambda_2, 0) &= \max\left\{\frac{1-\lambda_2}{4^\delta(4-\alpha)} + \frac{5(1-\alpha)^2\lambda_2^3}{2^{3\delta}(2-\alpha)^3}\right\} \\ &= \max\left\{\frac{1}{4^\delta(4-\alpha)}, \frac{5(1-\alpha)^2}{2^{3\delta}(2-\alpha)^3}\right\}\end{aligned}$$

and

$$\max_{0\le\lambda_3\le1} H(0, \lambda_3) = \max_{0\le\lambda_3\le1} \frac{1-\lambda_3}{4^\delta(4-\alpha)} = \frac{1}{4^\delta(4-\alpha)}.$$

Thus

$$\max_{\substack{0\le\lambda_2\le1 \\ 0\le\lambda_3\le1}} H(\lambda_2, \lambda_3) = \max\left\{\frac{1}{4^\delta(4-\alpha)}, \frac{5(1-\alpha)^2}{2^{3\delta}(2-\alpha)^3}\right\}.$$

The maximum of the above two terms can be found as in the case for $|b_3|$. We see that the sign of the expression

$$\frac{5(1-\alpha)^2}{2^{3\delta}(2-\alpha)^3} - \frac{1}{4^\delta(4-\alpha)}$$

is same as the sign of the cubic polynomial $P(\alpha) = a(\delta)\alpha^3 - 6a(\delta)\alpha^2 - 3b(\delta)\alpha + 4c(\delta)$, where $a(\delta) = 2^{3\delta} - 5 \cdot 4^\delta$, $b(\delta) = 15 \cdot 4^\delta - 4 \cdot 2^{3\delta}$ and $c(\delta) = 5 \cdot 4^\delta - 2 \cdot 2^{3\delta}$. We observe that

$$c(\delta)\begin{cases} \ge 0 & \text{if } \delta \le \delta_1 \\ < 0 & \text{if } \delta > \delta_1 \end{cases}; \qquad \left(\delta_1 = \frac{\log 5}{\log 2} - 1\right),$$

$$b(\delta)\begin{cases} \ge 0 & \text{if } \delta \le \delta_2 \\ < 0 & \text{if } \delta > \delta_2 \end{cases}; \qquad \left(\delta_2 = \delta_1 + \frac{\log 3}{\log 2} - 1\right)$$

and

$$a(\delta)\begin{cases} \le 0 & \text{if } \delta \le \delta_3 \\ > 0 & \text{if } \delta > \delta_3 \end{cases}; \qquad \left(\delta_3 = \frac{\log 5}{\log 2}\right).$$

Moreover, $\delta_1 < \delta_2 < \delta_3$. Also the quadratic polynomial $P'(\alpha) = 3\Big(a(\delta)\alpha^2 - 4a(\delta)\alpha - b(\delta)\Big)$ has roots at $2 \pm \sqrt{4 + \frac{b}{a}}$.

(c) (i) *The case* $0 \le \delta \le \delta_1$: In this case $c(\delta) \ge 0$, $b(\delta) \ge 0$ and $a(\delta) \le 0$. Note that both the roots of $P'(\alpha)$ are complex numbers and $P'(0) = -3b(\delta) \le 0$. Therefore $P'(\alpha) < 0$ for every real number and consequently, $P'(\alpha)$ is a decreasing function. Since $P(0) = 4c(\delta) \ge 0$ and $P(1) = -2^{3\delta} < 0$, the function $P(\alpha)$ has a unique root α_1 in the interval $0 < \alpha < 1$. Or equivalently, $P(\alpha) \geq 0$ for $0 < \alpha \leq \alpha_1$ and $P(\alpha) < 0$ if $\alpha_1 < \alpha < 1$. Thus

$$|b_4| \leq \begin{cases} \frac{5(1-\alpha)^3}{2^{3\delta}(2-\alpha)^3}; & (0 \leq \alpha \leq \alpha_1), \\ \\ \frac{(1-\alpha)}{4^\delta(4-\alpha)}; & (\alpha_1 \leq \alpha < 1). \end{cases}$$

We get the estimate (3.7).

(ii) *The case* $\delta > \delta_1$: We shall show below, separately, that if $\delta_1 < \delta \leq \delta_2$ or $\delta_2 < \delta \leq \delta_3$ or $\delta_3 < \delta$ then $P(\alpha) < 0$ in $0 \leq \alpha < 1$.
First suppose that $\delta_1 < \delta \leq \delta_2$. Then $c(\delta) < 0$, $b(\delta) \geq 0$ and $a(\delta) < 0$. Thus, as in case of (c)(i), $P'(\alpha)$ has only complex roots and $P'(0) < 0$. Therefore $P(\alpha)$ is a monotonic decreasing function in $0 \leq \alpha < 1$. Since $P(0) < 0$, we get that $P(\alpha) < 0$ for $0 \leq \alpha < 1$.
Next if $\delta_2 < \delta \leq \delta_3$, then $c(\delta) < 0$, $b(\delta) < 0$ and $a(\delta) < 0$. Therefore, $P'(\alpha)$ has two real roots: one is negative and the other is greater than 2. The condition $P'(0) > 0$ gives that $P'(\alpha) > 0$ in $0 \leq \alpha < 1$. Therefore $P(\alpha)$ is a monotonic increasing function in $0 \leq \alpha < 1$. Since $P(1) = -2^{3\delta} < 0$, we get that $P(\alpha) < 0$ in $0 \leq \alpha < 1$.
Lastly, if $\delta > \delta_3$ then $c(\delta) < 0$, $b(\delta) < 0$ and $a(\delta) > 0$. Hence $P'(\alpha)$ has only complex roots and the condition $P'(0) = -3b(\delta) > 0$ gives $P'(\alpha) > 0$ for every real α. Consequently $P(\alpha)$ is a monotonic increasing function. Since $P(1) < 0$, we get that $P(\alpha) < 0$ in $0 \leq \alpha < 1$.
Since $P(\alpha) < 0$ for $0 \leq \alpha < 1$, we have

$$|b_4| \leq \frac{(1-\alpha)}{4^\delta(4-\alpha)}; \qquad (0 \leq \alpha < 1).$$

This is precisely the estimate (3.9). We note that for the function $f_2(z)$ defined by (3.11)

$$\left(\frac{z}{f_2(z)}\right)^4 = 1 + \frac{4(1-\alpha)}{2^\delta(2-\alpha)}z + \frac{20(1-\alpha)^2}{2.2^{2\delta}(2-\alpha)^2}z^2 + \frac{20(1-\alpha)^3}{2^{3\delta}(2-\alpha)^3}z^3 + \cdots.$$

Therefore

$$|b_4| = \frac{C_3^{(-4)}}{4} = \frac{5(1-\alpha)^3}{2^{3\delta}(2-\alpha)^3}.$$

This shows sharpness of the estimate (3.7) with $0 \leq \alpha \leq \alpha_1$. Similarly, for the function $f_4(z)$ defined by

$$f_4(z) = z - \frac{(1-\alpha)}{4^\delta(4-\alpha)}z^4; \qquad (z \in \mathcal{U}, 0 \leq \alpha < 1, 0 \leq \delta < \infty) \tag{3.15}$$

we have

$$|b_4| = \frac{C_3^{(-4)}}{4} = \frac{(1-\alpha)}{4^\delta(4-\alpha)}$$

We get sharpness in (3.7) with $\alpha_1 \leq \alpha < 1$ and in (3.9). The proof of Theorem 3.1 is complete.

□

Theorem 3.2. *Let the function f, given by (1.1), be in $\mathcal{S}_\delta(\alpha)$ $(0 \le \alpha < 1, \delta > 0)$ and $f^{-1}(w)$ be given by (3.1). Then for each n there exist positive numbers ε_n, δ_n and t_n such that*

$$(3.16)\qquad |b_n| \le \begin{cases} \frac{2}{n2^{(n-1)\delta}}\binom{2n-3}{n-2}\left(\frac{1-\alpha}{2-\alpha}\right)^{n-1}; & (0 \le \alpha \le \varepsilon_n, 0 \le \delta \le \delta_n) \\ \frac{1-\alpha}{n^\delta(n-\alpha)}; & (1-t_n \le \alpha < 1, \delta > 0). \end{cases}$$

The estimate (3.16) is sharp.

Proof. We follow the lines of the proof of Theorem 3.1. Write

$$\begin{aligned} h(z) &= \left(\frac{z}{f(z)}\right)^n \\ &= (1 - a_2 z - a_3 z^2 - \cdots)^{-n} \qquad (a_n \ge 0,\ n = 2, 3, \dots) \\ &= 1 + \sum_{k=1}^{\infty} C_k^{(-n)} z^k \end{aligned}$$

and observe that $b_n = \frac{C_{n-1}^{(-n)}}{n}$. Now taking $p = -n$ and $d_k = -a_{k+1}$ in Lemma 2.1, we get

$$C_{k+1}^{(-n)} = \sum_{j=0}^{k}\left[n + \frac{(1-n)j}{k+1}\right] a_{k+2-j} C_j^{(-n)}.$$

Since $f \in \mathcal{S}_\delta(\alpha)$, we get

$$(3.17)\qquad a_n = \frac{(1-\alpha)}{n^\delta(n-\alpha)}\lambda_n; \qquad \left(0 \le \lambda_n \le 1,\ \sum_{n=2}^{\infty}\lambda_n \le 1\right).$$

Therefore

$$(3.18)\qquad C_{k+1}^{(-n)} = \sum_{j=0}^{k}\left[n + \frac{(1-n)j}{k+1}\right]\frac{(1-\alpha)\lambda_{k+2-j}}{(k+2-j)^\delta(k+2-j-\alpha)} C_j^{(-n)}.$$

In order to establish (3.16), we wish to show that for each $n = 2, 3, \dots$ there exist positive real numbers ε_n and δ_n such that $C_{n-1}^{(-n)}$ is maximized when $\lambda_2 = 1$ for $0 \le \alpha \le \varepsilon_n$ and $0 \le \delta \le \delta_n$. Using (3.18) we get

$$C_1^{(-n)} = \frac{n(1-\alpha)}{2^\delta(2-\alpha)}\lambda_2 C_0^{(-n)} = \frac{n(1-\alpha)}{2^\delta(2-\alpha)}\lambda_2$$

so that

$$(3.19)\qquad C_1^{(-n)} \le \frac{n(1-\alpha)}{2^\delta(2-\alpha)} = d_1^{(-n)} \qquad \text{(say)}.$$

Thus $C_1^{(-n)}$ is maximized when $\lambda_2 = 1$. Write

$$d_j^{(-n)} = \max_{f \in \mathcal{S}_\delta(\alpha)} C_j^{(-n)} \qquad (1 \le j \le n-1).$$

Assume that $C_j^{(-n)}$ $(1 \le j \le n-2)$ is maximized for $\lambda_2 = 1$ when $\alpha > 0$ and $\delta > 0$ are sufficiently small. It follows from (3.17) that $\lambda_2 = 1$ implies $\lambda_j = 0$ for every $j \ge 3$. Therefore using (3.18) and (3.19) we get

$$C_2^{(-n)} \leq \binom{n+1}{2} \frac{(1-\alpha)}{2^\delta(2-\alpha)} d_1^{(-n)}$$
$$= \binom{n+2-1}{2} \frac{1}{2^{2\delta}} \left(\frac{1-\alpha}{2-\alpha}\right)^2 = d_2^{(-n)} \qquad \text{(say)}.$$

Assume that

(3.20) $$d_j^{(-n)} = \binom{n+j-1}{j} \frac{1}{2^{j\delta}} \left(\frac{1-\alpha}{2-\alpha}\right)^j \qquad (0 \leq j \leq n-2).$$

Again, using (3.18), we get

(3.21) $$\begin{aligned} d_{n-1}^{(-n)} &= \max_{f \in \mathcal{S}_\delta(\alpha)} C_{n-1}^{(-n)} \\ &= \max_{f \in \mathcal{S}_\delta(\alpha)} \left(\sum_{j=0}^{n-2} (n-j) \frac{(1-\alpha)\lambda_{n-j}}{(n-j)^\delta (n-j-\alpha)} C_j^{(-n)} \right) \\ &\leq \max_{0 \leq j \leq n-2} \left\{ \frac{(n-j)(1-\alpha)}{(n-j)^\delta (n-j-\alpha)} C_j^{(-n)} \right\} \left(\sum_{j=0}^{n-2} \lambda_{n-j} \right) \\ &\leq \max_{0 \leq j \leq n-2} \left\{ \frac{(n-j)(1-\alpha)}{(n-j)^\delta (n-j-\alpha)} d_j^{(-n)} \right\}. \end{aligned}$$

Write

$$A_j(\alpha,\delta) = \frac{(n-j)(1-\alpha)}{(n-j)^\delta (n-j-\alpha)} d_j^{(-n)}; \qquad (j = 0, 1, 2, \dots, (n-2)).$$

Substituting $d_0^{(-n)} = 1$ and the value of $d_1^{(-n)}$ from (3.19), we get

$$A_0(\alpha,\delta) = \frac{n(1-\alpha)}{n^\delta(n-\alpha)} \quad \text{and} \quad A_1(\alpha,\delta) = \frac{n(n-1)(1-\alpha)^2}{2^\delta (n-1)^\delta (n-1-\alpha)(2-\alpha)}.$$

Now $A_0(\alpha,\delta) < A_1(\alpha,\delta)$ $(n \geq 2$ and $0 \leq \delta \leq 2)$ if and only if

(3.22) $$\frac{1}{n^\delta (n-1)(n-\alpha)(1-\alpha)} < \frac{1}{2^\delta (n-1)^\delta (n-1-\alpha)(2-\alpha)}.$$

The above inequality (3.22) is true, because $(n-1-\alpha) < (n-\alpha),\ (1-\alpha) < (2-\alpha)$ and the maximum value of $\left(\frac{n}{2}\right)^\delta (n-1)^{1-\delta}$ is equal to 1 $(n \geq 2,\ 0 \leq \delta \leq 2)$. Also by Lemma 2.3, there exist positive real numbers ε_n and δ_n such that $A_j(\alpha,\delta) < A_k(\alpha,\delta)$ $(0 \leq \alpha \leq \varepsilon_n,\ 0 \leq \delta \leq \delta_n,\ 1 \leq j < k \leq n-2)$. Therefore it follows from (3.21) that the maximum $C_{n-1}^{(-n)}$ occurs at $j = n-2$. Substituting the value of $d_{n-2}^{(-n)}$, from (3.20) in (3.21) we get

$$d_{n-1}^{(-n)} = \frac{2(1-\alpha)}{2^\delta(2-\alpha)} d_{n-2}^{(-n)} = \frac{2}{2^{(n-1)\delta}} \binom{2n-3}{n-2} \left(\frac{1-\alpha}{2-\alpha}\right)^{n-1}$$
$$(0 \leq \alpha \leq \varepsilon_n,\ 0 \leq \delta \leq \delta_n,\ n = 2, 3, \dots).$$

Therefore

$$|b_n| = \frac{C_{n-1}^{(-n)}}{n} \leq \frac{2}{n 2^{(n-1)\delta}} \binom{2n-3}{n-2} \left(\frac{1-\alpha}{2-\alpha}\right)^{n-1};$$
$$(0 \leq \alpha \leq \varepsilon_n,\ 0 \leq \delta \leq \delta_n,\ n = 2, 3, \dots).$$

The above is precisely the first assertion of (3.16). In order to prove the other case of (3.16), we first observe that in the degenerate case $\alpha = 1$ we have $\mathcal{S}_\delta(\alpha) = \{z\}$.Therefore $C_j^{(-n)} \to 0$ as $\alpha \to 1^-$ for every $j = 1, 2, 3, \ldots$. Hence there exists a positive real number t_n $(0 \leq t_n \leq 1)$ such that

$$\frac{n}{n^\delta(n-\alpha)} \geq \frac{(n-j)}{(n-j)^\delta(n-j-\alpha)} C_j^{(-n)} \qquad (j = 1, 2, \ldots, 1 - t_n \leq \alpha < 1).$$

Thus the maximum of (3.21) occurs at $j = 0$ and we get $d_{n-1}^{(-n)} = \frac{n(1-\alpha)}{n^\delta(n-\alpha)}$ or equivalently

$$|b_n| \leq \frac{C_{n-1}^{(-n)}}{n} = \frac{(1-\alpha)}{n^\delta(n-\alpha)}.$$

This last estimate is precisely the assertion of (3.16) with $(1 - t_n \leq \alpha < 1,\ \delta > 0)$.

We observe that the $(n-1)^{th}$ coefficient of the function $\left(\frac{z}{f_2(z)}\right)^n$, where $f_2(z)$ is defined by (3.11), is equal to

$$\frac{2}{2^{(n-1)\delta}} \binom{2n-3}{n-2} \left(\frac{1-\alpha}{2-\alpha}\right)^{n-1}.$$

Similarly, the $(n-1)^{th}$ coefficient of the function $\left(\frac{z}{f_n(z)}\right)^n$, where $f_n(z)$ is defined by

$$z - \frac{(1-\alpha)}{n^\delta(n-\alpha)} z^n, \qquad (z \in \mathcal{U}, 0 \leq \alpha < 1, 0 \leq \delta < 1)$$

is equal to

$$\frac{n(1-\alpha)}{n^\delta(n-\alpha)}.$$

Therefore the estimate (3.16) is sharp. The proof of Theorem 3.2 is complete. □

Theorem 3.3. *Let the function f given by* (1.1)*, be in* $\mathcal{S}_\delta(\alpha)$ $(0 \leq \alpha < 1, \delta > 0)$ *and* $f^{-1}(w)$ *be given by* (3.2)*. For fixed α and δ* $(0 \leq \alpha < 1,\ \delta > 0)$ *let* $B_n(\alpha, \delta) = \max_{f \in \mathcal{S}_\delta(\alpha)} |b_n|$*. Then*

$$B_n(\alpha, \delta) \leq \frac{1}{n} \cdot \frac{2^{n\delta}(2-\alpha)^n}{[2^\delta(2-\alpha) - (1-\alpha)]^n}. \tag{3.23}$$

Proof. Since $f \in \mathcal{S}_\delta(\alpha)$, by Definition 1.1 we have $\sum_{n=2}^\infty \frac{n^\delta(n-\alpha)}{(1-\alpha)} |a_n| \leq 1$.

Therefore $\frac{2^\delta(2-\alpha)}{(1-\alpha)} \sum_{n=2}^\infty |a_n| \leq 1$ or equivalently

$$\sum_{n=2}^\infty |a_n| \leq \frac{(1-\alpha)}{2^\delta(2-\alpha)}.$$

This gives

$$\begin{aligned} |f(z)| &= \left| z + \sum_{n=2}^\infty a_n z^n \right| \\ &\geq |z| - |z^2| \left(\sum_{n=2}^\infty |a_n| \right) \\ &\geq r - r^2 \frac{(1-\alpha)}{2^\delta(2-\alpha)}, \qquad (|z| = r). \end{aligned} \tag{3.24}$$

Now using the above estimate (3.24) we have

$$\begin{aligned}|b_n| &= \left|\frac{1}{2in\pi}\int_{|z|=r}\frac{1}{(f(z))^n}dz\right| \\ &\leq \frac{1}{2n\pi}\int_{|z|=r}\frac{1}{|f(z)|^n}|dz| \\ &\leq \frac{1}{n}\left(\frac{1}{r-\frac{r^2(1-\alpha)}{2^\delta(2-\alpha)}}\right)^n.\end{aligned}$$

We observe that the function $F(r)$ where

$$F(r) = \left(\frac{1}{r-\frac{r^2(1-\alpha)}{2^\delta(2-\alpha)}}\right)^n$$

is an increasing function of r $(0 \leq \alpha < 1,\ \delta > 0)$ in the interval $0 \leq r < 1$. Therefore

$$|b_n| \leq \frac{1}{n}\left(\frac{1}{1-\frac{(1-\alpha)}{2^\delta(2-\alpha)}}\right)^n.$$

Consequently,

$$B_n(\alpha,\delta) \leq \frac{1}{n}\frac{2^{n\delta}(2-\alpha)^n}{[2^\delta(2-\alpha)-(1-\alpha)]^n}.$$

The proof of Theorem 3.3 is complete. □

REFERENCES

[1] J.T.P. CAMPSCHROERER, Coefficients of the inverse of a convex function, Report 8227, Nov. 1982, Department of Mathematics, Catholic University, Nijmegen, The Netherlands, (1982).

[2] L. DE BRANGES, A proof of the Bieberbach conjecture, *Acta. Math.*, **154**(1-2) (1985), 137–152.

[3] P.L. DUREN, *Univalent Functions*, Volume 259, Grunlehren der Mathematischen Wissenchaften, Bd., Springer-Verlag, New York, (1983).

[4] A.W. GOODMAN, Univalent functions and nonanalytic curves, *Proc. Amer. Math. Soc.*, **8** (1957), 591–601.

[5] W.K. HAYMAN, *Multivalent Functions*, Cambridge University Press, Second edition, (1994).

[6] W.E. KIRWAN AND G. SCHOBER, Inverse coefficients for functions of bounded boundary rotations, *J. Anal. Math.*, **36** (1979), 167–178.

[7] J. KRZYZ, R.J. LIBERA AND E. ZLOTKIEWICZ, Coefficients of inverse of regular starlike functions, *Ann. Univ. Mariae. Curie-Skłodowska*, Sect.A, **33** (1979), 103–109.

[8] V. KUMAR, Quasi-Hadamard product of certain univalent functions, *J. Math. Anal. Appl.*, **126** (1987), 70–77.

[9] R.J. LIBERA AND E.J. ZLOTKIEWICZ, Early coefficients of the inverse of a regular convex function, *Proc. Amer. Math. Soc.*, **85**(2) (1982), 225–230.

[10] K. LÖWNER, Untersuchungen über schlichte konforme Abbildungen des Einheitskreises I, *Math. Ann.*, **89** (1923), 103–121.

[11] A.K. MISHRA, Quasi-Hadamard product of analytic functions related to univalent functions, *Math. Student*, **64**(1-4) (1995), 221–225.

[12] A.K. MISHRA AND M. CHOUDHURY, A class of multivalent functions with negative Taylor coefficients, *Demonstratio Math.*, **XXVIII**(1) (1995), 223–234.

[13] A.K. MISHRA AND M.K. DAS, Fractional integral operators and distortion theorems for a class of multivalent functions with negative coefficients, *J. Anal.*, **4** (1996), 185–199.

[14] M. POURAHMADI, Taylor expansion of exp $\left(\sum_{k=0}^{\infty} a_k z^k\right)$ and some applications, *Amer. Math. Monthly*, **91**(5) (1984), 303–307.

[15] H. SILVERMAN, Univalent functions with negative coefficients, *Proc. Amer. Math. Soc.*, **51** (1975), 109–116.

[16] H. SILVERMAN, Coefficient bounds for inverses of classes of starlike functions, *Complex Variables*, **12** (1989), 23–31.

[17] E.C. TITCHMARSH, *A Theory of Functions*, Oxford University Press, Second Edition, (1968).