

SOME PROPERTIES OF ANALYTIC FUNCTIONS DEFINED BY A LINEAR OPERATOR

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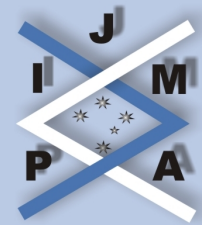
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Key words: Analytic functions, Hadamard product, Linear operator.

Abstract: The object of the present paper is to derive some properties of analytic functions defined by the Carlson - Shaffer linear operator $L(a, c)f(z)$.

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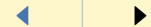
**Some Properties of
Analytic Functions**

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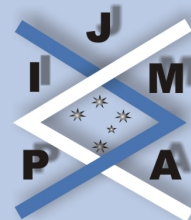
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1. Introduction and Definitions

Let \mathcal{A} denote the class of functions of the form:

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$. For two functions $f(z)$ and $g(z)$ given by

$$(1.2) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

their Hadamard product (or convolution) is defined by

$$(1.3) \quad (f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Define the function $\phi(a, c; z)$ by

$$(1.4) \quad \phi(a, c; z) := \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1},$$

$$(a \in \mathbb{R}; c \in \mathbb{R} \setminus \mathbb{Z}_0^-; \mathbb{Z}_0^- := \{0, -1, -2, \dots\}, z \in \mathcal{U}),$$

where $(\lambda)_n$ is the Pochhammer symbol given in terms of Gamma functions,

$$\begin{aligned} (\lambda)_n &:= \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} \\ &= \begin{cases} 1, & n = 0, \\ \lambda(\lambda + 1)(\lambda + 2) \dots (\lambda + n - 1), & n \in \mathbb{N} : \{1, 2, \dots\}. \end{cases} \end{aligned}$$



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Corresponding to the function $\phi(a, c; z)$, Carlson and Shaffer [1] introduced a linear operator $L(a, c) : \mathcal{A} \rightarrow \mathcal{A}$ by

$$(1.5) \quad L(a, c)f(z) := \phi(a, c; z) * f(z),$$

or, equivalently, by

$$(1.6) \quad L(a, c)f(z) := z + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} a_{n+1} z^{n+1} \quad (z \in \mathcal{U}).$$

It follows from (1.6) that

$$(1.7) \quad z(L(a, c)f(z))' = aL(a + 1, c)f(z) - (a - 1)L(a, c)f(z),$$

and

$$\begin{aligned} L(1, 1)f(z) &= f(z), & L(2, 1)f(z) &= zf'(z), \\ L(3, 1)f(z) &= zf'(z) + \frac{1}{2}z^2f''(z). \end{aligned}$$

Many properties of analytic functions defined by the Carlson-Shaffer linear operator were studied by (among others) Owa and Srivastava [10], Ding [5], Kim and Lee [6], Ravichandran *et al.* ([8, 9]), Shanmugam *et al.* [7] and Frasin ([2, 3]).

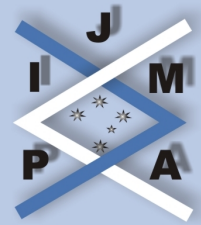
In this paper we shall derive some properties of analytic functions defined by the linear operator $L(a, c)f(z)$.

In order to prove our main results, we recall the following lemma:

Lemma 1.1 ([4]). *Let $\Phi(u, v)$ be a complex valued function,*

$$\Phi : D \rightarrow \mathbb{C}, \quad (D \subset \mathbb{C}^2; \mathbb{C} \text{ is the complex plane}),$$

and let $u = u_1 + iu_2$ and $v = v_1 + iv_2$. Suppose that the function $\Phi(u, v)$ satisfies:



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(i) $\Phi(u, v)$ is continuous in D ;

(ii) $(1, 0) \in D$ and $\operatorname{Re}(\Phi(1, 0)) > 0$;

(iii) $\operatorname{Re}(\Phi(iu_2, v_1)) \leq 0$ for all $(iu_2, v_1) \in D$ and such that $v_1 \leq -(1 + u_2^2)/2$.

Let $p(z) = 1 + p_1z + p_2z^2 + \dots$ be regular in U such that $(p(z), zp'(z)) \in D$ for all $z \in \mathcal{U}$. If $\operatorname{Re}(\Phi(p(z), zp'(z))) > 0$ ($z \in \mathcal{U}$), then $\operatorname{Re}(p(z)) > 0$ ($z \in \mathcal{U}$).



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2. Main Results

Theorem 2.1. Let $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}$, $(\operatorname{Re}(\beta) \geq 0)$, $(\alpha - \beta) \in \mathbb{R}$, and suppose that

$$(2.1) \quad \operatorname{Re} \left\{ \frac{L(a, c)f(z)}{L(a+1, c)f(z)} \left[\alpha - \beta \frac{(a+1)L(a+2, c)f(z)}{L(a+1, c)f(z)} \right] + a\alpha \right\} > \gamma$$

for some $\gamma(\gamma < (\alpha - \beta)(a + 1))$ and $2(\alpha - \beta) + \operatorname{Re}(\beta) \neq 0$, then

$$(2.2) \quad \operatorname{Re} \left\{ \frac{L(a, c)f(z)}{L(a+1, c)f(z)} \right\} > \frac{2\gamma - 2a(\alpha - \beta) + \operatorname{Re}(\beta)}{2(\alpha - \beta) + \operatorname{Re}(\beta)} \quad (z \in \mathcal{U}).$$

Proof. Define the function $p(z)$ by

$$(2.3) \quad \frac{L(a, c)f(z)}{L(a+1, c)f(z)} := \delta + (1 - \delta)p(z),$$

where

$$(2.4) \quad \delta = \frac{2\gamma - 2a(\alpha - \beta) + \operatorname{Re}(\beta)}{2(\alpha - \beta) + \operatorname{Re}(\beta)} < 1.$$

Then $p(z) = 1 + b_1z + b_2z^2 + \dots$ is regular in \mathcal{U} . It follows from (2.3) that

$$(2.5) \quad \frac{z(L(a, c)f(z))'}{L(a, c)f(z)} - \frac{z(L(a+1, c)f(z))'}{L(a+1, c)f(z)} = \frac{(1 - \delta)zp'(z)}{\delta + (1 - \delta)p(z)}$$

by making use of the familiar identity (1.7) in (2.5), we get

$$(2.6) \quad \frac{(a+1)L(a+2, c)f(z)}{L(a+1, c)f(z)} = 1 + \frac{a}{\delta + (1 - \delta)p(z)} - \frac{(1 - \delta)zp'(z)}{\delta + (1 - \delta)p(z)}$$

or, equivalently,



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$$(2.7) \quad \frac{L(a, c)f(z)}{L(a+1, c)f(z)} \left[\alpha - \beta \frac{(a+1)L(a+2, c)f(z)}{L(a+1, c)f(z)} \right] + a\alpha \\ = (\delta + a)(\alpha - \beta) + (1 - \delta)(\alpha - \beta)p(z) + \beta(1 - \delta)zp'(z).$$

Therefore, we have

$$(2.8) \quad \operatorname{Re} \left\{ \frac{L(a, c)f(z)}{L(a+1, c)f(z)} \left[\alpha - \beta \frac{(a+1)L(a+2, c)f(z)}{L(a+1, c)f(z)} \right] + a\alpha - \gamma \right\} \\ = \operatorname{Re} \{ (\delta + a)(\alpha - \beta) - \gamma + (1 - \delta)(\alpha - \beta)p(z) + \beta(1 - \delta)zp'(z) \} > 0.$$

If we define the function $\Phi(u, v)$ by

$$(2.9) \quad \Phi(u, v) = (\delta + a)(\alpha - \beta) - \gamma + (1 - \delta)(\alpha - \beta)u + \beta(1 - \delta)v$$

with $u = u_1 + iu_2$ and $v = v_1 + iv_2$, then

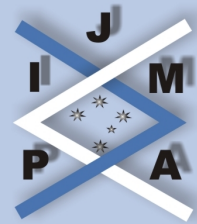
- (i) $\Phi(u, v)$ is continuous in $D = \mathbb{C}^2$;
- (ii) $(1, 0) \in D$ and $\operatorname{Re}(\Phi(1, 0)) = (\alpha - \beta)(a + 1) - \gamma > 0$;
- (iii) For all $(iu_2, v_1) \in D$ and such that $v_1 \leq -(1 + u_2^2)/2$,

$$\operatorname{Re}(\Phi(iu_2, v_1)) = \operatorname{Re}\{(\delta + a)(\alpha - \beta) - \gamma + \beta(1 - \delta)v_1\} \\ \leq (\delta + a)(\alpha - \beta) - \gamma - \frac{(1 - \delta)(1 + u_2^2) \operatorname{Re}(\beta)}{2} \\ = -\frac{(1 - \delta)u_2^2 \operatorname{Re}(\beta)}{2} \leq 0.$$

Therefore, the function $\Phi(u, v)$ satisfies the conditions in Lemma 1.1. Thus we have $\operatorname{Re}(p(z)) > 0$ ($z \in \mathcal{U}$), that is

$$\operatorname{Re} \left\{ \frac{L(a, c)f(z)}{L(a+1, c)f(z)} \right\} > \frac{2\gamma - 2a(\alpha - \beta) + \operatorname{Re}(\beta)}{2(\alpha - \beta) + \operatorname{Re}(\beta)}.$$

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Letting $\beta = -\bar{\alpha}$ in Theorem 2.1, we have

Corollary 2.2. Let $\alpha \in \mathbb{C}$, $(\operatorname{Re}(\alpha) < 0)$, and suppose that

$$(2.10) \quad \operatorname{Re} \left\{ \frac{L(a, c)f(z)}{L(a+1, c)f(z)} \left[\alpha + \bar{\alpha} \frac{(a+1)L(a+2, c)f(z)}{L(a+1, c)f(z)} \right] + a\alpha \right\} > \gamma$$

for some γ ($\gamma < 2(a+1)\operatorname{Re}(\alpha)$), then

$$(2.11) \quad \operatorname{Re} \left\{ \frac{L(a, c)f(z)}{L(a+1, c)f(z)} \right\} > \frac{2\gamma - (4a+1)\operatorname{Re}(\alpha)}{3\operatorname{Re}(\alpha)} \quad (z \in \mathcal{U}).$$

Letting $a = c = 1$ in Theorem 2.1, we have

Corollary 2.3. Let $\alpha, \beta \in \mathbb{C}$, $(\operatorname{Re}(\beta) \geq 0)$, $(\alpha - \beta) \in \mathbb{R}$, and suppose that

$$(2.12) \quad \operatorname{Re} \left\{ \frac{f(z)}{zf'(z)} \left[\alpha - 2\beta \left(1 + \frac{zf''(z)}{2f'(z)} \right) \right] + \alpha \right\} > \gamma$$

for some γ ($\gamma < 2(\alpha - \beta)$) and $2(\alpha - \beta) + \operatorname{Re}(\beta) \neq 0$, then

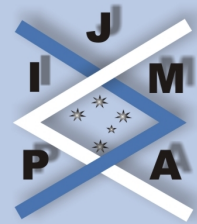
$$(2.13) \quad \operatorname{Re} \left\{ \frac{f(z)}{zf'(z)} \right\} > \frac{2\gamma - 2(\alpha - \beta) + \operatorname{Re}(\beta)}{2(\alpha - \beta) + \operatorname{Re}(\beta)} \quad (z \in \mathcal{U}).$$

Theorem 2.4. Let $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}$, $(\operatorname{Re}(\beta) \geq 0)$, $(\alpha - \beta) \in \mathbb{R}$, and suppose that

$$(2.14) \quad \operatorname{Re} \left\{ \frac{L(a, c)f(z)}{L(a+1, c)f(z)} \left[\alpha - \beta \frac{(a+1)L(a+2, c)f(z)}{L(a+1, c)f(z)} \right] + a\alpha \right\} < \gamma$$

for some γ ($\gamma > (\alpha - \beta)(a+1)$) and $2(\alpha - \beta) + \operatorname{Re}(\beta) \neq 0$, then

$$(2.15) \quad \operatorname{Re} \left\{ \frac{L(a, c)f(z)}{L(a+1, c)f(z)} \right\} < \frac{2\gamma - 2a(\alpha - \beta) + \operatorname{Re}(\beta)}{2(\alpha - \beta) + \operatorname{Re}(\beta)} \quad (z \in \mathcal{U}).$$



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Proof. Define the function $p(z)$ by

$$(2.16) \quad \frac{L(a, c)f(z)}{L(a+1, c)f(z)} := \delta + (1 - \delta)p(z),$$

where

$$(2.17) \quad \delta = \frac{2\gamma - 2a(\alpha - \beta) + \operatorname{Re}(\beta)}{2(\alpha - \beta) + \operatorname{Re}(\beta)} > 1.$$

Then $p(z) = 1 + b_1z + b_2z^2 + \dots$ is regular in \mathcal{U} . It follows from (2.16) that

$$(2.18) \quad \operatorname{Re} \left\{ \gamma - \frac{L(a, c)f(z)}{L(a+1, c)f(z)} \left[\alpha - \beta \frac{(a+1)L(a+2, c)f(z)}{L(a+1, c)f(z)} \right] - a\alpha \right\} \\ = \operatorname{Re} \{ \gamma - (\delta + a)(\alpha - \beta) - (1 - \delta)(\alpha - \beta)p(z) - \beta(1 - \delta)zp'(z) \} > 0.$$

If we define the function $\Phi(u, v)$ by

$$(2.19) \quad \Phi(u, v) = \gamma - (\delta + a)(\alpha - \beta) - (1 - \delta)(\alpha - \beta)u - \beta(1 - \delta)v$$

with $u = u_1 + iu_2$ and $v = v_1 + iv_2$, then

- (i) $\Phi(u, v)$ is continuous in $D = \mathbb{C}^2$;
- (ii) $(1, 0) \in D$ and $\operatorname{Re}(\Phi(1, 0)) = \gamma - (\alpha - \beta)(a + 1) > 0$;
- (iii) For all $(iu_2, v_1) \in D$ and such that $v_1 \leq -(1 + u_2^2)/2$,

$$\operatorname{Re}(\Phi(iu_2, v_1)) = \operatorname{Re} \{ \gamma - (\delta + a)(\alpha - \beta) - \beta(1 - \delta)v_1 \} \\ \leq \gamma - (\delta + a)(\alpha - \beta) + \frac{(1 - \delta)(1 + u_2^2) \operatorname{Re}(\beta)}{2} \\ = \frac{(1 - \delta)u_2^2 \operatorname{Re}(\beta)}{2} \leq 0,$$



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applying Lemma 1.1, we have $\operatorname{Re}(p(z)) > 0$ ($z \in \mathcal{U}$), that is

$$\operatorname{Re} \left\{ \frac{L(a, c)f(z)}{L(a+1, c)f(z)} \right\} < \frac{2\gamma - 2a(\alpha - \beta) + \operatorname{Re}(\beta)}{2(\alpha - \beta) + \operatorname{Re}(\beta)}.$$

□

Theorem 2.5. Let $a > -1$, $\mu > 0$, $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}$, ($\operatorname{Re}(\beta) \geq 0$), $(\alpha + \beta) \in \mathbb{R}$, and suppose that

$$(2.20) \quad \operatorname{Re} \left\{ \left(\frac{L(a+1, c)f(z)}{z} \right)^\mu \left[\alpha + \beta \frac{L(a+2, c)f(z)}{L(a+1, c)f(z)} \right] \right\} > \gamma$$

for some γ ($\gamma < \alpha + \beta$) and $2\mu(\alpha + \beta)(a+1) + \operatorname{Re}(\beta) \neq 0$, then

$$(2.21) \quad \operatorname{Re} \left\{ \left(\frac{L(a+1, c)f(z)}{z} \right)^\mu \right\} > \frac{2\mu\gamma(a+1) + \operatorname{Re}(\beta)}{2\mu(\alpha + \beta)(a+1) + \operatorname{Re}(\beta)} \quad (z \in \mathcal{U}).$$

Proof. Define the function $p(z)$ by

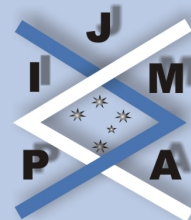
$$(2.22) \quad \left(\frac{L(a+1, c)f(z)}{z} \right)^\mu := \delta + (1 - \delta)p(z),$$

where

$$\delta = \frac{2\mu\gamma(a+1) + \operatorname{Re}(\beta)}{2\mu(\alpha + \beta)(a+1) + \operatorname{Re}(\beta)} > 1.$$

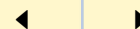
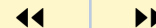
Then $p(z) = 1 + b_1z + b_2z^2 + \dots$ is regular in \mathcal{U} . Also, by a simple computation and by making use of the familiar identity (1.7) we find from (2.22) that

$$(2.23) \quad \frac{\beta L(a+2, c)f(z)}{L(a+1, c)f(z)} = \frac{\beta(1 - \delta)zp'(z)}{\mu(a+1)(\delta + (1 - \delta)p(z))} + \beta,$$



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and by using (2.22) and (2.23), we get

$$(2.24) \quad \left(\frac{L(a+1, c)f(z)}{z} \right)^\mu \left(\frac{\beta L(a+2, c)f(z)}{L(a+1, c)f(z)} + \alpha \right) \\ = \frac{\beta(1-\delta)zp'(z)}{\mu(a+1)} + (\alpha + \beta)(\delta + (1-\delta)p(z)).$$

Therefore, we have

$$(2.25) \quad \operatorname{Re} \left\{ \left(\frac{L(a+1, c)f(z)}{z} \right)^\mu \left(\frac{\beta L(a+2, c)f(z)}{L(a+1, c)f(z)} + \alpha \right) - \gamma \right\} \\ = \operatorname{Re} \left\{ (\alpha + \beta)\delta - \gamma + (1-\delta)(\alpha + \beta)p(z) + \frac{\beta(1-\delta)}{\mu(a+1)} zp'(z) \right\} > 0.$$

If we define the function $\Phi(u, v)$ by

$$(2.26) \quad \Phi(u, v) = (\alpha + \beta)\delta - \gamma + (1-\delta)(\alpha + \beta)u + \frac{\beta(1-\delta)}{\mu(a+1)}v$$

with $u = u_1 + iu_2$ and $v = v_1 + iv_2$, then

- (i) $\Phi(u, v)$ is continuous in $D = \mathbb{C}^2$;
- (ii) $(1, 0) \in D$ and $\operatorname{Re}(\Phi(1, 0)) = (\alpha + \beta) - \gamma > 0$;
- (iii) For all $(iu_2, v_1) \in D$ and such that $v_1 \leq -(1 + u_2^2)/2$,

$$\operatorname{Re}(\Phi(iu_2, v_1)) = \operatorname{Re}\left\{ (\alpha + \beta)\delta - \gamma + \frac{\beta(1-\delta)}{\mu(a+1)}v_1 \right\} \\ \leq (\alpha + \beta)\delta - \gamma - \frac{(1-\delta)(1 + u_2^2) \operatorname{Re}(\beta)}{2\mu(a+1)} \\ = -\frac{(1-\delta)u_2^2 \operatorname{Re}(\beta)}{2\mu(a+1)} \leq 0.$$



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Therefore, the function $\Phi(u, v)$ satisfies the conditions in Lemma 1.1. Thus we have $\operatorname{Re}(p(z)) > 0$ ($z \in \mathcal{U}$), that is,

$$\operatorname{Re} \left\{ \left(\frac{L(a+1, c)f(z)}{z} \right)^\mu \right\} > \frac{2\mu\gamma(a+1) + \operatorname{Re}(\beta)}{2\mu(\alpha + \beta)(a+1) + \operatorname{Re}(\beta)} \quad (z \in \mathcal{U}).$$

□

Letting $\alpha = \bar{\beta}$ in Theorem 2.5, we have

Corollary 2.6. *Let $a > -1$, $\mu > 0$, $\beta \in \mathbb{C}$, ($\operatorname{Re}(\beta) > 0$), and suppose that*

$$(2.27) \quad \operatorname{Re} \left\{ \left(\frac{L(a+1, c)f(z)}{z} \right)^\mu \left[\alpha + \beta \frac{L(a+2, c)f(z)}{L(a+1, c)f(z)} \right] \right\} > \gamma$$

for some γ ($\gamma < 2 \operatorname{Re}(\beta)$), then

$$(2.28) \quad \operatorname{Re} \left\{ \left(\frac{L(a+1, c)f(z)}{z} \right)^\mu \right\} > \frac{2\mu\gamma(a+1) + \operatorname{Re}(\beta)}{(4\mu(a+1) + 1) \operatorname{Re}(\beta)} \quad (z \in \mathcal{U}).$$

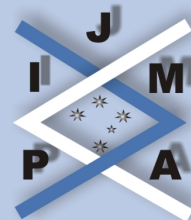
Letting $a = c = 1$ in Theorem 2.5, we have,

Corollary 2.7. *Let $\mu > 0$, $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}$, ($\operatorname{Re}(\beta) \geq 0$), $(\alpha + \beta) \in \mathbb{R}$, and suppose that*

$$(2.29) \quad \operatorname{Re} \left\{ (f'(z))^\mu \left[\alpha + \beta \left(1 + \frac{zf''(z)}{2f'(z)} \right) \right] \right\} > \gamma$$

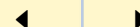
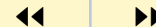
for some γ ($\gamma < \alpha + \beta$) and $4\mu(\alpha + \beta) + \operatorname{Re}(\beta) \neq 0$, then

$$(2.30) \quad \operatorname{Re} \left\{ (f'(z))^\mu \right\} > \frac{4\mu\gamma + \operatorname{Re}(\beta)}{4\mu(\alpha + \beta) + \operatorname{Re}(\beta)} \quad (z \in \mathcal{U}).$$



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Employing the same manner as in the proofs of Theorems 2.4 and 2.5, we have:

Theorem 2.8. Let $a > -1$, $\mu > 0$, $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}$, $(\operatorname{Re}(\beta) \geq 0)$, $(\alpha + \beta) \in \mathbb{R}$, and suppose that

$$(2.31) \quad \operatorname{Re} \left\{ \left(\frac{L(a+1, c)f(z)}{z} \right)^\mu \left[\alpha + \beta \frac{L(a+2, c)f(z)}{L(a+1, c)f(z)} \right] \right\} < \gamma$$

for some $\gamma (\gamma > \alpha + \beta)$ and $2\mu(\alpha + \beta)(a + 1) + \operatorname{Re}(\beta) \neq 0$, then

$$(2.32) \quad \operatorname{Re} \left\{ \left(\frac{L(a+1, c)f(z)}{z} \right)^\mu \right\} < \frac{2\mu\gamma(a+1) + \operatorname{Re}(\beta)}{2\mu(\alpha + \beta)(a+1) + \operatorname{Re}(\beta)} \quad (z \in \mathcal{U}).$$

Theorem 2.9. Let $\lambda > 0$, $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}$, $(\operatorname{Re}(\beta) \geq 0)$, $(\alpha + \beta) \in \mathbb{R}$, and suppose that

$$(2.33) \quad \operatorname{Re} \left\{ \left(\frac{L(a+1, c)f(z)}{L(a, c)f(z)} \right)^\lambda \times \left[\lambda\beta(a+1) \frac{L(a+2, c)f(z)}{L(a+1, c)f(z)} - a\lambda\beta \frac{L(a+1, c)f(z)}{L(a, c)f(z)} + \lambda\alpha \right] \right\} > \gamma$$

for some $\gamma (\gamma < \lambda(\alpha + \beta))$ and $2\lambda(\alpha + \beta) + \operatorname{Re}(\beta) \neq 0$, then

$$(2.34) \quad \operatorname{Re} \left(\frac{L(a+1, c)f(z)}{L(a, c)f(z)} \right)^\lambda > \frac{2\gamma + \operatorname{Re}(\beta)}{2\lambda(\alpha + \beta) + \operatorname{Re}(\beta)} \quad (z \in \mathcal{U}).$$

Proof. Define the function $p(z)$ by

$$(2.35) \quad \left(\frac{L(a+1, c)f(z)}{L(a, c)f(z)} \right)^\lambda := \delta + (1 - \delta)p(z),$$

where

$$(2.36) \quad \delta = \frac{2\gamma + \operatorname{Re}(\beta)}{2\lambda(\alpha + \beta) + \operatorname{Re}(\beta)} < 1.$$

Then $p(z) = 1 + b_1z + b_2z^2 + \dots$ is regular in \mathcal{U} . Also, by a simple computation and by making use of the familiar identity (1.7), we find from (2.35) that

$$(2.37) \quad \left(\frac{L(a+1, c)f(z)}{L(a, c)f(z)} \right)^\lambda \\ \times \left[\lambda\beta(a+1) \frac{L(a+2, c)f(z)}{L(a+1, c)f(z)} - a\lambda\beta \frac{L(a+1, c)f(z)}{L(a, c)f(z)} + \lambda\alpha \right] \\ = \lambda(\alpha + \beta)(\delta + (1 - \delta)p(z)) + \beta(1 - \delta)zp'(z)$$

Therefore, we have

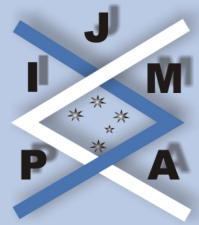
$$(2.38) \quad \operatorname{Re} \left\{ \left(\frac{L(a+1, c)f(z)}{L(a, c)f(z)} \right)^\lambda \right. \\ \left. \times \left[\lambda\beta(a+1) \frac{L(a+2, c)f(z)}{L(a+1, c)f(z)} - a\lambda\beta \frac{L(a+1, c)f(z)}{L(a, c)f(z)} + \lambda\alpha \right] - \gamma \right\} \\ = \operatorname{Re} \{ \lambda(\alpha + \beta)(\delta + (1 - \delta)p(z)) + \beta(1 - \delta)zp'(z) - \gamma \} > 0.$$

If we define the function $\Phi(u, v)$ by

$$(2.39) \quad \Phi(u, v) = \lambda\delta(\alpha + \beta) - \gamma + \lambda(1 - \delta)(\alpha + \beta)u + \beta(1 - \delta)v$$

with $u = u_1 + iu_2$ and $v = v_1 + iv_2$, then

- (i) $\Phi(u, v)$ is continuous in $D = \mathbb{C}^2$;
- (ii) $(1, 0) \in D$ and $\operatorname{Re}(\Phi(1, 0)) = \lambda(\alpha + \beta) - \gamma > 0$;



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(iii) For all $(iu_2, v_1) \in D$ and such that $v_1 \leq -(1 + u_2^2)/2$,

$$\begin{aligned} \operatorname{Re}(\Phi(iu_2, v_1)) &= \operatorname{Re}\{\lambda\delta(\alpha + \beta) - \gamma + \beta(1 - \delta)v_1\} \\ &\leq \lambda\delta(\alpha + \beta) - \gamma - \frac{(1 - \delta)(1 + u_2^2)}{2} \operatorname{Re}(\beta) \\ &= -\frac{(1 - \delta)u_2^2 \operatorname{Re}(\beta)}{2} \leq 0. \end{aligned}$$

Therefore, the function $\Phi(u, v)$ satisfies the conditions in Lemma 1.1. Thus we have

$$\operatorname{Re} \left(\frac{L(a+1, c)f(z)}{L(a, c)f(z)} \right)^\lambda > \frac{2\gamma + \operatorname{Re}(\beta)}{2\lambda(\alpha + \beta) + \operatorname{Re}(\beta)} \quad (z \in \mathcal{U}).$$

□

Letting $\alpha = \bar{\beta}$ in Theorem 2.9, we have:

Corollary 2.10. Let $\lambda > 0$, $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}$, $(\operatorname{Re}(\beta) > 0)$, $(\alpha + \beta) \in \mathbb{R}$, and suppose that

$$(2.40) \quad \operatorname{Re} \left\{ \left(\frac{L(a+1, c)f(z)}{L(a, c)f(z)} \right)^\lambda \times \left[\lambda\beta(a+1) \frac{L(a+2, c)f(z)}{L(a+1, c)f(z)} - a\lambda\beta \frac{L(a+1, c)f(z)}{L(a, c)f(z)} + \lambda\bar{\beta} \right] \right\} > \gamma$$

for some $\gamma(\gamma < \lambda(\alpha + \beta))$, then

$$(2.41) \quad \operatorname{Re} \left(\frac{L(a+1, c)f(z)}{L(a, c)f(z)} \right)^\lambda > \frac{2\gamma + \operatorname{Re}(\beta)}{(4\lambda + 1) \operatorname{Re}(\beta)} \quad (z \in \mathcal{U}).$$



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Letting $a = c = \lambda = 1$ in Theorem 2.9, we have:

Corollary 2.11. Let $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}$, $(\operatorname{Re}(\beta) \geq 0)$, $(\alpha + \beta) \in \mathbb{R}$, and suppose that:

$$(2.42) \quad \operatorname{Re} \left\{ \left(\frac{z f'(z)}{f(z)} \right) \left[(2\beta + \alpha) + \beta z \left(\frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)} \right) \right] \right\} > \gamma$$

for some $\gamma(\gamma < \alpha + \beta)$ and $2(\alpha + \beta) + \operatorname{Re}(\beta) \neq 0$, then $f(z)$ is starlike of order δ , where $\delta = \frac{2\gamma + \operatorname{Re}(\beta)}{2(\alpha + \beta) + \operatorname{Re}(\beta)}$.

Employing the same manner as in the proofs of Theorems 2.4 and 2.9, we have:

Theorem 2.12. Let $\lambda > 0$, $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}$, $(\operatorname{Re}(\beta) \geq 0)$, $(\alpha + \beta) \in \mathbb{R}$, and suppose that:

$$(2.43) \quad \operatorname{Re} \left\{ \left(\frac{L(a+1, c)f(z)}{L(a, c)f(z)} \right)^\lambda \times \left[\lambda\beta(a+1) \frac{L(a+2, c)f(z)}{L(a+1, c)f(z)} - a\lambda\beta \frac{L(a+1, c)f(z)}{L(a, c)f(z)} + \lambda\alpha \right] \right\} < \gamma$$

for some $\gamma(\gamma > \lambda(\alpha + \beta))$ and $2\lambda(\alpha + \beta) + \operatorname{Re}(\beta) \neq 0$, then

$$(2.44) \quad \operatorname{Re} \left(\frac{L(a+1, c)f(z)}{L(a, c)f(z)} \right)^\lambda < \frac{2\gamma + \operatorname{Re}(\beta)}{2\lambda(\alpha + \beta) + \operatorname{Re}(\beta)} \quad (z \in \mathcal{U}).$$

Theorem 2.13. Let $a > -1$, $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}$, $(\operatorname{Re}(\beta) \geq 0)$, $(\alpha - \beta) \in \mathbb{R}$, and suppose that

$$(2.45) \quad \operatorname{Re} \left\{ \frac{z}{L(a+1, c)f(z)} \left[(a+1)\alpha - \beta \frac{L(a+2, c)f(z)}{L(a+1, c)f(z)} \right] \right\} > \gamma$$



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for some $\gamma(\gamma < (\alpha - \beta)(a + 1))$, then

$$(2.46) \quad \operatorname{Re} \left\{ \frac{z}{L(a+1, c)f(z)} \right\} > \frac{2\gamma + \operatorname{Re}(\beta)}{2(a+1)(\alpha - \beta) + \operatorname{Re}(\beta)} \quad (z \in \mathcal{U}).$$

Proof. Define the function $p(z)$ by

$$(2.47) \quad \frac{z}{L(a+1, c)f(z)} := \delta + (1 - \delta)p(z),$$

where

$$(2.48) \quad \delta = \frac{2\gamma + \operatorname{Re}(\beta)}{2(a+1)(\alpha - \beta) + \operatorname{Re}(\beta)} > 1.$$

Then $p(z) = 1 + b_1z + b_2z^2 + \dots$ is regular in \mathcal{U} . Also, by a simple computation and by making use of the familiar identity (1.7), we find from (2.47) that

$$(2.49) \quad \frac{z}{L(a+1, c)f(z)} \left[(a+1)\alpha - \beta \frac{L(a+2, c)f(z)}{L(a+1, c)f(z)} \right] \\ = (a+1)(\alpha - \beta)(\delta + (1 - \delta)p(z)) + \beta(1 - \delta)zp'(z).$$

Therefore, we have

$$(2.50) \quad \operatorname{Re} \left\{ \frac{z}{L(a+1, c)f(z)} \left[(a+1)\alpha - \beta \frac{L(a+2, c)f(z)}{L(a+1, c)f(z)} \right] - \gamma \right\} \\ = \operatorname{Re} \{ (a+1)(\alpha - \beta)(\delta + (1 - \delta)p(z)) + \beta(1 - \delta)zp'(z) - \gamma \} > 0.$$

If we define the function $\Phi(u, v)$ by

$$(2.51) \quad \Phi(u, v) = \delta(a+1)(\alpha - \beta) - \gamma + (a+1)(\alpha - \beta)(1 - \delta)u + \beta(1 - \delta)v$$



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with $u = u_1 + iu_2$ and $v = v_1 + iv_2$, then

- (i) $\Phi(u, v)$ is continuous in $D = \mathbb{C}^2$;
- (ii) $(1, 0) \in D$ and $\operatorname{Re}(\Phi(1, 0)) = (\alpha - \beta)(a + 1) - \gamma > 0$;
- (iii) For all $(iu_2, v_1) \in D$ and such that $v_1 \leq -(1 + u_2^2)/2$,

$$\begin{aligned}\operatorname{Re}(\Phi(iu_2, v_1)) &= \operatorname{Re}\{\delta(a + 1)(\alpha - \beta) - \gamma + \beta(1 - \delta)v_1\} \\ &\leq \delta(a + 1)(\alpha - \beta) - \gamma - \frac{(1 - \delta)(1 + u_2^2)}{2} \operatorname{Re}(\beta) \\ &= -\frac{(1 - \delta)u_2^2 \operatorname{Re}(\beta)}{2} \leq 0.\end{aligned}$$

Therefore, the function $\Phi(u, v)$ satisfies the conditions in Lemma 1.1. Thus we have

$$\operatorname{Re}\left\{\frac{z}{L(a + 1, c)f(z)}\right\} > \frac{2\gamma + \operatorname{Re}(\beta)}{2(a + 1)(\alpha - \beta) + \operatorname{Re}(\beta)} \quad (z \in \mathcal{U}).$$

□

Letting $\beta = -\bar{\alpha}$ in Theorem 2.13, we have:

Corollary 2.14. Let $a > -1$, $a \neq 0$, $\alpha \in \mathbb{C}$, $(\operatorname{Re}(\alpha) < 0)$, and suppose that

$$(2.52) \quad \operatorname{Re}\left\{\frac{z}{L(a + 1, c)f(z)}\left[(a + 1)\alpha + \bar{\alpha}\frac{L(a + 2, c)f(z)}{L(a + 1, c)f(z)}\right]\right\} > \gamma$$

for some $\gamma(\gamma < 2(a + 1)\operatorname{Re}(\alpha))$, then

$$(2.53) \quad \operatorname{Re}\left\{\frac{z}{L(a + 1, c)f(z)}\right\} > \frac{2\gamma - \operatorname{Re}(\alpha)}{(4a + 3)\operatorname{Re}(\alpha)} \quad (z \in \mathcal{U}).$$

Letting $a = c = 1$ in Theorem 2.13, we have:



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Corollary 2.15. Let $a > -1$, $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}$, $(\operatorname{Re}(\beta) \geq 0)$, $(\alpha - \beta) \in \mathbb{R}$, and suppose that

$$(2.54) \quad \operatorname{Re} \left\{ \frac{1}{f'(z)} \left[2\alpha - \beta - \frac{zf''(z)}{2f'(z)}\beta \right] \right\} > \gamma$$

for some $\gamma(\gamma < 2(\alpha - \beta))$ and $4(\alpha - \beta) + \operatorname{Re}(\beta) \neq 0$, then

$$(2.55) \quad \operatorname{Re} \left\{ \frac{1}{f'(z)} \right\} > \frac{2\gamma + \operatorname{Re}(\beta)}{4(\alpha - \beta) + \operatorname{Re}(\beta)} \quad (z \in \mathcal{U}).$$

Employing the same manner as in the proofs of Theorem 2.4 and 2.13, we have:

Theorem 2.16. Let $a > -1$, $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}$, $(\operatorname{Re}(\beta) \geq 0)$, $(\alpha - \beta) \in \mathbb{R}$, and suppose that

$$(2.56) \quad \operatorname{Re} \left\{ \frac{z}{L(a+1, c)f(z)} \left[(a+1)\alpha - \beta \frac{L(a+2, c)f(z)}{L(a+1, c)f(z)} \right] \right\} < \gamma$$

for some $\gamma(\gamma > (\alpha - \beta)(a+1))$ and $2(a+1)(\alpha - \beta) + \operatorname{Re}(\beta) \neq 0$, then

$$(2.57) \quad \operatorname{Re} \left\{ \frac{z}{L(a+1, c)f(z)} \right\} < \frac{2\gamma + \operatorname{Re}(\beta)}{2(a+1)(\alpha - \beta) + \operatorname{Re}(\beta)} \quad (z \in \mathcal{U}).$$

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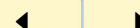
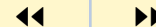
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