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ON A CONVOLUTION CONJECTURE OF BOUNDED FUNCTIONS

KRZYSZTOF PIEJKO, JANUSZ SOKÓŁ, AND JAN STANKIEWICZ

DEPARTMENT OF MATHEMATICS
RZESZÓW UNIVERSITY OF TECHNOLOGY
UL. W. POLA 2, 35-959 RZESZÓW, POLAND.
piejko@prz.rzeszow.pl

DEPARTMENT OF MATHEMATICS
RZESZÓW UNIVERSITY OF TECHNOLOGY
UL. W. POLA 2, 35-959 RZESZÓW, POLAND.
jsokol@prz.rzeszow.pl

INSTITUTE OF MATHEMATICS
UNIVERSITY OF RZESZÓW
UL. REJTANA 16A
35-959 RZESZÓW, POLAND.
jan.stankiewicz@prz.rzeszow.pl

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ABSTRACT. We consider the convolution $P(A,B) \star P(C,D)$ of the classes of analytic functions subordinated to the homographies $\frac{1+Az}{1-Bz}$ and $\frac{1+Cz}{1-Dz}$ respectively, where A, B, C, D are some complex numbers. In 1988 J. Stankiewicz and Z. Stankiewicz [11] showed that for certain A, B, C, D there exist X, Y such that $P(A,B) \star P(C,D) \subset P(X,Y)$. In this paper we verify the conjecture that $P(X,Y) \subset (A,B) \star P(C,D)$ for some A, B, C, D, X, Y.

Key words and phrases: Hadamard product, Convolution, Subordination, Bounded functions.

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1. INTRODUCTION

Let $\Delta = \{z \in \mathbb{C} : \ |z| < 1\}$ denote the open unit disc and let $\mathcal{H}$ be the class of functions regular in Δ. We will denote by $\mathcal{N}$ the class of functions $f \in \mathcal{H}$ normalized by $f(0) = 1$. The class of Schwarz functions Ω is the class of functions $\omega \in \mathcal{H}$, such that $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in \Delta$. We say that a function f is subordinate to a function g in Δ (and write $f \prec g$ or $f(z) \prec g(z)$) if there exists a function $\omega \in \Omega$ such that $f(z) = g(\omega(z))$; $z \in \Delta$. If the function g is univalent in Δ, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$. In this case we have $\omega(z) = g^{-1}(f(z))$.

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Let the functions f and g be of the form

$$f(z)=\sum_{n=0}^{\infty}a_nz^n,\qquad g(z)=\sum_{n=0}^{\infty}b_nz^n,\quad z\in\Delta.$$

We say that the Hadamard product of f and g is the function $f\star g$ if

$$(f\star g)(z)=f(z)\star g(z)=\sum_{n=0}^{\infty}a_nb_nz^n.$$

J. Hadamard [2] proved, that the radius of convergence of $f\star g$ is the product of the radii of convergence of the corresponding series f and g. The function $f\star g$ is also called the *convolution* of the functions f and g.

For the classes $Q_1\subset\mathcal{H}$ and $Q_2\subset\mathcal{H}$ the convolution $Q_1\star Q_2$ is defined as

$$Q_1\star Q_2=\{h\in\mathcal{H};\ h=f\star g,\ f\in Q_1,\ g\in Q_2\}.$$

The problem of connections between functions f, g and their convolution $f\star g$ or between Q_1,Q_2 and $Q_1\star Q_2$ has often been investigated. Many conjectures have been given, however, many of them have still not been verified.

In 1958, G. Pólya and I.J. Schoenberg [7] conjectured that the Hadamard product of two convex mappings is a convex mapping. In 1961 H.S. Wilf [12] gave the more general supposition that if F and G are convex mappings in Δ and f is subordinate to F, then the convolution $f\star G$ is subordinate to $F\star G$.

In 1973, S. Rusheweyh and T. Sheil-Small [9] proved both conjectures and more results of this type. Their very important results we may write as:

Theorem A. *If $f\in S^c$ and $g\in S^c$, then $f\star g\in S^c$, where S^c is the class of univalent and convex functions. Moreover $S^c\star S^c=S^c$.*

Theorem B. *If $F\in S^c$, $G\in S^c$ and $f\prec F$, then $f\star G\prec F\star G$.*

In 1985, S. Ruscheweyh and J. Stankiewicz [10] proved some generalizations of Theorems A and B:

Theorem C. *If the functions F and G are univalent and convex in Δ, then for all functions f and g, if $f\prec F$ and $g\prec G$ then $f\star g\prec F\star G$.*

For the given complex numbers A, B such that $A+B\neq 0$ and $|B|\leq 1$ let us denote

$$P(A,B)=\left\{f\in\mathcal{N}:\ f(z)\prec\frac{1+Az}{1-Bz}\right\}.$$

W. Janowski introduced the class $P(A,B)$ in [3] and considered it for some real A and B. If $A=B=1$ then the class $P(1,1)$ is the class of functions with positive real part (Carathéodory functions). Note, that for $|B|<1$ the class $P(A,B)$ is the class of bounded functions. In 1988 J. Stankiewicz and Z. Stankiewicz [11] investigated the convolution properties of the class $P(A,B)$ and proved the following theorem:

Theorem D. *If A,B,C,D are some complex numbers such that $A+B\neq 0$, $C+D\neq 0$, $|B|\leq 1$, $|D|\leq 1$, then*

$$P(A,B)\star P(C,D)\subset P(AC+AD+BC,BD),$$

moreover, if $|B|=1$ or $|D|=1$, then

$$P(A,B)\star P(C,D)=P(AC+AD+BC,BD).$$

The equality between the class $P(A,B) \star P(C,D)$ and the class $P(AC+AD+BC,BD)$ for $|B|<1$ and $|D|<1$ was an open problem. In [5] K. Piejko, J. Sokół and J. Stankiewicz proved that the above mentioned classes are different. In this paper we give an extension of this result. First we need two theorems.

Theorem E (G. Eneström [1], S. Kakeya [4])**.** *If* $a_0 > a_1 > \cdots > a_n > 0$, *where* $n \in \mathbb{N}$, *then the polynomial* $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ *has no roots in* $\overline{\Delta} = \{z \in \mathbb{C} : \ |z| \leq 1\}$.

Theorem F (W. Rogosinski [8])**.** *If the function* $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ *is subordinated to the function* $F(z) = \sum_{n=0}^{\infty} \beta_n z^n$ *in* Δ, *then* $\sum_{n=0}^{\infty} |\alpha_n|^2 \leq \sum_{n=0}^{\infty} |\beta_n|^2$.

2. MAIN RESULT

We prove the following theorem.

Theorem 2.1. *Let* A, B, C, D *be some complex numbers such that* $B + A \neq 0$, $C + D \neq 0$, $|B| < 1$, $|D| < 1$, *then there are not complex numbers* X, Y, $X + Y \neq 0$, $|Y| \leq 1$ *such that* $P(X,Y) \subset P(A,B) \star P(C,D)$.

Proof. As in [5], the proof will be divided into three steps. First we give a family of bounded functions ω^ν; $\nu = 1,2,3\ldots$ having special properties of coefficients. Afterwards we construct, using ω^ν, a function belonging to the class $P(X,Y)$ and finally we will show that such a function is not in the class $P(A,B) \star P(C,D)$.

Now we use a method of E. Landau [6] and find some functions ω^ν, $\nu = 1,2,3,\ldots$, which are basic in this proof. We observe that

$$\frac{1}{1-z} = \left(\frac{1}{\sqrt{1-z}}\right)^2 = \left(\sum_{k=0}^{\infty} p_k z^k\right)^2 = 1 + z + z^2 + z^3 + \cdots ,$$

where

$$p_0 = 1 \ \text{ and } \ p_k = \frac{1 \cdot 3 \cdot \cdots \cdot (2k-1)}{2 \cdot 4 \cdot \cdots \cdot 2k}; \quad k = 1,2,3,\ldots \tag{2.1}$$

For some $\nu \in \mathbb{N}$ we set

$$K_\nu(z) = \sum_{k=0}^{\nu} p_k z^k = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots + p_\nu z^\nu$$

and note that

$$K_\nu^2(z) = 1 + z + z^2 + \cdots + z^\nu + b_{\nu+1} z^{\nu+1} + \cdots + b_{2\nu} z^{2\nu},$$

where $b_{\nu+1}, \ldots, b_{2\nu} \in \mathbb{C}$. Let ω^ν be given by

$$\omega^\nu(z) = \frac{z^{\nu+1} K_\nu\left(\frac{1}{z}\right)}{K_\nu(z)} = z\frac{z^\nu + p_1 z^{\nu-1} + p_2 z^{\nu-2} + \cdots + p_\nu}{1 + p_1 z + p_2 z^2 + \cdots + p_\nu z^\nu}. \tag{2.2}$$

Since

$$p_k > p_k \frac{2k+1}{2k+2} = p_{k+1} \quad \text{where} \quad k = 0,1,2,3,\ldots,$$

then for $\nu \in \mathbb{N}$ we have $1 > p_1 > p_2 > p_3 > \cdots > p_\nu > 0$.

Applying Theorem E to the polynomial K_ν we obtain $K_\nu(z) \neq 0$ for $|z| \leq 1$, hence the function ω^ν is regular in Δ. Moreover $\omega^\nu(0) = 0$ and on the circle $|z| = 1$ we have

$$\left|\omega^\nu\left(e^{it}\right)\right| = \left|\frac{e^{(\nu+1)it} K_\nu\left(e^{-it}\right)}{K_\nu\left(e^{it}\right)}\right| = \frac{\left|\overline{K_\nu\left(e^{it}\right)}\right|}{\left|K_\nu\left(e^{it}\right)\right|} = 1; \quad t \in \mathbb{R}.$$

In this way we conclude that for $\nu \in \{1, 2, 3, \dots\}$

$$\omega^{\nu} \in \Omega. \tag{2.3}$$

Let, for a certain $\nu \in \mathbb{N}$, the function ω^{ν} be represented by following power series expansions: $\omega^{\nu}(z) = \gamma_1^{\nu} z + \gamma_2^{\nu} z^2 + \gamma_3^{\nu} z^3 + \cdots$ and let $s_n^{\nu}(z)$ denote the partial sum

$$s_n^{\nu}(z) = \sum_{k=1}^{n} \gamma_k^{\nu} z^k = \gamma_1^{\nu} z + \gamma_2^{\nu} z^2 + \gamma_3^{\nu} z^3 + \cdots + \gamma_n^{\nu} z^n$$

for all $n \in \{1, 2, 3, \dots, \nu + 1\}$.

Now we will estimate $s_n^{\nu}(1) = \gamma_1^{\nu} + \gamma_2^{\nu} + \gamma_3^{\nu} + \cdots + \gamma_n^{\nu}$. If we integrate on a circle $C : |z| = r$ in a counterclockwise direction with $0 < r < 1$, then we obtain

$$\int_C a z^m dz = 0 \quad \text{and} \quad \int_C \frac{a}{z} dz = 2\pi a i, \tag{2.4}$$

for all integers $m \neq -1$ and $a \in \mathbb{C}$. Hence

$$\begin{aligned}
s_n^{\nu}(1) &= \gamma_1^{\nu} + \gamma_2^{\nu} + \gamma_3^{\nu} + \cdots + \gamma_n^{\nu} \\
&= \frac{1}{2\pi i} \int_C \omega^{\nu}(z) \left(\frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots + \frac{1}{z^{n+1}} \right) dz \\
&= \frac{1}{2\pi i} \int_C \frac{\omega^{\nu}(z)}{z^{n+1}} \left(1 + z + z^2 + z^3 + \cdots + z^{n-1}\right) dz \\
&= \frac{1}{2\pi i} \int_C \frac{\omega^{\nu}(z)}{z^{n+1}} Q(z) dz,
\end{aligned}$$

where

$$Q(z) = 1 + z + z^2 + \cdots + z^{n-1} + d_n z_n + d_{n+1} z^{n+1} + \cdots + d_s z^s$$

is any polynomial whose first n terms are equal to $1 + z + z^2 + \cdots + z^{n-1}$.

If for $n \leq \nu + 1$ we set

$$Q(z) = K_{\nu}^2(z) = 1 + z + z^2 + \cdots + z^{n-1} + z^n + \cdots + z^{\nu} + b_{\nu+1} z^{\nu+1} + \cdots + b_{2\nu} z^{2\nu},$$

then we obtain

$$s_n^{\nu}(1) = \frac{1}{2\pi i} \int_C \frac{\omega^{\nu}(z)}{z^{n+1}} K_{\nu}^2(z) dz.$$

From (2.2) it follows that

$$\begin{aligned}
s_n^{\nu}(1) &= \frac{1}{2\pi i} \int_C \frac{\frac{z^{\nu+1} K_{\nu}\left(\frac{1}{z}\right)}{K_{\nu}(z)}}{z^{n+1}} K_{\nu}^2(z) dz \\
&= \frac{1}{2\pi i} \int_C z^{\nu-n} K_{\nu}\left(\frac{1}{z}\right) K_{\nu}(z) dz \\
&= \frac{1}{2\pi i} \int_C z^{\nu-n} \left(1 + p_1 \frac{1}{z} + p_2 \frac{1}{z^2} + \cdots + p_{\nu} \frac{1}{z^{\nu}}\right) \left(1 + p_1 z + p_2 z^2 + \cdots + p_{\nu} z^{\nu}\right) dz.
\end{aligned}$$

Using (2.4) we get

$$s_n^{\nu}(1) = p_{\nu-n+1} + p_{\nu-n+2} p_1 + p_{\nu-n+3} p_2 + \cdots + p_{\nu} p_{n-1},$$

where (p_k) are given by (2.1), and so

$$s_n^\nu(1) = \sum_{k=1}^{n} \gamma_k^\nu = \sum_{k=1}^{n} p_{\nu-n+k} p_{k-1}. \tag{2.5}$$

By the above we have $\gamma_1^\nu = s_1^\nu(1) = p_\nu$ and for all $\nu \in \{1, 2, 3, \dots\}$ and $n \in \{2, 3, 4, \dots, \nu+1\}$

$$\gamma_n^\nu = s_n^\nu(1) - s_{n-1}^\nu(1) = \sum_{k=1}^{n} p_{\nu-n+k} p_{k-1} - \sum_{k=1}^{n-1} p_{\nu-n+1+k} p_{k-1}.$$

Therefore

$$\gamma_n^\nu = \sum_{k=1}^{n-1} \left(p_{\nu-n+k} - p_{\nu-n+1+k}\right) p_{k-1} + p_\nu p_{n-1}. \tag{2.6}$$

The sequence (p_n) is positive and decreasing, then from (2.6) it follows that

$$\gamma_n^\nu > 0 \quad \text{for all} \quad \nu \in \mathbb{N} \quad \text{and} \quad n \in \{1, 2, 3, \dots, \nu+1\}. \tag{2.7}$$

We conclude from (2.5) that for $n = \nu + 1$

$$s_{\nu+1}^\nu(1) = \gamma_1^\nu + \gamma_2^\nu + \cdots + \gamma_\nu^\nu + \gamma_{\nu+1}^\nu = \sum_{k=1}^{\nu+1} p_{k-1} p_{k-1} = 1 + \sum_{k=1}^{\nu} p_k^2.$$

Since $\sum_{k=1}^{\infty} p_k^2 = \infty$ then

$$\lim_{\nu\to\infty} s_{\nu+1}^\nu(1) = \lim_{\nu\to\infty} \left(1 + \sum_{k=1}^{\nu} p_k^2\right) = +\infty. \tag{2.8}$$

Now using the properties of ω^ν we construct the function belonging to the class $P(X, Y)$ which is not in the class $P(A, B) \star P(C, D)$, where $|B| < 1$ and $|D| < 1$.

Since for $|x| = 1$ we have $P(A, B) = P(Ax, Bx)$, we can assume without loss of generality that $B \in [0; 1)$, $D \in [0; 1)$ and $Y \in [0; 1]$.

For fixed $\nu \in \mathbb{N}$ let h^ν be given by

$$h^\nu(z) = \frac{1 + X\omega^\nu(z)}{1 - Y\omega^\nu(z)}, \tag{2.9}$$

where $w^\nu \in \Omega$ is given by (2.2). It is clearly that $h^\nu \in P(X, Y)$. Suppose that there exist the functions $f \in P(A, B)$, $g \in P(C, D)$ such that

$$f(z) \star g(z) = h^\nu(z). \tag{2.10}$$

Let the functions f and g have the following form:

$$f(z) = \frac{1 + A\omega_1(z)}{1 - B\omega_1(z)} = 1 + (A + B)\frac{\omega_1(z)}{1 - B\omega_1(z)}$$

and

$$g(z) = \frac{1 + C\omega_2(z)}{1 - D\omega_2(z)} = 1 + (C + D)\frac{\omega_2(z)}{1 - D\omega_2(z)},$$

where $\omega_1, \omega_2 \in \Omega$. For simplicity of notation we write

$$f(z) = 1 + (A + B)\tilde{f}(z), \qquad g(z) = 1 + (C + D)\tilde{g}(z),$$

where

$$\tilde{f}(z) = \frac{\omega_1(z)}{1 - B\omega_1(z)} \quad \text{and} \quad \tilde{g}(z) = \frac{\omega_2(z)}{1 - D\omega_2(z)}. \tag{2.11}$$

Using these notations we can rewrite (2.10) as

$$\left[1+(A+B)\tilde{f}(z)\right]\star[1+(C+D)\tilde{g}(z)]=\frac{1+X\omega^{\nu}(z)}{1-Y\omega^{\nu}(z)}$$

and so

$$\tilde{f}(z)\star\tilde{g}(z)=\frac{X+Y}{(A+B)(C+D)}\tilde{h}^{\nu}(z), \tag{2.12}$$

where $\tilde{h}^{\nu}(z)=\frac{\omega^{\nu}(z)}{1-BD\omega^{\nu}(z)}$.

Let the functions $\tilde{h}^{\nu}$, $\tilde{f}$ and $\tilde{g}$ have the following expansions in Δ:

$$\tilde{h}^{\nu}(z)=\sum_{n=1}^{\infty}c_n^{\nu}z^n,\quad \tilde{f}(z)=\sum_{n=1}^{\infty}a_nz^n,\quad \tilde{g}(z)=\sum_{n=1}^{\infty}b_nz^n. \tag{2.13}$$

From (2.11) it follows, that $\tilde{f}(z)\prec\frac{z}{1-Bz}$ and $\tilde{g}(z)\prec\frac{z}{1-Dz}$, and hence by Theorem F (and since $0\le B<1$ and $0\le D<1$) we obtain

$$\sum_{n=1}^{\infty}|a_n|^2\le\frac{1}{1-B^2}\quad\text{and}\quad\sum_{n=1}^{\infty}|b_n|^2\le\frac{1}{1-D^2}.$$

Let us note, that

$$\sum_{n=1}^{\infty}\left(|a_n|^2+|b_n|^2\right)=\sum_{n=1}^{\infty}\left[(|a_n|-|b_n|)^2+2|a_n||b_n|\right]\le\frac{1}{1-|B|^2}+\frac{1}{1-|D|^2}.$$

From (2.12) and (2.13) we obtain

$$a_nb_n=\frac{X+Y}{(A+B)(C+D)}c_n^{\nu},\quad\text{for}\quad n=1,2,3,\dots,$$

therefore

$$\sum_{n=1}^{\infty}(|a_n|-|b_n|)^2\le\frac{1}{1-B^2}+\frac{1}{1-D^2}-\left|\frac{2(X+Y)}{(A+B)(C+D)}\right|\sum_{n=1}^{\infty}|c_n^{\nu}|. \tag{2.14}$$

Now we observe that

$$\begin{aligned}\sum_{n=1}^{\infty}c_n^{\nu}z^n&=\tilde{h}^{\nu}(z)=\frac{\omega^{\nu}(z)}{1-Y\omega^{\nu}(z)}\\&=\omega^{\nu}(z)+Y\left[\omega^{\nu}(z)\right]^2+Y^2\left[\omega^{\nu}(z)\right]^3+\cdots\\&=\left(\gamma_1^{\nu}z+\gamma_2^{\nu}z^2+\cdots\right)+Y\left(\gamma_1^{\nu}z+\gamma_2^{\nu}z^2+\cdots\right)^2+Y^2\left(\gamma_1^{\nu}z+\gamma_2^{\nu}z^2+\cdots\right)^3+\cdots\\&=\gamma_1^{\nu}z+\left(\gamma_2^{\nu}+Y(\gamma_1^{\nu})^2\right)z^2+\left(\gamma_3^{\nu}+2Y\gamma_1^{\nu}\gamma_2^{\nu}+Y^2(\gamma_1^{\nu})^3\right)z^3+\cdots.\end{aligned}$$

Since $Y\in[0;1]$ and since (2.7) we have

$$\begin{aligned}\sum_{n=1}^{\infty}|c_n^{\nu}|&=|\gamma_1^{\nu}|+\left|\gamma_2^{\nu}+Y(\gamma_1^{\nu})^2\right|+\left|\gamma_3^{\nu}+2Y\gamma_1^{\nu}\gamma_2^{\nu}+Y^2(\gamma_1^{\nu})^3\right|+\cdots\\&\ge\sum_{n=1}^{\nu+1}|c_n^{\nu}|\ge\gamma_1^{\nu}+\gamma_2^{\nu}+\gamma_3^{\nu}+...+\gamma_{\nu+1}^{\nu}=s_{\nu+1}^{\nu}(1).\end{aligned}$$

From the above we have for all $\nu\in\{1,2,3,\dots\}$

$$\sum_{n=1}^{\infty}|c_n^{\nu}|\ge s_{\nu+1}^{\nu}(1). \tag{2.15}$$

Combining (2.14) and (2.15) we obtain

$$\sum_{n=1}^{\infty}(|a_n|-|b_n|)^2 \le \frac{1}{1-B^2}+\frac{1}{1-D^2}-\left|\frac{2(X+Y)}{(A+B)(C+D)}\right| s_{\nu+1}^{\nu}(1). \tag{2.16}$$

From (2.8) it follows that we are able to choose a suitable ν such that the right side of (2.16) is negative. In this way (2.16) follows the contradiction and the proof is complete. □

From Theorem 2.1 and Theorem D we immediately have the following

Corollary 2.2. *The classes* $P(A,B)\star P(C,D)$ *and* $P(AD+AC+BC,BD)$ *are equal if and only if* $|B|=1$ *or* $|D|=1$.

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