ON THE MAXIMUM ROW AND COLUMN SUM NORM OF GCD AND RELATED MATRICES

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Abstract: We estimate the maximum row and column sum norm of the $n \times n$ matrix, whose ij entry is $(i,j)^s/[i,j]^r$, where $r, s \in \mathbb{R}$, (i,j) is the greatest common divisor of i and j and $[i,j]$ is the least common multiple of i and j.

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1. Introduction

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers, and let f be an arithmetical function. Let $(S)_f$ denote the $n \times n$ matrix having f evaluated at the greatest common divisor (x_i, x_j) of x_i and x_j as its ij entry, that is, $(S)_f = (f((x_i, x_j)))$. Analogously, let $[S]_f$ denote the $n \times n$ matrix having f evaluated at the least common multiple $[x_i, x_j]$ of x_i and x_j as its ij entry, that is, $[S]_f = (f([x_i, x_j]))$. The matrices $(S)_f$ and $[S]_f$ are referred to as the GCD and LCM matrices on S associated with f. H. J. S. Smith [15] calculated $\det(S)_f$ when S is a factor-closed set and $\det[S]_f$ in a more special case. Since Smith, a large number of results on GCD and LCM matrices have been presented in the literature. For general accounts see e.g. [8, 9, 12, 14].

Norms of GCD matrices have not been discussed much in the literature. Some results for the ℓ_p norm are reported in [1, 6, 7], see also the references in [6]. In this paper we consider the maximum row sum norm in a similar way as we considered the ℓ_p norm in [6]. Since the matrices in this paper are symmetric, all the results also hold for the maximum column sum norm.

The maximum row sum norm of an $n \times n$ matrix M is defined as

$$\|\|M\|\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |m_{ij}|.$$

Let $r, s \in \mathbb{R}$. Let A denote the $n \times n$ matrix, whose i, j entry is given as

$$a_{ij} = \frac{(i,j)^s}{[i,j]^r}, \tag{1.1}$$

where (i, j) is the greatest common divisor of i and j and $[i, j]$ is the least common multiple of i and j. For $s = 1, r = 0$ and $s = 0, r = -1$, respectively, the matrix

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A is the GCD and the LCM matrix on $\{1, 2, \ldots, n\}$. For $s = 1, r = 1$ the matrix A is the Hadamard product of the GCD matrix and the reciprocal LCM matrix on $\{1, 2, \ldots, n\}$. In this paper we estimate the maximum row sum norm of the matrix A given in (1.1) for all $r, s \in \mathbb{R}$.

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2. Preliminaries

In this section we review the basic results on arithmetical functions needed in this paper. For more comprehensive treatments of arithmetical functions we refer to [2, 13, 14].

The Dirichlet convolution $f * g$ of two arithmetical functions f and g is defined as

$$(f * g)(n) = \sum_{d|n} f(d)g(n/d).$$

Let N^u, $u \in \mathbb{R}$, denote the arithmetical function defined as $N^u(n) = n^u$ for all $n \in \mathbb{Z}^+$, and let E denote the arithmetical function defined as $E(n) = 1$ for all $n \in \mathbb{Z}^+$. The divisor function σ_u, $u \in \mathbb{R}$, is defined as

$$\sigma_u(n) = \sum_{d|n} d^u = (N^u * E)(n). \tag{2.1}$$

It is known that if $0 \leq u < 1$, then

$$\sigma_u(n) = O(n^{u+\epsilon}) \tag{2.2}$$

for all $\epsilon > 0$ (see [5]),

$$\sigma_1(n) = O(n \log\log n) \tag{2.3}$$

(see [4, 11, 13]), and if $u > 1$, then

$$\sigma_u(n) = O(n^u) \tag{2.4}$$

(see [3, 4, 13]).

The Jordan totient function $J_k(n)$, $k \in \mathbb{Z}^+$, is defined as the number of k-tuples $a_1, a_2, \ldots, a_k \pmod{n}$ such that the greatest common divisor of $a_1, a_2, \ldots, a_k$ and

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n is 1. By convention, $J_k(1) = 1$. The Möbius function μ is the inverse of E under the Dirichlet convolution. It is well known that $J_k = N^k * \mu$. This suggests we define $J_u = N^u * \mu$ for all $u \in \mathbb{R}$. Since μ is the inverse of E under the Dirichlet convolution, we have

$$n^u = \sum_{d|n} J_u(d). \tag{2.5}$$

It is easy to see that

$$J_u(n) = n^u \prod_{p|n} (1 - p^{-u}).$$

We thus have

$$0 \le J_u(n) \le n^u \quad \text{for } u \ge 0. \tag{2.6}$$

The following estimates for the summatory function of N^u are well known (see [2]):

$$\sum_{k \le n} k^{-u} = O(1) \quad \text{if } u > 1, \tag{2.7}$$

$$\sum_{k \le n} k^{-1} = O(\log n), \tag{2.8}$$

$$\sum_{k \le n} k^{-u} = O(n^{1-u}) \quad \text{if } u < 1. \tag{2.9}$$

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3. Results

In Theorems 3.1 – 3.5 we estimate the maximum row sum norm of the matrix A given in (1.1). Their proofs are based on the formulas in Section 2 and the following observations.

Since $(i,j)[i,j] = ij$, we have for all r, s

$$\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} \frac{(i,j)^s}{[i,j]^r} = \max_{1\le i\le n} \sum_{j=1}^{n} \frac{(i,j)^{r+s}}{i^r j^r}. \tag{3.1}$$

From (2.5) we obtain

$$\|A\|_\infty = \max_{1\le i\le n} \frac{1}{i^r} \sum_{j=1}^{n} \frac{1}{j^r} \sum_{d\mid(i,j)} J_{r+s}(d)$$

$$= \max_{1\le i\le n} \frac{1}{i^r} \sum_{d\mid i} J_{r+s}(d) \sum_{\substack{j=1 \\ d\mid j}}^{n} \frac{1}{j^r}$$

$$= \max_{1\le i\le n} \frac{1}{i^r} \sum_{d\mid i} \frac{J_{r+s}(d)}{d^r} \sum_{j=1}^{[n/d]} \frac{1}{j^r}. \tag{3.2}$$

Theorem 3.1. *Suppose that* $r > 1$.

1. *If* $s \ge r$, *then* $\|A\|_\infty = O(n^{s-r})$.
2. *If* $s < r$, *then* $\|A\|_\infty = O(1)$.

Proof. Let $r > 1$ and $s \ge 0$. Then, by (3.2) and (2.7),

$$\|A\|_\infty = O(1) \max_{1\le i\le n} \frac{1}{i^r} \sum_{d\mid i} \frac{J_{r+s}(d)}{d^r}.$$

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Since $r + s \geq 0$, according to (2.6) and (2.1),

$$|||A|||_\infty = O(1) \max_{1\leq i\leq n} \frac{\sigma_s(i)}{i^r}.$$

Now, if $s \geq r > 1$, then on the basis of (2.4),

$$|||A|||_\infty = O(1) \max_{1\leq i\leq n} i^{s-r} = O(n^{s-r}).$$

If $0 \leq s < r$, then

$$|||A|||_\infty = O(1) \max_{1\leq i\leq n} i^{s-r+\epsilon} = O(1).$$

Let $r > 1$ and $s < 0$. Then

$$|||A|||_\infty \leq \max_{1\leq i\leq n} \sum_{j=1}^{n} \frac{1}{j^r} = O(1).$$

□

Theorem 3.2. *Suppose that* $r = 1$.

1. *If* $s > 1$, *then* $|||A|||_\infty = O(n^{s-1} \log n)$.
2. *If* $s = 1$, *then* $|||A|||_\infty = O(\log n \ \log\log n)$.
3. *If* $s < 1$, *then* $|||A|||_\infty = O(\log n)$.

Proof. From (3.2) with $r = 1$ we obtain

$$|||A|||_\infty = \max_{1\leq i\leq n} \frac{1}{i} \sum_{d|i} \frac{J_{s+1}(d)}{d} \sum_{j=1}^{[n/d]} \frac{1}{j}.$$

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By (2.8),

$$|||A|||_\infty = O(\log n) \max_{1\le i\le n} \frac{1}{i} \sum_{d|i} \frac{J_{s+1}(d)}{d}.$$

Since $s \ge 0$, on the basis of (2.6) and (2.1),

$$|||A|||_\infty = O(\log n) \max_{1\le i\le n} \frac{\sigma_s(i)}{i}.$$

If $s > 1$, then according to (2.4),

$$|||A|||_\infty = O(\log n) \max_{1\le i\le n} i^{s-1} = O(n^{s-1}\log n).$$

If $s = 1$, then according to (2.3),

$$|||A|||_\infty = O(\log n)O(\log\log n) = O(\log n \ \log\log n).$$

If $0 \le s < 1$, then according to (2.2),

$$|||A|||_\infty = O(\log n) \max_{1\le i\le n} i^{s-1+\epsilon} = O(\log n).$$

If $s < 0$, then according to (3.1),

$$|||A|||_\infty \le \max_{1\le i\le n} \sum_{j=1}^{n} \frac{1}{j} = O(\log n).$$

□

Remark 1. Let $\|M\|_1$ denote the sum norm (or ℓ_1 norm) of an $n \times n$ matrix M, that is

$$\|M\|_1 = \sum_{i=1}^{n}\sum_{j=1}^{n} |m_{ij}|.$$

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It is known [6, Theorem 3.2(1)] that

$$\left\|\left((i,j)^s/[i,j]\right)\right\|_1 = O(n^s \log^2 n), \ s \geq 1. \tag{3.3}$$

Since $\|M\|_1 \leq n\,\|\|M\|\|_\infty$ for all $n \times n$ matrices M (see [10]), we obtain from Theorem 3.2(1,2) an improvement on (3.3) as

$$\left\|\left((i,j)^s/[i,j]\right)\right\|_1 = O(n^s \log n), \ s > 1, \tag{3.4}$$

$$\left\|\left((i,j)/[i,j]\right)\right\|_1 = O(n \log n \ \log\log n). \tag{3.5}$$

Theorem 3.3. *Suppose that* $r < 1$.

1. *If* $s > 2 - r$, *then* $\|\|A\|\|_\infty = O(n^{s-r})$.
2. *If* $s = 2 - r$, *then* $\|\|A\|\|_\infty = O(n^{2-2r} \log\log n)$.
3. *If* $\max\{1-r, 1\} \leq s < 2 - r$, *then* $\|\|A\|\|_\infty = O(n^{s-r+\epsilon})$ *for all* $\epsilon > 0$.
4. *If* $1 - r \leq s < 1$, *then* $\|\|A\|\|_\infty = O(n^{1-r})$.

Proof. Let $r < 1$. By (3.2) and (2.9),

$$\|\|A\|\|_\infty = O(n^{1-r}) \max_{1\leq i\leq n} \frac{1}{i^r} \sum_{d|i} \frac{J_{r+s}(d)}{d}.$$

Since $r + s \geq 0$, by (2.6) and (2.1),

$$\|\|A\|\|_\infty = O(n^{1-r}) \max_{1\leq i\leq n} \frac{\sigma_{r+s-1}(i)}{i^r}.$$

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If $s > 2 - r$ or $r + s - 1 > 1$, then according to (2.4),

$$|||A|||_\infty = O(n^{1-r}) \max_{1 \le i \le n} i^{s-1}.$$

Since $s - 1 \ge 0$, we have

$$|||A|||_\infty = O(n^{s-r}).$$

If $s = 2 - r$ or $r + s - 1 = 1$, then according to (2.3),

$$|||A|||_\infty = O(n^{1-r}) \max_{1 \le i \le n} i^{1-r} \log\log i.$$

Since $1 - r > 0$, we have

$$|||A|||_\infty = O(n^{2-2r} \log\log n).$$

If $1 - r \le s < 2 - r$ or $0 \le r + s - 1 < 1$, then according to (2.2),

$$|||A|||_\infty = O(n^{1-r}) \max_{1 \le i \le n} i^{s-1+\epsilon}. \tag{3.6}$$

If $s \ge 1$ in (3.6), we obtain $|||A|||_\infty = O(n^{s-r+\epsilon})$. If $s < 1$ in (3.6), we obtain $|||A|||_\infty = O(n^{1-r})$. □

Corollary 3.4. *Suppose that $r = 0$.*

1. *If $s > 2$, then $|||A|||_\infty = O(n^s)$.*
2. *If $s = 2$, then $|||A|||_\infty = O(n^2 \log\log n)$.*
3. *If $1 \le s < 2$, then $|||A|||_\infty = O(n^{s+\epsilon})$ for all $\epsilon > 0$. In particular, for $s = 1$,*

$$\left|\left|\left| \big((i,j)\big) \right|\right|\right|_\infty = O(n^{1+\epsilon}) \text{ for all } \epsilon > 0. \tag{3.7}$$

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Remark 2. Let $\|M\|_2$ denote the ℓ_2 norm of an $n \times n$ matrix M, that is

$$\|M\|_2 = \sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}^2.$$

It is known [6, Theorem 3.2(1)] that

$$\left\|\left((i,j)^{3/2}/[i,j]^{1/2}\right)\right\|_2 = O(n^{3/2}\log n). \tag{3.8}$$

Since $\|M\|_2 \le \sqrt{n}\,\|M\|_\infty$ for all $n \times n$ matrices M (see [10]), we obtain from Theorem 3.3(2) an improvement on (3.8) as

$$\left\|\left((i,j)^{3/2}/[i,j]^{1/2}\right)\right\|_2 = O(n^{3/2}\log\log n). \tag{3.9}$$

In Theorem 3.5 we treat the remaining cases of r and s in the most elementary way.

Theorem 3.5.

1. *If* $0 \le r < 1$ *and* $s \le 0$, *then* $\|A\|_\infty = O(n^{1-r})$.
2. *If* $r < 0$ *and* $s \le 0$, *then* $\|A\|_\infty = O(n^{1-2r})$.
3. *If* $0 \le r < 1$, $s > 0$ *and* $r + s < 1$, *then* $\|A\|_\infty = O(n^{1+s-r})$.
4. *If* $r < 0$, $s > 0$ *and* $r + s < 1$, *then* $\|A\|_\infty = O(n^{1+s-2r})$.

Proof. Let $0 \le r < 1$ and $s \le 0$. Then, according to (3.1) and (2.9)

$$\|A\|_\infty \le \max_{1\le i\le n} \sum_{j=1}^{n} \frac{1}{j^r} = O(n^{1-r}).$$

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Let $r < 0$ and $s \leq 0$. Then, according to (3.1) and the inequality $[i,j] < n^2$

$$|||A|||_\infty \leq \max_{1\leq i\leq n} \sum_{j=1}^{n} [i,j]^{-r} < \max_{1\leq i\leq n} \sum_{j=1}^{n} n^{-2r} = O(n^{1-2r}).$$

Let $0 \leq r < 1$, $s > 0$ and $r + s < 1$. Then, according to (3.1) and (2.9)

$$|||A|||_\infty \leq n^s \max_{1\leq i\leq n} \sum_{j=1}^{n} \frac{1}{j^r} = O(n^{1+s-r}).$$

Let $r < 0$, $s > 0$ and $r + s < 1$. Then, according to (3.1) and the inequality $[i,j] < n^2$

$$|||A|||_\infty \leq \max_{1\leq i\leq n} \sum_{j=1}^{n} \frac{n^s}{n^{2r}} = O(n^{1+s-2r}).$$

□

Remark 3. Applying [6, Theorem 3.3] and the inequality $|||M|||_\infty \leq \sqrt{n}\|M\|_2$ for all $n \times n$ matrices M (see [10]) a partial improvement on Theorem 3.5(4) of the present paper as

(a) if $r < 0$, $s > 0$ and $1/2 < r + s < 1$, then $|||A|||_\infty = O(n^{1+s-r})$,

(b) if $r < 0$, $s > 0$ and $r + s = 1/2$, then $|||A|||_\infty = O(n^{-2r+3/2} \log^{1/2} n)$,

(c) if $r < 0$, $s > 1/2$ and $r + s < 1/2$, then $|||A|||_\infty = O(n^{-2r+3/2})$.

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