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THE BEST CONSTANTS FOR A DOUBLE INEQUALITY IN A TRIANGLE

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Dedicated to Professor Bi-Cheng Yang on the occasion of his 63rd birthday.

ABSTRACT. In this short note, by using some of Chen's theorems and classic analysis, we obtain a double inequality for triangle and give a positive answer to a problem posed by Yang and Yin [6].

Key words and phrases: Inequality; Best Constant; Triangle.

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1. INTRODUCTION AND MAIN RESULTS

For $\triangle ABC$, let a, b, c denote the side-lengths, A, B, C the angles, s the semi-perimeter, R the circumradius and r the inradius, respectively.

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In 1957, Kooistra (see [1]) built the following double inequality for any triangle:

$$2 < \cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} \leq \frac{3\sqrt{3}}{2}. \tag{1.1}$$

In 2000, Yang and Yin [6] considered a new bound of inequality (1.1) and posed a problem as follows:

Problem 1.1. Determine the best constant μ such that

$$\left(\frac{3\sqrt{3}}{2}\right)^{\mu} \cdot \left(\frac{s}{R}\right)^{1-\mu} \leq \cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} \tag{1.2}$$

holds for any $\triangle ABC$.

In this short note, we solve the above problem and obtain the following result.

Theorem 1.1. *Let*

$$\lambda \geq \lambda_0 = 1$$

and

$$\mu \leq \mu_0 = \frac{2\ln(2-\sqrt{2}) + \ln 2}{4\ln 2 - 3\ln 3} \approx 0.7194536993.$$

Then the double inequality

$$\left(\frac{3\sqrt{3}}{2}\right)^{\mu} \cdot \left(\frac{s}{R}\right)^{1-\mu} \leq \cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} \leq \left(\frac{3\sqrt{3}}{2}\right)^{\lambda} \cdot \left(\frac{s}{R}\right)^{1-\lambda} \tag{1.3}$$

holds for any $\triangle ABC$, while the constants λ_0 and μ_0 are both the best constant for inequality (1.3).

Remark 1. When $\lambda_0 = 1$, the right hand of inequality (1.3) is just the right hand of inequality (1.1).

Remark 2. It is not difficult to demonstrate that:

$$\begin{cases} \left(\frac{3\sqrt{3}}{2}\right)^{\mu_0} \cdot \left(\frac{s}{R}\right)^{1-\mu_0} < 2 \left(0 < \frac{s}{R} < 2^{\frac{1}{1-\mu_0}} \left(\frac{2}{3\sqrt{3}}\right)^{\frac{\mu_0}{1-\mu_0}}\right), \\ \left(\frac{3\sqrt{3}}{2}\right)^{\mu_0} \cdot \left(\frac{s}{R}\right)^{1-\mu_0} \geq 2 \left(2^{\frac{1}{1-\mu_0}} \left(\frac{2}{3\sqrt{3}}\right)^{\frac{\mu_0}{1-\mu_0}} \leq \frac{s}{R} \leq \frac{3\sqrt{3}}{2}\right). \end{cases}$$

2. PRELIMINARY RESULTS

In order to establish our main theorem, we shall require the following lemmas.

Lemma 2.1 (see [3, 4, 5])**.** *If the inequality $s \geq (>) f(R, r)$ holds for any isosceles triangle whose top-angle is greater than or equal to $\frac{\pi}{3}$, then the inequality $s \geq (>) f(R, r)$ holds for any triangle.*

Lemma 2.2 (see [2, 3])**.** *The homogeneous inequality*

$$s \geq (>) f(R, r) \tag{2.1}$$

holds for any acute-angled triangle if and only if it holds for any acute isosceles triangle whose top-angle $A \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right)$ *with* $2r \leq R < (\sqrt{2}+1)r$ *and any right-angled triangle with* $R \geq (\sqrt{2}+1)r$.

For the convenience of our readers, we give below the proof by Chen in [2, 3].

Proof. Let $\odot O$ denote the circumcircle of $\triangle ABC$. Necessity is obvious from Lemma 2.1. Thus we only need to prove the sufficiency. It is well known that $R \geq 2r$ for any acute-angled triangle. So we consider the following two cases:

(i) When $2r \leq R < (\sqrt{2}+1)r$: In this case , we can construct an isosceles triangle $A_1B_1C_1$ whose circumcircle is also $\odot O$ and the top-angle of $\triangle A_1B_1C_1$(see Figure 2.1) is

$$A_1 = 2\arcsin \frac{1}{2}\left(1 + \sqrt{1 - \frac{2r}{R}}\right).$$

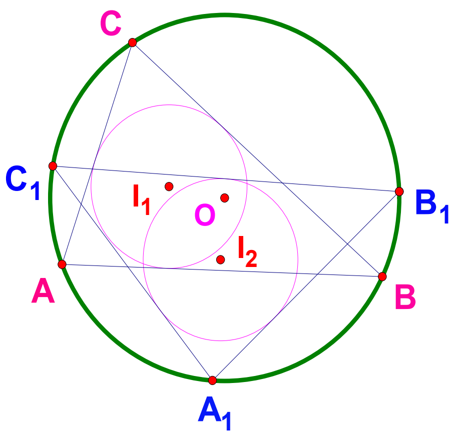

Figure 2.1:

It is easy to see that (see [4, 5]):

$$R_1 = R, \quad r_1 = r, \quad s_1 \leq s \quad \text{and} \quad \frac{\pi}{3} \leq A_1 < \frac{\pi}{2}.$$

Thus we have

$$s \geq s_1 \geq f(R_1, r_1) = f(R, r). \tag{2.2}$$

because the inequality (2.1) holds for any acute isosceles triangle whose top-angle $A \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right)$.

(ii) When $R \geq (\sqrt{2}+1)r$: In this case we can construct a right-angled triangle $A_2B_2C_2$ whose inscribed circle is also $\odot I$ and the length of its hypotenuse is $c_2 = 2R$ (see Figure 2.2). This implies that

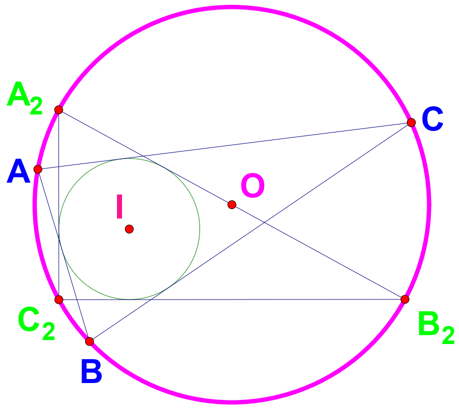

Figure 2.2:

$$r_2 = \frac{1}{2}(a_2+b_2-c_2) = r, \quad R_2 = \frac{1}{2}c_2 = R,$$
$$s_2 = \frac{1}{2}(a_2+b_2+c_2) = 2R_2 + r_2 = 2R + r < s.$$

Thus we have the inequality (2.2) since the inequality (2.1) holds for any right-angled triangle.

□

Lemma 2.3 (see [2, 3])**.** *The homogeneous inequality* (2.1) *holds for any acute-angled triangle if and only if*

(2.3) $$\sqrt{(1-x)(3+x)^3} \geq (>) f(2, 1-x^2) \qquad \left(0 \leq x < \sqrt{2}-1\right),$$

and

(2.4) $$5 - x^2 \geq (>) f(2, 1-x^2) \qquad \left(\sqrt{2}-1 \leq x < 1\right).$$

Proof. Since the inequality (2.1) is homogeneous, we may assume $R = 2$ without losing generality.

(i) When $2r \leq R < (\sqrt{2}+1)r$: By Lemma 2.2, we only need to consider the isosceles triangle whose top-angle $A \in [\frac{\pi}{3}, \frac{\pi}{2})$. Let

$$t = \sin\frac{A}{2} \in \left[\frac{1}{2}, \frac{\sqrt{2}}{2}\right).$$

Then we have (see [4, 5])

(2.5) $$r = 4t(1-t) \quad \text{and} \quad s = 4(1+t)\sqrt{1-t^2}.$$

Let $x = 2t - 1$. Then the inequality (2.1) is just the inequality (2.3).

(ii) When $R \geq (\sqrt{2}+1)r$: We only need to consider a right-angled triangle. Let

$$r = \frac{2r}{R} = 4t(1-t) \in \left(0, \sqrt{2}-1\right) \qquad \left(\frac{\sqrt{2}}{2} \leq t < 1\right).$$

Thus we have

$$s = 2R + r = 4 + 4t(1-t). \tag{2.6}$$

Let $x = 2t - 1$. Then the inequality (2.1) is just the inequality (2.4).

This completes the proof Lemma 2.3. □

Lemma 2.4 ([3, 4, 5])**.** *The homogeneous inequality*

$$s \leq (<) f(R, r) \tag{2.7}$$

holds for any triangle if and only if it holds for any isosceles triangle whose top-angle $A \in (0, \frac{\pi}{3}]$*, or the following inequality holds*

$$\sqrt{(1-x)(3+x)^3} \leq (<) f\left(2, 1-x^2\right) \qquad (-1 < x \leq 0). \tag{2.8}$$

Lemma 2.5 (see [2, 3])**.** *The homogeneous inequality* (2.7) *holds for any acute-angled triangle if and only if it holds for any isosceles triangle whose top-angle* $A \in (0, \frac{\pi}{3}]$*, or the inequality* (2.8) *holds.*

Proof. As acute-angled triangles include all isosceles triangles whose top-angle is less than or equal to $\frac{\pi}{3}$, Lemma 2.5 straightforwardly follows from Lemma 2.4 and Lemma 2.1. □

Lemma 2.6. *Define*

$$G_1(x) := \frac{2\ln(1-x) + 2\ln(1+x)}{3\ln(1-x) + 3\ln(3+x) + 2\ln(1+x) - 3\ln 3}. \tag{2.9}$$

Then G_1 *is decreasing on* $(-1, \sqrt{2}-1)$*, and*

$$\lim_{x\to(\sqrt{2}-1)^-} G_1(x) = \frac{2\ln(2-\sqrt{2}) + \ln 2}{4\ln 2 - 3\ln 3} < G_1(x) < 1 = \lim_{x\to -1^+} G_1(x). \tag{2.10}$$

Proof. Let G_1' be the derivative of G_1. It is easy to see that

$$G_1'(x) = \frac{4xg_1(x)}{(3\ln(1-x) + 3\ln(3+x) + 2\ln(1+x) - 3\ln 3)^2(1-x^2)(3+x)} \tag{2.11}$$

with

$$g_1(x) := (x-1)\ln(1-x) - 3(x+3)[\ln(3+x) - \ln 3] + 2(x+1)\ln(1+x). \tag{2.12}$$

Moreover, we know that

$$g_1'(x) = \ln(1-x) - 3\ln(3+x) + 2\ln(1+x) + 3\ln 3 \tag{2.13}$$

and

$$g_1''(x) = \frac{-8x}{(1-x^2)(3+x)}. \tag{2.14}$$

Now we show that G_1 is decreasing on $\left(-1, \sqrt{2}-1\right)$.

(i) It is easy to see that $g_1''(x) \geq 0$ when $-1 < x \leq 0$, and g_1' is increasing on $(-1, 0]$. Thus, $g_1'(x) \leq g_1'(0) = 0$, and g_1 is decreasing on $(-1, 0]$. Therefore, $g_1(x) \geq g_1(0) = 0$, and $G_1'(x) \leq 0$. This means that G_1 is decreasing on $(-1, 0]$.

(ii) It is easy to see that $g_1''(x) \leq 0$ when $0 \leq x < \sqrt{2}-1$, and g_1' is decreasing on $\left[0, \sqrt{2}-1\right)$. Thus, $g_1'(x) \leq g_1'(0) = 0$, and g_1 is decreasing on $\left[0, \sqrt{2}-1\right)$. Therefore, $g_1(x) \leq g_1(0) = 0$, and $G_1'(x) \leq 0$. This means that G_1 is decreasing on $\left[0, \sqrt{2}-1\right)$.

Combining (i) and (ii), it follows that G_1 is decreasing on $(-1, \sqrt{2}-1)$ and (2.10) holds, and hence the proof is complete. □

Lemma 2.7. *Define*

$$G_2(x) := \frac{2\ln(1-x^2)}{2\ln(5-x^2) + 2\ln(1-x^2) - 3\ln 3}. \tag{2.15}$$

Then G_2 is increasing on $\left[\sqrt{2}-1, 1\right)$, and

$$G_2(x) \geq G_2(\sqrt{2}-1) = \frac{2\ln(2-\sqrt{2}) + \ln 2}{4\ln 2 - 3\ln 3}. \tag{2.16}$$

Proof. Let G_2' be the derivative of G_2. It is easy to see that

$$G_2'(x) = \frac{4xg_2(x)}{(2\ln(5-x^2) + 2\ln(1-x^2) - 3\ln 3)^2(1-x^2)(5-x^2)}, \tag{2.17}$$

$$g_2'(x) = -2xh_2(x),$$

and

$$h_2'(x) = \frac{-16x}{(1-x^2)(5-x^2)}; \tag{2.18}$$

where

$$g_2(x) := 2\left(1-x^2\right)\left[\ln(1-x) + \ln(1+x)\right] + 2\left(x^2-5\right)\ln\left(5-x^2\right) + 3\left(5-x^2\right)\ln 3,$$

and

$$h_2(x) = 2\ln\left(1-x^2\right) - 2\ln\left(5-x^2\right) + 3\ln 3. \tag{2.19}$$

Thus it follows that $h_2'(x) < 0$ when $\sqrt{2}-1 \leq x < 1$, and h_2 is decreasing on $\left[\sqrt{2}-1, 1\right)$, and

$$h_2(x) \leq h_2(\sqrt{2}-1) = \ln\frac{27}{34+24\sqrt{2}} < 0.$$

Therefore $g_2'(x) > 0$, and g_2 is increasing on $\left[\sqrt{2}-1, 1\right)$, and

$$g_2(x) \geq g_2(\sqrt{2}-1) = 6(\sqrt{2}+1)\ln 3 - 8\ln 2 - 8\sqrt{2}\ln\left(\sqrt{2}+1\right) > 0.$$

This means that $G_2'(x) > 0$, and G_2 is increasing on $\left[\sqrt{2}-1, 1\right)$, and (2.16) holds. The proof of Lemma 2.7 is thus completed. □

3. THE PROOF OF THEOREM 1.1

Proof. (i) The first inequality of (1.3) is equivalent to

$$\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\ge\left(\frac{3\sqrt{3}}{2}\right)^{\mu}\cdot(\sin A+\sin B+\sin C)^{1-\mu} \tag{3.1}$$

with application to the well known identity

$$\sin A+\sin B+\sin C=\frac{s}{R}.$$

Taking

$$A\to\pi-2A,\quad B\to\pi-2B\quad\text{and}\quad C\to\pi-2C,$$

then inequality (3.1) is equivalent to

$$\sin A+\sin B+\sin C\ge\left(\frac{3\sqrt{3}}{2}\right)^{\mu}\cdot(\sin 2A+\sin 2B+\sin 2C)^{1-\mu} \tag{3.2}$$

for an acute-angled triangle ABC.

By the well known identities

$$\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C,$$

and

$$\sin A\sin B\sin C=\frac{rs}{2R^2},$$

the inequality (3.2) can be written as follows:

$$\frac{s}{R}\ge\left(\frac{3\sqrt{3}}{2}\right)^{\mu}\cdot\left(\frac{2rs}{R^2}\right)^{1-\mu}\Longleftrightarrow\left(\frac{s}{R}\right)^{\mu}\ge\left(\frac{3\sqrt{3}}{2}\right)^{\mu}\cdot\left(\frac{2r}{R}\right)^{1-\mu}. \tag{3.3}$$

Furthermore, by Lemma 2.3, the inequality (3.3) holds if and only if the following two inequalities

$$\left(\frac{\sqrt{(1-x)(3+x)^3}}{2}\right)^{\mu}\ge\left(\frac{3\sqrt{3}}{2}\right)^{\mu}\left(1-x^2\right)^{1-\mu}\qquad\left(0\le x<\sqrt{2}-1\right) \tag{3.4}$$

and

$$\left(\frac{5-x^2}{2}\right)^{\mu}\ge\left(\frac{3\sqrt{3}}{2}\right)^{\mu}\left(1-x^2\right)^{1-\mu}\qquad\left(\sqrt{2}-1\le x<1\right) \tag{3.5}$$

hold. In other words,

$$\mu\le\min_{0\le x<1}G(x) \tag{3.6}$$

where

$$G(x)=\begin{cases}G_1(x) & \left(0\le x<\sqrt{2}-1\right),\\ G_2(x) & \left(\sqrt{2}-1\le x<1\right),\end{cases} \tag{3.7}$$

while $G_1(x)$ and $G_2(x)$ are defined by (2.9) and (2.15) respectively.

By Lemma 2.6 and Lemma 2.7, it follows that

$$\min_{0\le x<1}G(x)=G\left(\sqrt{2}-1\right).$$

Thus the first inequality of (1.3) holds, and the best constant μ for inequality (1.3) is

$$\mu_0 = \frac{2\ln\left(2-\sqrt{2}\right)+\ln 2}{4\ln 2 - 3\ln 3}.$$

(ii) By applying a similar method to (i), it follows that the second inequality of (1.3) is equivalent to

$$\left(\frac{s}{R}\right)^{\lambda} \le \left(\frac{3\sqrt{3}}{2}\right)^{\lambda} \cdot \left(\frac{2r}{R}\right)^{1-\lambda}. \tag{3.8}$$

By Lemma 2.5, the inequality (3.8) holds if and only if the following inequality holds:

$$\left(\frac{\sqrt{(1-x)(3+x)^3}}{2}\right)^{\lambda} \le \left(\frac{3\sqrt{3}}{2}\right)^{\lambda} (1-x^2)^{1-\lambda} \qquad (-1 < x \le 0)\,, \tag{3.9}$$

or equivalently,

$$\lambda \ge \sup_{-1<x\le 0} G_1(x), \tag{3.10}$$

where $G_1(x)$ is given by (2.9).

By Lemma 2.6, it follows that $\lambda \ge 1$. Moreover, the second inequality of (1.3) holds when $\lambda_0 = 1$. Thus the second inequality of (1.3) holds and the best constant λ for inequality (1.3) is $\lambda_0 = 1$. The proof of Theorem 1.1 is hence completed. □

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