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A NOTE ON OSTROWSKI LIKE INEQUALITIES

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Abstract. The aim of this note is to establish new Ostrowski like inequalities by using a fairly elementary analysis.

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1. Introduction

In an elegant note [5], A.M. Ostrowski proved the following interesting and useful inequality (see also [3, p. 468]):

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right](b-a)\,\|f'\|_\infty , \tag{1.1}$$

for all $x \in [a,b]$, where $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b), whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty = \sup_{x\in(a,b)} |f'(x)| < \infty$.

In the last few years, the study of such inequalities has been the focus of great attention to many researchers and a number of papers have appeared which deal with various generalizations, extensions and variants, see [2, 3, 6] and the references given therein. Inspired and motivated by the recent work going on related to the inequality (1.1), in the present note , we establish new inequalities of the type (1.1) involving two functions and their derivatives. An interesting feature of our results is that they are presented in an elementary way and provide new estimates on these types of inequalities.

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2. Main Results

Our main result is given in the following theorem.

Theorem 2.1. *Let* $f, g : [a, b] \to \mathbb{R}$ *be continuous functions on* $[a, b]$ *and differentiable on* (a, b)*, whose derivatives* $f', g' : (a, b) \to \mathbb{R}$ *are bounded on* (a, b) *, i.e.,* $\|f'\|_\infty = \sup_{x\in(a,b)} |f'(x)| < \infty$, $\|g'\|_\infty = \sup_{x\in(a,b)} |g'(x)| < \infty$. *Then*

$$\left| f(x) g(x) - \frac{1}{2(b-a)} \left[g(x) \int_a^b f(y)\, dy + f(x) \int_a^b g(y)\, dy \right] \right| \leq \frac{1}{2} \left\{ |g(x)| \|f'\|_\infty + |f(x)| \|g'\|_\infty \right\} \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a), \tag{2.1}$$

for all $x \in [a, b]$.

Proof. For any $x, y \in [a, b]$ we have the following identities:

$$f(x) - f(y) = \int_y^x f'(t)\, dt \tag{2.2}$$

and

$$g(x) - g(y) = \int_y^x g'(t)\, dt. \tag{2.3}$$

Multiplying both sides of (2.2) and (2.3) by $g(x)$ and $f(x)$ respectively and adding we get

$$2f(x) g(x) - [g(x) f(y) + f(x) g(y)] = g(x) \int_y^x f'(t)\, dt + f(x) \int_y^x g'(t)\, dt. \tag{2.4}$$

Integrating both sides of (2.4) with respect to y over $[a, b]$ and rewriting we have

$$\begin{aligned} f(x) g(x) &- \frac{1}{2(b-a)} \left[g(x) \int_a^b f(y)\, dy + f(x) \int_a^b g(y)\, dy \right] \\ &= \frac{1}{2(b-a)} \int_a^b \left\{ g(x) \int_y^x f'(t)\, dt + f(x) \int_y^x g'(t)\, dt \right\} dy. \end{aligned} \tag{2.5}$$

From (2.5) and using the properties of modulus we have

$$\begin{aligned} &\left| f(x) g(x) - \frac{1}{2(b-a)} \left[g(x) \int_a^b f(y)\, dy + f(x) \int_a^b g(y)\, dy \right] \right| \\ &\leq \frac{1}{2(b-a)} \int_a^b \left\{ |g(x)| \|f'\|_\infty |x-y| + |f(x)| \|g'\|_\infty |x-y| \right\} dy \\ &= \frac{1}{2(b-a)} \left\{ |g(x)| \|f'\|_\infty + |f(x)| \|g'\|_\infty \right\} \left[\frac{(x-a)^2 + (b-x)^2}{2} \right] \\ &= \frac{1}{2} \left\{ |g(x)| \|f'\|_\infty + |f(x)| \|g'\|_\infty \right\} \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a). \end{aligned}$$

The proof is complete. □

Remark 2.2. We note that, by taking $g(x) = 1$ and hence $g'(x) = 0$ in Theorem 2.1, we recapture the well known Ostrowski's inequality in (1.1).

Integrating both sides of (2.5) with respect to x over $[a,b]$, rewriting the resulting identity and using the properties of modulus, we obtain the following Grüss type inequality:

$$\begin{aligned}(2.6)\quad &\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)\right| \\ &\qquad \le \frac{1}{2(b-a)^2}\int_a^b \left[\int_a^b \{|g(x)|\,\|f'\|_\infty + |f(x)|\,\|g'\|_\infty\}\,|x-y|\,dy\right]dx.\end{aligned}$$

For other inequalities of the type (2.6), see the book [3], where many other references are given.

A slight variant of Theorem 2.1 is embodied in the following theorem.

Theorem 2.3. *Let f, g, f', g' be as in Theorem 2.1. Then*

$$\begin{aligned}(2.7)\quad &\left|f(x)\,g(x) - \frac{1}{b-a}\left[g(x)\int_a^b f(y)\,dy\right.\right. \\ &\qquad \left.\left.+ f(x)\int_a^b g(y)\,dy\right] + \frac{1}{b-a}\int_a^b f(y)\,g(y)\,dy\right| \\ &\qquad\qquad \le \frac{1}{b-a}\,\|f'\|_\infty\,\|g'\|_\infty\left[\frac{(x-a)^3+(b-x)^3}{3}\right],\end{aligned}$$

for all $x \in [a,b]$.

Proof. From the hypotheses, the identities (2.2) and (2.3) hold. Multiplying the left and right sides of (2.2) and (2.3) we get

$$(2.8)\quad f(x)\,g(x) - [g(x)\,f(y) + f(x)\,g(y)] + f(y)\,g(y) = \left\{\int_y^x f'(t)\,dt\right\}\left\{\int_y^x g'(t)\,dt\right\}.$$

Integrating both sides of (2.8) with respect to y over $[a,b]$ and rewriting we have

$$\begin{aligned}(2.9)\quad &f(x)\,g(x) - \frac{1}{b-a}\left[g(x)\int_a^b f(y)\,dy\right. \\ &\qquad \left.+ f(x)\int_a^b g(y)\,dy\right] + \frac{1}{b-a}\int_a^b f(y)\,g(y)\,dy \\ &\qquad\qquad = \frac{1}{b-a}\int_a^b\left\{\int_y^x f'(t)\,dt\right\}\left\{\int_y^x g'(t)\,dt\right\}dy.\end{aligned}$$

From (2.9) and using the properties of modulus we obtain

$$\begin{aligned}&\left|f(x)\,g(x) - \frac{1}{b-a}\left[g(x)\int_a^b f(y)\,dy\right.\right. \\ &\qquad\qquad \left.\left.+ f(x)\int_a^b g(y)\,dy\right] + \frac{1}{b-a}\int_a^b f(y)\,g(y)\,dy\right| \\ &\quad \le \frac{1}{b-a}\,\|f'\|_\infty\,\|g'\|_\infty\int_a^b |x-y|^2\,dy \\ &\quad = \frac{1}{b-a}\,\|f'\|_\infty\,\|g'\|_\infty\left[\frac{(x-a)^3+(b-x)^3}{3}\right].\end{aligned}$$

The proof is complete. □

Remark 2.4. Integrating both sides of (2.9) with respect to x over $[a, b]$, rewriting the resulting identity, using the properties of modulus and by elementary calculations we get

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)\right| \leq \frac{1}{12}(b-a)^2 \|f'\|_\infty \|g'\|_\infty . \tag{2.10}$$

Here, it is to be noted that the inequality (2.10) is the well known Čebyšev inequality (see [4, p. 297]).

References

[1] S.S. DRAGOMIR, Some integral inequalities of Grüss type, *Indian J. Pure and Appl. Math.*, **31** (2000), 379–415.

[2] S.S.DRAGOMIR AND Th.M. RASSIAS (Eds.), *Ostrowski Type Inequalities and Applications in Numerical Integration*, Kluwer Academic Publishers, Dordrecht , 2002.

[3] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Inequalities for Functions and their Integrals and Derivatives*, Kluwer Academic Publishers, Dordrecht, 1994.

[4] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Drodrecht, 1993.

[5] A.M. OSTROWSKI, Über die Absolutabweichung einer differentiebaren Funktion von ihrem Integralmitelwert, *Comment. Math. Helv.*, **10** (1938), 226–227.

[6] B.G. PACHPATTE, On a new generalization of Ostrowski's inequality, *J. Inequal. Pure and Appl. Math.*, **5**(2) (2004), Art. 36. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=378`]