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ON THE ERDÖS-DEBRUNNER INEQUALITY

VANIA MASCIONI
DEPARTMENT OF MATHEMATICAL SCIENCES
BALL STATE UNIVERSITY
MUNCIE, IN 47306-0490, USA
vdm@cs.bsu.edu
URL: http://www.cs.bsu.edu/homepages/vdm/

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ABSTRACT. We confirm two recent conjectures of W. Janous and thereby state the best possible form of the Erdös-Debrunner inequality for triangles.

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Fix a triangle ABC and, on each of the sides BC, CA, AB fix arbitrary interior points D, E, F. Label the areas of the resulting triangles DEF, AEF, BDF, CED as F_0, F_1, F_2, F_3. F_0 is thus the area of the central triangle, while the other three are the areas of the "corner" triangles. The Erdös-Debrunner inequality states that at least one of the corner triangles has no greater area than the central triangle:

$$\min\{F_1, F_2, F_3\} \leq F_0. \tag{1}$$

Walther Janous [1] generalized (1), proving that

$$M_{-1}(F_1, F_2, F_3) \leq F_0, \tag{2}$$

where $M_{-1}(F_1, F_2, F_3)$ denotes the harmonic mean of the areas F_1, F_2, F_3 (for notation and properties of general power means, see the standard reference [2]). Moreover, Janous [1] also proves that if an inequality of the form

$$M_p(F_1, F_2, F_3) \leq F_0 \tag{3}$$

should generally hold (with $p \geq -1$) then we must necessarily have

$$-1 \leq p \leq -\frac{\ln(3/2)}{\ln(2)}.$$

Prompted by these results, Janous formulates the following conjecture

Conjecture 1 (Janous [1]). *The best possible value of p for which* (3) *generally holds is* $p = -\frac{\ln(3/2)}{\ln(2)}$.

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In this note we will confirm this conjecture, and thereby state the best possible form of the Erdös-Debrunner inequality as a theorem:

Theorem 2. *It is always true that*

$$M_p(F_1, F_2, F_3) \le F_0$$

with $p = -\frac{\ln(3/2)}{\ln(2)}$*, and this value of* p *is best possible, in the sense that with any greater* p *there are examples that contradict the inequality.*

In [1] Janous develops a useful notation to simplify the Erdös-Debrunner problem, and we will adopt it as our starting point. First, he selects $t, u, v > 0$ so that the sides BC, CA, AB are divided by the points D, E, F in the ratios $t : 1-t$, $u : 1-u$, $v : 1-v$. Then, defining

$$x = \frac{t}{1-u}, \quad y = \frac{u}{1-v}, \quad \frac{v}{1-t},$$

and setting $q := -p$, Janous shows that the inequality (3) for $p < 0$ is equivalent to

$$f(x, y, z) \ge 3, \tag{4}$$

where f is defined by

$$f(x, y, z) := \left(\frac{1}{z} + x - 1\right)^q + \left(\frac{1}{x} + y - 1\right)^q + \left(\frac{1}{y} + z - 1\right)^q. \tag{5}$$

Here we require that

$$x, y, z > 0, \quad \frac{1}{z} + x - 1 \ge 0 \quad \frac{1}{x} + y - 1 \ge 0 \quad \frac{1}{y} + z - 1 \ge 0. \tag{6}$$

This new x, y, z notation and the related conditions, and the fact that we are only interested in exponents q with $\ln(3/2)/\ln(2) \le q < 1$, is all we need to know. In reference to the function f, Janous formulates a second "minor" conjecture:

Conjecture 3 (Janous [1])**.** *Under conditions* (6) *and for any* $q > 0$*, the minimum of* $f(x, y, z)$ *is attained at points satisfying* $xyz = 1$.

To prove Theorem 2 we would only need to consider the smallest possible q. However, we will start with a proof of this conjecture for the relevant interval of exponents $\ln(3/2)/\ln(2) \le q < 1$.

Lemma 4. *Under the conditions* (6) *and if* $\ln(3/2)/\ln(2) \le q < 1$*, the function* $f(x, y, z)$ *can only attain a minimum at* (x, y, z) *when* $xyz = 1$.

Proof. The inequalities in (6) define a region in $\mathbb{R}^3$, and we first want to consider points on its boundary. That is, we first assume that one of the last three inequalities is actually an identity; without loss of generality, we assume that

$$\frac{1}{y} + z - 1 = 0.$$

Thus, since $z = (y-1)/y$ and since $z > 0$, we conclude that $y > 1$. The function f defined in (5) simplifies to

$$g(x, y) := \left(\frac{1}{y-1} + x\right)^q + \left(\frac{1}{x} + y - 1\right)^q.$$

After the change of variables $s = x^q$, $t = \frac{1}{(y-1)^q}$, $p = 1/q$ this takes the more symmetric form

$$g(s, t) := (s^p + t^p)^{1/p} + \left(\frac{1}{s^p} + \frac{1}{t^p}\right)^{1/p}. \tag{7}$$

Using the definition of general power means, we can rewrite g as

$$g(s,t) = 2^{1/p}\left(M_p(s,t) + \frac{1}{M_{-p}(s,t)}\right).$$

Thus, estimating both summands within parentheses via the geometric mean $M_0(s,t)$, we get

$$g(s,t) \geq 2^{1/p}\left(M_0(s,t) + \frac{1}{M_0(s,t)}\right) \geq 2^{1+1/p} = 2^{1+q},$$

because of the well-known inequality $a + 1/a \geq 2$. We can now see, working backwards through the previous steps, that the minimum 2^{1+q} can only be attained if $s = t$, which in turn means that $x = 1/(y-1)$. Therefore,

$$xyz = \frac{1}{y-1} y \frac{y-1}{y} = 1$$

as claimed. Further, we notice that 2^{1+q} is greater than or equal to 3, where equality holds when $q = \ln(3/2)/\ln(2)$.

Next, we will look for the extrema of f under the set of strict conditions

$$x, y, z > 0, \quad \frac{1}{z} + x - 1 > 0 \quad \frac{1}{x} + y - 1 > 0 \quad \frac{1}{y} + z - 1 > 0, \tag{8}$$

which together define an *open* region in $\mathbb{R}^3$. The extrema in this region must occur where the gradient of f vanishes. We compute the partial derivative with respect to x, and obtain

$$\frac{\partial f}{\partial x} = q\left(\frac{1}{z} + x - 1\right)^{q-1} - q\left(\frac{1}{x} + y - 1\right)^{q-1}\frac{1}{x^2}.$$

The condition $\frac{\partial f}{\partial x} = 0$ can be rewritten as (remembering that $\ln(3/2)/\ln(2) \leq q < 1$)

$$\left(\frac{1}{x} + y - 1\right)^q = \left(\frac{1}{z} + x - 1\right)^q \frac{1}{x^{2q/(1-q)}}. \tag{9}$$

By permuting the variables x, y, z cyclically, we obtain from (9) the corresponding equations equivalent to $\frac{\partial f}{\partial y} = 0$ and $\frac{\partial f}{\partial z} = 0$, that is,

$$\left(\frac{1}{y} + z - 1\right)^q = \left(\frac{1}{x} + y - 1\right)^q \frac{1}{y^{2q/(1-q)}} \tag{10}$$

and

$$\left(\frac{1}{z} + x - 1\right)^q = \left(\frac{1}{y} + z - 1\right)^q \frac{1}{z^{2q/(1-q)}}. \tag{11}$$

It should be now clear that the product of the three equations (9), (10), (11) implies $xyz = 1$ in this case, too. The lemma is thus proved. □

Proof of Theorem 2. Using Lemma 4, finding the minimum of f becomes a two-variable problem after setting $z = 1/xy$. Accordingly, we consider a new function

$$h(x,y) := (xy + x - 1)^q + \left(\frac{1}{x} + y - 1\right)^q + \left(\frac{1}{y} + \frac{1}{xy} - 1\right)^q,$$

and henceforth we will also fix q to be $\ln(3/2)/\ln(2)$, recalling Janous' proof that the inequality is invalid for $q < \ln(3/2)/\ln(2)$. Our ultimate target is to show that with $q = \ln(3/2)/\ln(2)$ and under conditions (6) the minimum of h is 3 (see (4) and replace z with $1/xy$ in (6)).

Now, if any of the last three inequalities in (6) is an identity, the proof of Lemma 4 already shows that the minimum of h is 2^{q+1}, and this number is identical to 3 given the choice $q =$

$\ln(3/2)/\ln(2)$. We thus want to examine possible extrema of h under the more restrictive conditions

$$x, y > 0, \quad xy + x - 1 > 0 \quad \frac{1}{x} + y - 1 > 0 \quad \frac{1}{y} + \frac{1}{xy} - 1 > 0 \tag{12}$$

which result from (8) after replacing z with $1/xy$. Rewriting (12) as

$$x, y > 0, \quad y + 1 > \frac{1}{x} \quad \frac{1}{x} + y > 1 \quad 1 + \frac{1}{x} > y \tag{13}$$

it follows that $1/x$, y and 1 must be the lengths of the three sides of a triangle. After the change of variables $s = 1/x$, $t = y$, h can be written as

$$h(s,t) = \left(\frac{1 + t - s}{s}\right)^q + (s + t - 1)^q + \left(\frac{1 + s - t}{t}\right)^q, \tag{14}$$

where the quantities $s, t, 1$ are the sides of a (non-degenerate) triangle.

If we now look at

$$H(a,b,c) := \left(\frac{b + c - a}{a}\right)^q + \left(\frac{c + a - b}{b}\right)^q + \left(\frac{a + b - c}{c}\right)^q, \tag{15}$$

where a, b, c are the sides of a triangle, and realize that the function H is invariant under a common scaling of a, b, c, we see that the problem of minimizing $h(s,t)$ in (14) is equivalent to minimizing $H(a,b,c)$ in (15). Let us now use elementary trigonometric relations to rewrite H as a function of the angles α, β, γ (defined as the angles opposite the sides of length a, b, c). The result is

$$H(\alpha,\beta,\gamma) = 2^q \left[\left(\frac{\sin(\beta/2)\sin(\gamma/2)}{\sin(\alpha/2)}\right)^q + \left(\frac{\sin(\gamma/2)\sin(\alpha/2)}{\sin(\beta/2)}\right)^q + \left(\frac{\sin(\alpha/2)\sin(\beta/2)}{\sin(\gamma/2)}\right)^q\right].$$

Since we are dealing with (positive) angles satisfying $\alpha + \beta + \gamma = \pi$, we have $\sin(\gamma/2) = \cos\left((\alpha + \beta)/2\right)$, and so a further dose of trigonometry transforms H into a function of the two variables α, β which we nevertheless call $H(\alpha,\beta)$, since the value is the same:

$$H(\alpha,\beta) = 2^q \left(\sin(\alpha/2)^{2q} + \sin(\beta/2)^{2q}\right)\left(\cot(\alpha/2)\cot(\beta/2) - 1\right)^q + \frac{1}{(\cot(\alpha/2)\cot(\beta/2) - 1)^q}.$$

Next, using the identity $\sin^2(\xi) = 1/(1 + \cot^2(\xi))$ we can express H as a function of $\cot(\alpha/2)$ and $\cot(\beta/2)$. After one more change of variables, namely $u = \cot(\alpha/2))$ and $v = \cot(\beta/2)$, we obtain our final expression for H:

$$H(u,v) = 2^q \left[\left(\frac{1}{(1 + u^2)^q} + \frac{1}{(1 + v^2)^q}\right)(uv - 1)^q + \frac{1}{(uv - 1)^q}\right] \tag{16}$$

where u and v are only required to be positive and such that $uv > 1$. We are now able to minimize (16) with traditional methods. Any critical point in the open domain specified must satisfy the conditions $\frac{\partial H}{\partial u} = \frac{\partial H}{\partial v} = 0$. To spare the reader the rather unpleasant complete calculation of these partial derivatives, let us just state that, for some function $M(u,v)$ (whose details are not needed here), we have

$$\frac{1}{q2^q(uv - 1)^q}\frac{\partial H}{\partial u} = -2u\frac{1}{(1 + u^2)^{\ln(3)/\ln(2)}} + vM(u,v)$$

and
$$\frac{1}{q2^q(uv-1)^q}\frac{\partial H}{\partial v} = -2v\frac{1}{(1+v^2)^{\ln(3)/\ln(2)}} + uM(u,v).$$
If both partial derivatives are zero, we can solve the resulting equations for $M(u,v)$, eliminate $M(u,v)$, and obtain

(17)
$$\frac{u}{v}\frac{1}{(1+u^2)^{\ln(3)/\ln(2)}} = \frac{v}{u}\frac{1}{(1+v^2)^{\ln(3)/\ln(2)}}.$$

Introducing the function
$$\phi(z) := \frac{z}{(1+z)^{\ln(3)/\ln(2)}},$$
condition (17) simplifies to
$$\phi(u^2) = \phi(v^2).$$
We first consider the case where $u \neq v$. The function ϕ is easily seen to be strictly increasing for $z \in [0, \ln(2)/\ln(3/2)]$ and strictly decreasing for $z > \ln(2)/\ln(3/2)$. $u \neq v$ implies that $u^2 < 1/q < v^2$. Since we assume that $uv > 1$, we also have $1/v^2 < u^2$ (and thus $\phi(1/v^2) < \phi(u^2)$). Now, elementary algebra shows that
$$\phi(1/v^2) = \phi(v^2)v^{2(\ln(3)/\ln(2)-1)}.$$
Since $v^2 > 1/q > 1$, this implies that
$$\phi(1/v^2) > \phi(v^2) = \phi(u^2),$$
which is a contradiction. Therefore, the case $u \neq v$ is impossible, and we are left with the analysis of the "isosceles" case $u = v$. Indeed, backtracking through our last change of variables, $u = v$ means that $\alpha = \beta$, and thus $a = b$ in the original expression (15) for $H(a,b,c)$. Thus, we should consider the function $h(s,t)$ from (14), for the case when $s = t$ (and $2s > 1$, to preserve the triangle condition). Our last task is thus to minimize

(18)
$$h(s,s) = 2\frac{1}{s^q} + (2s-1)^q$$

for $s \in (1/2, \infty)$. An analysis of the derivative of $h(s,s)$ shows that it has exactly two zeros for $s > 1/2$, and since the function initially increases (with infinite derivative at $s = 1/2$), the second critical point, at $s = 1$, must be a minimum, which corresponds to the equilateral case. When $s = 1$, $h(1,1) = 3$. This and $h(1/2,1/2) = 3$ complete the proof. □

Remark 5. Based on our proof, the following corollary can be stated, which is a consequence of $H(a,b,c) \geq 3$ and the general power means inequality:

Corollary 6. *Let $p \geq \ln(3/2)/\ln(2)$ be an arbitrary real number. Then for all triangles with sides a, b and c and semi-perimeter s the inequality*
$$\left(\frac{s-a}{a}\right)^p + \left(\frac{s-b}{b}\right)^p + \left(\frac{s-c}{c}\right)^p \geq \frac{3}{2^p}$$
is valid.

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