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YOUNG'S INEQUALITY IN COMPACT OPERATORS – THE CASE OF EQUALITY

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Abstract

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Abstract

If a and b are compact operators acting on a complex separable Hilbert space, and if $p, q \in (1, \infty)$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, then there exists a partial isometry u such that the initial space of u is $(\ker(|ab^*|))^\perp$ and

$$u|ab^*|u^* \leq \frac{1}{p}|a|^p + \frac{1}{q}|b|^q.$$

Furthermore, if $|ab^*|$ is injective, then the operator u in the inequality above can be taken as a unitary. In this paper, we discuss the case of equality of this Young's inequality, and obtain a characterization for compact normal operators.

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Contents

1. Introduction

Operator and matrix versions of classical inequalities are of considerable interest, and there is an extensive body of literature treating this subject; see, for example, [1] – [4], [6] – [11]. In one direction, many of the operator inequalities to have come under study are inequalities between the norms of operators. However, a second line of research is concerned with inequalities arising from the partial order on Hermitian operators acting on a Hilbert space. It is in this latter direction that this paper aims.

A fundamental inequality between positive real numbers is the arithmetic-geometric mean inequality, which is of interest herein, as is its generalisation in the form of Young's inequality.

For the positive real numbers a, b, the arithmetic-geometric mean inequality says that

$$\sqrt{ab} \leq \frac{1}{2}(a+b).$$

Replacing a, b by their squares, this could be written in the form

$$ab \leq \frac{1}{2}(a^2+b^2).$$

R. Bhatia and F. Kittaneh [3] extended the arithmetic-geometric mean inequality to positive (semi-definite) matrices a, b in the following manner: for any $n \times n$ positive matrices a, b, there is an $n \times n$ unitary matrix u such that

$$u|ab^*|u^* \leq \frac{1}{2}(a^2+b^2).$$

(The modulus $|y|$ is defined by

$$|y| = (y^* y)^{\frac{1}{2}}.$$

for any $n \times n$ complex matrix y.) We note that the product ab of two positive matrices a and b is not necessarily positive.

Young's inequality is a generalisation of the arithmetic-geometric mean inequality: for any positive real numbers a, b, and any $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$ab \leq \frac{1}{p} a^p + \frac{1}{q} b^q.$$

T. Ando [2] showed Young's inequality admits a matrix-valued version analogous to the Bhatia–Kittaneh theorem: if $p, q \in (1, \infty)$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, then for any pair a, b of $n \times n$ complex matrices, there is a unitary matrix u such that

$$u|ab^*|u^* \leq \frac{1}{p}|a|^p + \frac{1}{q}|b|^q.$$

Although finite-rank operators are norm-dense in the set of all compact operators acting on a fixed Hilbert space, the Ando–Bhatia–Kittaneh inequalities, like most matrix inequalities, do not immediately carry over to compact operators via the usual approximation methods, and consequently only a few of the fundamental matrix inequalities are known to hold in compact operators.

J. Erlijman, D. R. Farenick, and the author [4] developed a technique through which the Ando–Bhatia–Kittaneh results extend to compact operators, and established the following version of Young's inequality.

Theorem 1.1. *If a and b are compact operators acting on a complex separable Hilbert space, and if $p, q \in (1, \infty)$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, then there is a partial isometry u such that the initial space of u is $(\ker(|ab*|))^{\perp}$ and*

$$u|ab^*|u^* \leq \frac{1}{p}|a|^p + \frac{1}{q}|b|^q.$$

Furthermore, if $|ab^*|$ is injective, then the operator u in the inequality above can be taken to be a unitary.

Theorem 1.1 is made in a special case as a corollary below.

Corollary 1.2. *If a and b are positive compact operators with trivial kernels, and if $t \in [0, 1]$, then there is a unitary u such that*

$$u|a^t b^{1-t}|u^* \leq ta + (1-t)b.$$

The proof of the following Theorem 1.3 is very straightforward.

Theorem 1.3. *If A is a commutative C^*-algebra with multiplicative identity, and if $p, q \in (1, \infty)$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$|ab^*| \leq \frac{1}{p}|a|^p + \frac{1}{q}|b|^q$$

for all $a, b \in A$. Furthermore, if the equality holds, then

$$|b| = |a|^{p-1}.$$

2. An Example

We give an example here for convenience.

We illustrate that, in general, we do not have

$$|ab^*| \le \frac{1}{p}|a|^p + \frac{1}{q}|b|^q.$$

But, for this example, there exists a unitary u such that

$$u|ab^*|u^* \le \frac{1}{2}(|a|^p + |b|^q).$$

Example 2.1. *If* $a = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ *and* $b = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, *then* a *and* b *are (semi-definite) positive and*

$$\frac{1}{2}(a^2 + b^2) = \begin{pmatrix} 3 & 1 \\ 1 & \frac{1}{2} \end{pmatrix},$$

and

$$|ab^*| = |ab| = \begin{pmatrix} \frac{\sqrt{10}}{2} & \frac{\sqrt{10}}{2} \\ \frac{\sqrt{10}}{2} & \frac{\sqrt{10}}{2} \end{pmatrix}.$$

However,

$$c = \frac{1}{2}(a^2 + b^2) - |ab| = \begin{pmatrix} 3 - \frac{\sqrt{10}}{2} & 1 - \frac{\sqrt{10}}{2} \\ 1 - \frac{\sqrt{10}}{2} & 3 - \frac{\sqrt{10}}{2} \end{pmatrix}$$

is not a (semi-definite) positive matrix, i.e., $c = \frac{1}{2}(a^2 + b^2) - |ab| \ge 0$ *does not hold. (In fact, the determinant of* c *satisfies that* $\det(c) < 0$*). So, we do not have*

$$|ab| \le \frac{1}{2}(a^2 + b^2).$$

But the spectrum of $|ab|$ *is*

$$\sigma(|ab|) = \left\{\sqrt{10}, 0\right\},$$

the spectrum of $\frac{1}{2}(a^2 + b^2)$ *is*

$$\sigma\left(\frac{1}{2}(a^2 + b^2)\right) = \left\{\frac{7}{2}, 1\right\}.$$

Therefore, there exists a unitary matrix u *such that*

$$u|ab|u^* \leq \frac{1}{2}(a^2 + b^2).$$

We compute the unitary matrix u *as follows.*

Taking unitary matrices

$$v = \frac{1}{\sqrt{5}}\begin{pmatrix} -2 & 1 \\ -1 & -2 \end{pmatrix}$$

and

$$w = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

we then have

$$v\left(\frac{1}{2}(a^2 + b^2)\right)v^* = \begin{pmatrix} \frac{7}{2} & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$w|ab|w^* = \begin{pmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore

$$w|ab|w^* \le v \left(\frac{1}{2}(a^2 + b^2) \right) v^*.$$

By taking a unitary matrix

$$u = v^* w = \frac{1}{\sqrt{10}} \begin{pmatrix} -3 & -1 \\ -1 & 3 \end{pmatrix},$$

we get

$$u|ab|u^* \le \frac{1}{2}(a^2 + b^2).$$

3. The Case Of Equality In Commuting Normal Operators

In this section, we discuss the cases of equality in Young's inequality.

Assume that H denotes a complex, separable Hilbert space of finite or infinite dimension. The inner product of vectors $\xi, \eta \in H$ is denoted by $\langle \xi, \eta \rangle$, and the norm of $\xi \in H$ is denoted by $||\xi||$.

If $x : H \to H$ is a linear transformation, then x is called an operator (on H) if x is also continuous with respect to the norm-topology on H. The complex algebra of all operators on H is denoted by $B(H)$, which is a C^*-algebra. We use x^* to denote the adjoint of $x \in B(H)$.

An operator x on H is said to be Hermitian if $x^* = x$. A Hermitian operator x is positive if $\sigma(x) \subseteq \mathbb{R}_0^+$, where $\sigma(x)$ is the spectrum of x, and $\mathbb{R}_0^+$ is the set of non-negative numbers. Equivalently, $x \in B(H)$ is positive if and only if $\langle x\xi, \xi \rangle \geq 0$ for all $\xi \in H$. If $a, b \in B(H)$ are Hermitian, then $a \leq b$ shall henceforth denote that $b - a$ is positive.

Lemma 3.1. *If $a, b \in B(H)$ are normal and commuting, where $B(H)$ is the complex algebra of all continuous linear operators on H, then*

$$|a||b| = |b||a|,$$

and $|a||b|$ is positive.

Proof. We obviously have

$$a^*b^* = b^*a^*.$$

And by the Fuglede theorem [5] we get

$$a^*b = ba^*, \quad ab^* = b^*a.$$

On the other hand, if $c, d \in B(H)$ with c,d positive and commuting, then

$$c^{1/2}d^{1/2} \cdot c^{1/2}d^{1/2} = c^{1/2}c^{1/2} \cdot d^{1/2}d^{1/2} = cd.$$

Hence

$$(cd)^{1/2} = c^{1/2}d^{1/2}.$$

Therefore

$$\begin{aligned} |a||b| &= (a^*a)^{1/2}(b^*b)^{1/2} \\ &= (a^*ab^*b)^{1/2} \\ &= (b^*b)^{1/2}(a^*a)^{1/2} \\ &= |b||a|. \end{aligned}$$

Which implies that $|a||b|$ is positive and

$$|a||b| = (|a||b|)^* = |b||a|.$$

(In fact, $|a||b|$ is the positive square root of the positive operator a^*ab^*b). $\square$

Lemma 3.2. *If $a, b \in B(H)$are normal operators such that $ab = ba$, then the following statements are equivalent:*

(i) the kernel of $|ab^*|$: $\ker(|ab^*|) = \{0\}$;

(ii) a *and* b *are injective and have dense range.*

Proof. (i) $\rightarrow$ (ii). Let $b = w|b|$ be the polar decomposition of b. By observation we have

$$||a||b|| = (|b||a|^2|b|)^{1/2}.$$

Thus, because the closures of the ranges of a positive operator and its square root are equal, the closures of the ranges of $|b||a|^2|b|$and $||a||b||$ are the same. Moreover, as $w^*w||a||b|| = ||a||b||$, we have that

$$f(w|b||a|^2|b|w^*) = wf(|b||a|^2|b|)w^*, \tag{3.1}$$

for all polynomials f. Choose $\delta > 0$ so that $\sigma(|b||a|^2|b|) \subseteq [0, \delta]$. By the Weierstrass approximation theorem, there is a sequence of polynomials f_n such that $f_n(t) \rightarrow \sqrt{t}(n \rightarrow \infty)$ uniformly on $[0, \delta]$. Thus, from (3.1) and functional calculus,

$$(w|b||a|^2|b|w^*)^{1/2} = w(|b||a|^2|b|)^{1/2}w^* = w||a||b||w^*.$$

Let $a = v|a|$ be the polar decomposition of a. Then the left-hand term in the equalities above expands as follows:

$$(w|b||a|^2|b|w^*)^{1/2} = w(|b||a|v^*v|a||b|)^{1/2}w^* = (ba^*ab^*)^{1/2} = |ab^*|.$$

Thus,

$$|ab^*| = w||a||b||w^*.$$

Because a and b are commuting normal, from Lemma 3.1 $|a||b| = |b||a|$ and $|a||b|$ is positive. This implies that

$$|ab^*| = w|a||b|w^*.$$

If $\xi \in \ker(w^*)$, then $\xi \in \ker(|ab^*|)$. Hence $\ker(w^*) = \{0\}$, which means that the range of $\operatorname{ran}(w) = H$. Hence, w is unitary. By the theorem on polar decomposition [5, p. 75], b is injective and has dense range.

Let $a = v|a|$ be the polar decomposition of a. We know that $ab = ba$ implies that $ab^* = b^*a$ (again, by Fuglede theorem). Therefore, we can interchange the role of a and b in the previous paragraph to obtain: a^* is injective and has dense range. Thus, a is injective and has dense range.

(ii) $\rightarrow$ (i). From the hypothesis we have polar decompositions $a = v|a|$, $b = w|b|$, where v and w are unitary [5, p. 75]. Therefore, $\ker(|a|) = \ker(|b|) = \{0\}$. Because

$$|ab^*| = w|a||b|w^*$$

and w is unitary, we have

$$\ker(ab^*) = \{0\}.$$

□

Lemma 3.3. *If $x \in B(H)$ is positive, compact, and injective, and if $x \le u^*xu$ for some unitary u, then u is diagonalisable and commutes with x.*

Proof. Because x is injective, the Hilbert space H is the direct sum of the eigenspaces of x:

$$H = \sum\nolimits^{\oplus}_{\lambda\in\sigma_p(x)} \ker(x - \lambda 1).$$

Let

$$\sigma_p(x) = \{\lambda_1, \lambda_2, \dots\},$$

where $\lambda_1 > \lambda_2 > \cdots > 0$ are the (distinct) eigenvalues of x, listed in descending order. Our first goal is to prove that $\ker(x - \lambda_j 1)$ is invariant under u and u^* for every positive integer j; we shall do so by induction.

Start with λ_1; note that $\lambda_1 = ||x||$.

If $\xi \in \ker(x - \lambda_1 1)$ is a unit vector, then

$$\begin{aligned}
\lambda_1 &= \lambda_1 \langle \xi, \xi \rangle \\
&= \langle \lambda_1 \xi, \xi \rangle \\
&= \langle x\xi, \xi \rangle \\
&\leq \langle u^* x u \xi, \xi \rangle \\
&= \langle x u \xi, u\xi \rangle \\
&\leq ||x|| \cdot ||u\xi||^2 = \lambda_1.
\end{aligned}$$

Thus,

$$\langle x u\xi, u\xi \rangle = \lambda_1 = \max\{\langle x\eta, \eta \rangle : ||\eta|| = 1\}.$$

Which means that $u\xi$ is an eigenvector of x corresponding to the eigenvalue λ_1. Then,

$$u\xi \in \ker(x - \lambda_1 1).$$

Because $\ker(x - \lambda_1 1)$ is finite-dimensional and u is unitary, we have that

$$u : \ker(x - \lambda_1 1) \to \ker(x - \lambda_1 1)$$

is an isomorphism. Furthermore, $U|_{\ker(x-\lambda_1 1)}$ is diagonalisable because

$$\dim(\ker(x - \lambda_1 1)) < \infty,$$

where $U|_{\ker(x-\lambda_1 1)}$ is the restriction of U in the subspace $\ker(x-\lambda_1 1)$. Hence, $\ker(x-\lambda_1 1)$ is invariant under u^* (because $\ker(x-\lambda_1 1)$ has a finite orthonormal basis of eigenvectors of u), which means that if

$$\eta \in \ker(x-\lambda_1 1),$$

then

$$u\eta \in \ker(x-\lambda_1 1).$$

Now choose λ_2, and pick up a unit vector $\xi \in \ker(x-\lambda_2 1)$.

Note that

$$\lambda_2 = \max\{\langle x\eta, \eta\rangle : ||\eta|| = 1, \eta \in \ker(x-\lambda_1 1)^{\perp}\}.$$

Using the arguments of the previous paragraph,

$$\lambda_2 \leq \langle x\xi, \xi\rangle \leq \langle xu\xi, u\xi\rangle \leq \lambda_2.$$

(Because $u\xi$ is a unit vector orthogonal to $\ker(x-\lambda_1 1)$). Hence, by the minimum maximum principle,

$$u\xi \in \ker(x-\lambda_2 1).$$

So

$$u : \ker(x-\lambda_2 1) \to \ker(x-\lambda_2 1)$$

is an isomorphism, $\ker(x-\lambda_2 1)$ has an orthonormal basis of eigenvectors of u.

And if $\eta \in \ker(x-\lambda_1 1) \oplus \ker(x-\lambda_2 1)$, then

$$u\eta \in \ker(x-\lambda_1 1) \oplus \ker(x-\lambda_2 1).$$

Inductively, assume that u leaves $\ker(x-\lambda_j 1)$ invariant for all $1 \le j \le k$, and look at λ_{k+1}. By the arguments above,

$$\left(\sum_{1\le j\le k}^{\oplus} \ker(x-\lambda_j 1)\right)^{\perp}$$

is also invariant under u. Hence, if $\xi \in \ker(x-\lambda_{k+1}1)$ is a unit vector, then

$$\begin{aligned}\lambda_{k+1} &= \langle x\xi, \xi\rangle \\ &\le \langle xu\xi, u\xi\rangle \\ &\le \max\left\{\langle x\eta,\eta\rangle : ||\eta|| = 1, \eta \in \left(\sum_{1\le j\le k}^{\oplus} \ker(x-\lambda_j 1)\right)^{\perp}\right\} \\ &= \lambda_{k+1}.\end{aligned}$$

By the minimum-maximum principle, $u\xi$ is an eigenvector of x corresponding to λ_{k+1}. Hence,

$$\ker(x-\lambda_{k+1}1)$$

is invariant under u and u^*. This completes the induction process.

What these arguments show is that H has an orthonormal basis $\{\phi\}_{j=1}^{\infty}$ of eigenvectors of both x and u; hence

$$xu\phi_j = ux\phi_j,$$

for each positive integer j. Consequently,

$$xu\xi = ux\xi, \ \forall \xi \in H.$$

meaning that

$$xu = ux.$$

$\square$

Below is a major result of this paper

Theorem 3.4. *Assume that $a, b \in B(H)$ are commuting compact normal operators, each being injective and having dense ranges. If there exists a unitary u such that:*

$$u|ab^*|u^* = \frac{1}{p}|a|^p + \frac{1}{q}|b|^q,$$

for some $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$|b| = |a|^{p-1}.$$

Proof. By the hypothesis, if $b = w|b|$ is the polar decomposition of b, then $\ker(|ab^*|) = \{0\}$, (Lemma 3.2) and w is unitary ([5, p. 75]). Moreover,

$$|ab^*| = w|a||b|w^*,$$

as a and b are commuting normals (noting that $|a||b|$ is positive from Lemma 3.1). Thus $u|ab^*|u^* = \frac{1}{p}|a|^p + \frac{1}{q}|b|^q$ becomes

$$uw|a||b|w^*u^* = \frac{1}{p}|a|^p + \frac{1}{q}|b|^q. \tag{3.2}$$

By Theorem 1.3, and because $|a||b| = |b||a|$ (Lemma 3.2), we get

$$\frac{1}{p}|a|^p + \frac{1}{q}|b|^q \geq |a||b|.$$

Hence from (3.2)

$$uw|a||b|w^*u^* = \frac{1}{p}|a|^p + \frac{1}{q}|b|^q \geq |a||b|. \tag{3.3}$$

Because uw is unitary (since w is unitary from the proof of Lemma 3.2), and because $|a||b|$ is positive, Lemma 3.3 yields

$$|a||b| = uw|a||b|w^*u^*.$$

Hence, (3.2) becomes

$$|a||b| = \frac{1}{p}|a|^p + \frac{1}{q}|b|^q. \tag{3.4}$$

Let

$$\lambda_1(|a|) \geq \lambda_2(|a|) \geq \cdots > 0$$

and

$$\lambda_1(|b|) \geq \lambda_2(|b|) \geq \cdots > 0$$

be the eigenvalues of $|a|$ and $|b|$. Because $|a|$ and $|b|$ belong to a commutative C^*-algebra, the spectra of $|a||b|$ and $\frac{1}{p}|a|^p+\frac{1}{q}|b|^q$ are determined from the spectra of $|a|$ and $|b|$, i.e., for each positive integer k,

$$\lambda_k(|a||b|) = \lambda_k(|a|)\lambda_k(|b|),$$

and

$$\lambda_k\left(\frac{1}{p}|a|^p + \frac{1}{q}|b|^q\right) = \frac{1}{p}\lambda_k(|a|)^p + \frac{1}{q}\lambda_k(|b|)^q.$$

Therefore, the equation (3.4) implies that for every k

$$\lambda_k(|a|)\lambda_k(|b|) = \frac{1}{p}\lambda_k(|a|)^p + \frac{1}{q}\lambda_k(|b|)^q.$$

This is equality in the (scalar) Young's inequality, and hence for every k

$$\lambda_k(|b|) = \lambda_k(|a|)^{p-1}$$

which yields (note that a and b are normal operators)

$$|b| = |a|^{p-1}.$$

$\square$

From Theorem 3.4 we immediately have

Corollary 3.5. *If a and b are positive commuting compact operators such that $|ab|$ is injective, and if there is an isometry $v \in B(H)$ for which*

$$u|a^t b^{1-t}|u^* = ta + (1-t)b$$

for some $t \in [0,1]$, then

$$b = a^{t-1}.$$

Theorem 3.6. *Assume that $a, b \in B(H)$ are commuting compact normal operators, each being injective and having dense range. If*

$$|b| = |a|^{p-1},$$

then there exists a unitary u such that:

$$u|ab^*|u^* = \frac{1}{p}|a|^p + \frac{1}{q}|b|^q,$$

for $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By the hypothesis, it is easy to get

$$|a||b| = \frac{1}{p}|a|^p + \frac{1}{q}|b|^q,$$

we note that $|a||b|$ is positive here.

If $b = w|b|$ is the polar decomposition of b, then $\ker(|ab^*|) = \{0\}$ (Lemma 3.2), w is unitary ([5, p. 75]), and

$$|ab^*| = w|a||b|w^*.$$

Let $u = w^*$. Then

$$u|ab^*|u^* = \frac{1}{p}|a|^p + \frac{1}{q}|b|^q.$$

□

Corollary 3.7. *If a and b are positive commuting compact operators such that ab is injective, and if there exists $t \in [0, 1]$ such that*

$$b = a^{t-1},$$

then there is an isometry $v \in B(H)$ for which

$$u|a^t b^{1-t}|u^* = ta + (1 - t)b.$$

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