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LOWER BOUNDS FOR THE SPECTRAL NORM

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ABSTRACT. Let $\mathbf{A}$ be a complex $m \times n$ matrix. We find simple and good lower bounds for its spectral norm $\|\mathbf{A}\| = \max\{ \|\mathbf{Ax}\| \mid \mathbf{x} \in \mathbb{C}^n, \|\mathbf{x}\| = 1 \}$ by choosing $\mathbf{x}$ smartly. Here $\| \cdot \|$ applied to a vector denotes the Euclidean norm.

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1. INTRODUCTION

Throughout this paper, $\mathbf{A}$ denotes a complex $m \times n$ matrix ($m, n \geq 2$). We denote by $\|\mathbf{A}\|$ its spectral norm or largest singular value.

The singular values of $\mathbf{A}$ are square roots of the eigenvalues of $\mathbf{A}^*\mathbf{A}$. Since much is known about bounds for eigenvalues of Hermitian matrices, we may apply this knowledge to $\mathbf{A}^*\mathbf{A}$ to find bounds for singular values, but the bounds so obtained are very complicated in general. However, as we will see, we can find simple and good lower bounds for $\|\mathbf{A}\|$ by choosing $\mathbf{x}$ smartly in the variational characterization of the largest eigenvalue of $\mathbf{A}^*\mathbf{A}$,

$$\|\mathbf{A}\| = \max\left\{ (\mathbf{x}^*\mathbf{A}^*\mathbf{Ax})^{1/2} \;\middle|\; \mathbf{x} \in \mathbb{C}^n,\, \mathbf{x}^*\mathbf{x} = 1 \right\}, \tag{1.1}$$

or, equivalently, in the definition

$$\|\mathbf{A}\| = \max\left\{ \|\mathbf{Ax}\| \;\middle|\; \mathbf{x} \in \mathbb{C}^n,\, \|\mathbf{x}\| = 1 \right\}, \tag{1.1$'$}$$

where $\| \cdot \|$ applied to a vector denotes the Euclidean norm.

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249-04

Our earlier papers [3] and [4] are based on somewhat similar ideas to find lower bounds for the spread and numerical radius of $\mathbf{A}$.

2. SIMPLE BOUNDS

Consider the partition $\mathbf{A} = (\mathbf{a}_1 \dots \mathbf{a}_n)$ of $\mathbf{A}$ into columns. For $H \subseteq N = \{1, \dots, n\}$, denote by $\mathbf{A}_H$ the block of the columns $\mathbf{a}_h$ with $h \in H$. We accept also the empty block $\mathbf{A}_\emptyset$.

Throughout this paper, we let I ($\neq \emptyset$), K, and L be disjoint subsets of N satisfying $N = I \cup K \cup L$. Since multiplication by permutation matrices does not change singular values, we can reorder the columns, and so we are allowed to assume that

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_I & \mathbf{A}_K & \mathbf{A}_L \end{pmatrix}.$$

Then

$$\mathbf{A}^*\mathbf{A} = \begin{pmatrix} \mathbf{A}_I^*\mathbf{A}_I & \mathbf{A}_I^*\mathbf{A}_K & \mathbf{A}_I^*\mathbf{A}_L \\ \mathbf{A}_K^*\mathbf{A}_I & \mathbf{A}_K^*\mathbf{A}_K & \mathbf{A}_K^*\mathbf{A}_L \\ \mathbf{A}_L^*\mathbf{A}_I & \mathbf{A}_L^*\mathbf{A}_K & \mathbf{A}_L^*\mathbf{A}_L \end{pmatrix}.$$

We denote $\mathbf{e}_H = \sum_{h\in H} \mathbf{e}_h$, where $\mathbf{e}_h$ is the h'th standard basis vector of $\mathbb{C}^n$.

We choose $\mathbf{x} = \mathbf{e}_I \big/ \sqrt{|I|}$ in (1.1), where $|\cdot|$ stands for the number of the elements. Then

$$\|\mathbf{A}\| \geq \left(\frac{\operatorname{su} \mathbf{A}_I^*\mathbf{A}_I}{|I|} \right)^{\frac{1}{2}}, \tag{2.1}$$

where su denotes the sum of the entries. Hence

$$\|\mathbf{A}\| \geq \max_{I\neq\emptyset} \left(\frac{\operatorname{su} \mathbf{A}_I^*\mathbf{A}_I}{|I|} \right)^{\frac{1}{2}} = \max_{I\neq\emptyset} \frac{1}{\sqrt{|I|}} \left\| \sum_{i\in I} \mathbf{a}_i \right\|, \tag{2.2}$$

and, restricting to $I = \{1\}, \dots, \{n\}$,

$$\|\mathbf{A}\| \geq \max_i \|\mathbf{a}_i\| = \max_i \left(\sum_j |a_{ji}|^2 \right)^{\frac{1}{2}}, \tag{2.3}$$

and also, restricting to $I = N$,

$$\|\mathbf{A}\| \geq \left(\frac{\operatorname{su} \mathbf{A}_I^*\mathbf{A}_I}{n} \right)^{\frac{1}{2}} = \left(\frac{|r_1|^2 + \cdots + |r_n|^2}{n} \right)^{\frac{1}{2}}, \tag{2.4}$$

where $r_1, \dots, r_n$ are the row sums of $\mathbf{A}$.

3. IMPROVED BOUNDS

To improve (2.2), choose

$$\mathbf{x} = \frac{\mathbf{e}_I + z\mathbf{e}_K}{\sqrt{|I| + |K|}},$$

where $z \in \mathbb{C}$ satisfies $|z| = 1$. Then

$$\begin{aligned} \|\mathbf{A}\| &\geq \frac{1}{\sqrt{|I| + |K|}} \left[\operatorname{su} \begin{pmatrix} \mathbf{A}_I^*\mathbf{A}_I & z\mathbf{A}_I^*\mathbf{A}_K \\ \overline{z}\mathbf{A}_K^*\mathbf{A}_I & \mathbf{A}_K^*\mathbf{A}_K \end{pmatrix} \right]^{\frac{1}{2}} \\ &= \frac{1}{\sqrt{|I| + |K|}} \left[\left\| \sum_{i\in I} \mathbf{a}_i \right\|^2 + \left\| \sum_{k\in K} \mathbf{a}_k \right\|^2 + 2 \operatorname{Re} \left(z \sum_{i\in I} \sum_{k\in K} \mathbf{a}_i^*\mathbf{a}_k \right) \right]^{\frac{1}{2}}, \end{aligned}$$

which, for $z = \overline{w}/|w|$ if $w = \sum_{i\in I}\sum_{k\in K} \mathbf{a}_i^*\mathbf{a}_k \neq 0$, and z arbitrary if $w = 0$, implies

$$\|\mathbf{A}\| \geq \frac{1}{\sqrt{|I|+|K|}} \left(\left\| \sum_{i\in I} \mathbf{a}_i \right\|^2 + \left\| \sum_{k\in K} \mathbf{a}_k \right\|^2 + 2 \left| \sum_{i\in I}\sum_{k\in K} \mathbf{a}_i^*\mathbf{a}_k \right| \right)^{\frac{1}{2}}. \tag{3.1}$$

Hence

$$\|\mathbf{A}\| \geq \max_{I\neq\emptyset,\, I\cap K=\emptyset} \frac{1}{\sqrt{|I|+|K|}} \left(\left\| \sum_{i\in I} \mathbf{a}_i \right\|^2 + \left\| \sum_{k\in K} \mathbf{a}_k \right\|^2 + 2 \left| \sum_{i\in I}\sum_{k\in K} \mathbf{a}_i^*\mathbf{a}_k \right| \right)^{\frac{1}{2}}, \tag{3.2}$$

and, restricting to $I, K = \{1\}, \dots, \{n\}$,

$$\|\mathbf{A}\| \geq \frac{1}{\sqrt{2}} \max_{i\neq k} \left(\|\mathbf{a}_i\|^2 + \|\mathbf{a}_k\|^2 + 2|\mathbf{a}_i^*\mathbf{a}_k| \right)^{\frac{1}{2}}. \tag{3.3}$$

It is well-known that the largest eigenvalue of a principal submatrix of a Hermitian matrix is less or equal to that of the original matrix. So, computing the largest eigenvalue of

$$\begin{pmatrix} \mathbf{a}_i^*\mathbf{a}_i & \mathbf{a}_i^*\mathbf{a}_k \\ \mathbf{a}_k^*\mathbf{a}_i & \mathbf{a}_k^*\mathbf{a}_k \end{pmatrix}$$

improves (3.3) to

$$\|\mathbf{A}\| \geq \frac{1}{\sqrt{2}} \max_{i\neq k} \left\{ \|\mathbf{a}_i\|^2 + \|\mathbf{a}_k\|^2 + \left[\left(\|\mathbf{a}_i\|^2 - \|\mathbf{a}_k\|^2 \right)^2 + 4|\mathbf{a}_i^*\mathbf{a}_k|^2 \right]^{\frac{1}{2}} \right\}^{\frac{1}{2}}. \tag{3.4}$$

4. FURTHER IMPROVEMENTS

Let $\mathbf{B}$ be a Hermitian $n \times n$ matrix with largest eigenvalue λ. If $\mathbf{0} \neq \mathbf{x} \in \mathbb{C}^n$, then

$$\lambda \geq \frac{\mathbf{x}^*\mathbf{B}\mathbf{x}}{\mathbf{x}^*\mathbf{x}}. \tag{4.1}$$

We replace $\mathbf{x}$ with $\mathbf{B}\mathbf{x}$ (assumed nonzero) and ask whether the bound so obtained,

$$\lambda \geq \frac{\mathbf{x}^*\mathbf{B}^3\mathbf{x}}{\mathbf{x}^*\mathbf{B}^2\mathbf{x}}, \tag{4.2}$$

is better. In other words, is

$$\frac{\mathbf{x}^*\mathbf{B}^3\mathbf{x}}{\mathbf{x}^*\mathbf{B}^2\mathbf{x}} \geq \frac{\mathbf{x}^*\mathbf{B}\mathbf{x}}{\mathbf{x}^*\mathbf{x}}$$

generally valid? The answer is yes if $\mathbf{B}$ is nonnegative definite (and $\mathbf{B}\mathbf{x} \neq \mathbf{0}$). In fact, the function

$$f(p) = \frac{\mathbf{x}^*\mathbf{B}^{p+1}\mathbf{x}}{\mathbf{x}^*\mathbf{B}^p\mathbf{x}} \qquad (p \geq 0)$$

is then increasing. We omit the easy proof but note that several interesting questions arise if we instead of nonnegative definiteness assume symmetry and nonnegativity, see [2] and its references.

If $\mathbf{B} = \mathbf{A}^*\mathbf{A}$, then (4.1) implies

$$\|\mathbf{A}\| \geq \frac{\|\mathbf{A}\mathbf{x}\|}{\|\mathbf{x}\|}, \tag{4.1$'$}$$

and (4.2) implies a better bound

$$\|\mathbf{A}\| \geq \frac{\|\mathbf{A}\mathbf{A}^*\mathbf{A}\mathbf{x}\|}{\|\mathbf{A}^*\mathbf{A}\mathbf{x}\|}. \tag{4.2$'$}$$

Since (2.2) is obtained by choosing in (4.1′) $\mathbf{x} = \mathbf{e}_I$ and maximizing over I, we get a better bound by applying (4.2′) instead. Because

$$\mathbf{A}^*\mathbf{A}\mathbf{e}_I = \mathbf{A}^* \sum_{i \in I} \mathbf{a}_i = \left(\mathbf{a}_1^* \sum_{i \in I} \mathbf{a}_i \quad \dots \quad \mathbf{a}_n^* \sum_{i \in I} \mathbf{a}_i \right)^{\mathsf{T}}$$

and

$$\mathbf{A}\mathbf{A}^*\mathbf{A}\mathbf{e}_I = \left(\sum_j \mathbf{a}_j \mathbf{a}_j^* \right) \sum_{i \in I} \mathbf{a}_i,$$

we have

$$\begin{aligned}(4.3)\quad &\|\mathbf{A}\| \\ &\geq \max \left\{ \left\| \left(\sum_j \mathbf{a}_j \mathbf{a}_j^* \right) \sum_{i \in I} \mathbf{a}_i \right\| \bigg/ \left\| \left(\mathbf{a}_1^* \sum_{i \in I} \mathbf{a}_i \;\dots\; \mathbf{a}_n^* \sum_{i \in I} \mathbf{a}_i \right)^{\mathsf{T}} \right\| \;\middle|\; I \neq \emptyset, \sum_{i \in I} \mathbf{a}_i \neq \mathbf{0} \right\}.\end{aligned}$$

Hence, restricting to $I = \{1\}, \dots, \{n\}$ and assuming all the $\mathbf{a}_i$'s nonzero,

$$\|\mathbf{A}\| \geq \max_i \left\| \left(\sum_j \mathbf{a}_j \mathbf{a}_j^* \right) \mathbf{a}_i \right\| \bigg/ \left\| (\mathbf{a}_1^*\mathbf{a}_i \;\dots\; \mathbf{a}_n^*\mathbf{a}_i)^{\mathsf{T}} \right\|, \tag{4.4}$$

and, restricting to $I = N$ and assuming the row sum vector $\mathbf{r} = (r_1 \dots r_n)^{\mathsf{T}}$ nonzero,

$$\|\mathbf{A}\| \geq \left\| \left(\sum_j \mathbf{a}_j \mathbf{a}_j^* \right) \mathbf{r} \right\| \bigg/ \left\| (\mathbf{a}_1^*\mathbf{r} \;\dots\; \mathbf{a}_n^*\mathbf{r})^{\mathsf{T}} \right\|. \tag{4.5}$$

5. EXPERIMENTS

We studied our bounds by random matrices of order 10. In the case of (2.2), (3.2), and (4.3), to avoid big complexities, we did not maximize over sets but studied only $I = \{1, 3, 5, 7, 9\}$, $K = \{2, 4, 6, 8, 10\}$. We considered various types of matrices (positive, normal, etc.). For each type, we performed one hundred experiments and computed means (m) and standard deviations (s) of

$$\frac{\|\mathbf{A}\| - \text{bound}}{\text{bound}}.$$

For positive symmetric matrices, (4.5) was by far the best with very surprising success: $m = 0.0000215$, $s = 0.0000316$. For all the remaining types, (4.4) was the best also with surprising success. We mention a few examples.

Type	m	s
Normal	0.0463	0.0227
Positive	0.0020	0.0012
Real	0.0261	0.0151
Complex	0.0429	0.0162

For positive matrices, also the very simple bound (2.4) was surprisingly good: $m = 0.0150$, $s = 0.0067$.

6. CONCLUSIONS

Our bounds (2.1), (2.3), (2.4), (3.1), (3.3), (3.4), (4.4), and (4.5) have complexity $O(n^2)$. Also the bounds (2.2), (3.2), and (4.3) have this complexity if we do not include all the subsets of N but only some suitable subsets. Our best $O(n^2)$ bounds seem to be in general better than all the $O(n^2)$ bounds we have found from the literature (e.g., the bound of [1, Theorem 3.7.15]).

One natural way [5, 6] of finding a lower bound for $\|\mathbf{A}\|$ is to compute the Wolkowicz-Styan [7] lower bound for the largest eigenvalue of $\mathbf{A}^*\mathbf{A}$ and to take the square root. The bound so obtained [5, 6] is fairly simple but seems to be in general worse than many of our bounds and has complexity $O(n^3)$.

7. REMARK

As $\|\mathbf{A}\| = \|\mathbf{A}^\mathsf{T}\| = \|\mathbf{A}^*\|$, all our results remain valid if we take row vectors instead of column vectors, and column sums instead of row sums. We can also do both and choose the better one.

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