

SOME RESULTS RELATED TO A CONJECTURE OF R. BRÜCK

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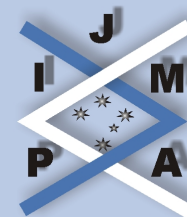
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Key words: Meromorphic function; Shared value; Small function.

Abstract: In this paper, we investigate the uniqueness problems of meromorphic functions that share a small function with its differential polynomials, and give some results which are related to a conjecture of R. Brück and improve some results of Liu, Gu, Lahiri and Zhang, and also answer some questions of Kit-Wing Yu.

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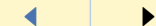
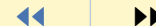
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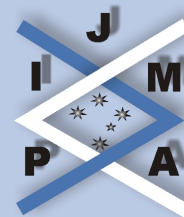
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1. Introduction and Results

In this paper a meromorphic function will mean meromorphic in the whole complex plane. We say that two meromorphic functions f and g share a finite value a IM (ignoring multiplicities) when $f - a$ and $g - a$ have the same zeros. If $f - a$ and $g - a$ have the same zeros with the same multiplicities, then we say that f and g share the value a CM (counting multiplicities). It is assumed that the reader is familiar with the standard symbols and fundamental results of Nevanlinna theory, as found in [5] and [15]. For any non-constant meromorphic function f , we denote by $S(r, f)$ any quantity satisfying

$$\lim_{r \rightarrow \infty} \frac{S(r, f)}{T(r, f)} = 0,$$

possibly outside of a set of finite linear measure in \mathbb{R} . Suppose that $a(z)$ is a meromorphic function, we say that $a(z)$ is a small function of f , if $T(r, a) = S(r, f)$.

Let l be a non-negative integer or infinite. For any $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_l(a, f)$ the set of all a -points of f where an a -point of multiplicity m is counted m times if $m \leq l$ and $l + 1$ times if $m > l$. If $E_l(a, f) = E_l(a, g)$, we say that f and g share the value a with weight l (see [6]).

We say that f and g share (a, l) if f and g share the value a with weight l . It is easy to see that f and g share (a, l) implies f and g share (a, p) for $0 \leq p \leq l$. Also we note that f and g share a value a IM or CM if and only if f and g share $(a, 0)$ or (a, ∞) respectively (see [6]).

L.A. Rubel and C.C. Yang [9], E. Mues and N. Steinmetz [8], G. Gundersen [3] and L.-Z. Yang [10], J.-H. Zheng and S.P. Wang [18], and many other authors have obtained elegant results on the uniqueness problems of entire functions that share values CM or IM with their first or k -th derivatives. In the aspect of only one CM value, R. Brück [1] posed the following conjecture.



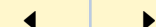
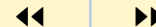
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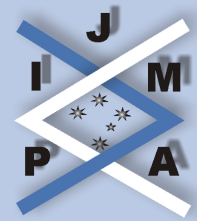
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Conjecture 1.1. Let f be a non-constant entire function. Suppose that $\rho_1(f)$ is not a positive integer or infinite, if f and f' share one finite value a CM, then

$$\frac{f' - a}{f - a} = c$$

for some non-zero constant c , where $\rho_1(f)$ is the first iterated order of f which is defined by

$$\rho_1(f) = \limsup_{r \rightarrow \infty} \frac{\log \log T(r, f)}{\log r}.$$

R. Brück also showed in the same paper that the conjecture is true if $a = 0$ or $N\left(r, \frac{1}{f'}\right) = S(r, f)$ (no growth condition in the later case). Furthermore in 1998, G.G. Gundersen and L.Z. Yang [4] proved that the conjecture is true if f is of finite order, and in 1999, L. Z. Yang [11] generalized their results to the k -th derivatives. In 2004, Z.-X. Chen and K. H. Shon [2] proved that the conjecture is true for entire functions of first iterated order $\rho_1 < 1/2$. In 2003, Kit-Wing Yu [16] considered the case that a is a small function, and obtained the following results.

Theorem A. *Let f be a non-constant entire function, let k be a positive integer, and let a be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. If $f - a$ and $f^{(k)} - a$ share the value 0 CM and $\delta(0, f) > \frac{3}{4}$, then $f \equiv f^{(k)}$.*

Theorem B. *Let f be a non-constant, non-entire meromorphic function, let k be a positive integer, and let a be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. If f and a do not have any common pole, and if $f - a$ and $f^{(k)} - a$ share the value 0 CM and $4\delta(0, f) + 2(8 + k)\Theta(\infty, f) > 19 + 2k$, then $f \equiv f^{(k)}$.*

In the same paper, Kit-Wing Yu [16] posed the following questions.

Problem 1. *Can a CM shared value be replaced by an IM shared value in Theorem A?*



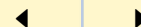
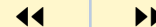
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Problem 2. Is the condition $\delta(0, f) > \frac{3}{4}$ sharp in Theorem A?

Problem 3. Is the condition $4\delta(0, f) + 2(8+k)\Theta(\infty, f) > 19 + 2k$ sharp in Theorem B?

Problem 4. Can the condition “ f and a do not have any common pole” be deleted in Theorem B?

In 2004, Liu and Gu [7] obtained the following results.

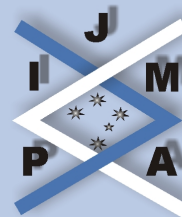
Theorem C. Let $k \geq 1$ and let f be a non-constant meromorphic function, and let a be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. If $f - a$ and $f^{(k)} - a$ share the value 0 CM, $f^{(k)}$ and a do not have any common poles of the same multiplicities and

$$2\delta(0, f) + 4\Theta(\infty, f) > 5,$$

then $f \equiv f^{(k)}$.

Theorem D. Let $k \geq 1$ and let f be a non-constant entire function, and let a be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. If $f - a$ and $f^{(k)} - a$ share the value 0 CM and $\delta(0, f) > \frac{1}{2}$, then $f \equiv f^{(k)}$.

Let p be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $N_p\left(r, \frac{1}{f-a}\right)$ the counting function of the zeros of $f - a$ with the multiplicities less than or equal to p , and by $N_{(p+1)}\left(r, \frac{1}{f-a}\right)$ the counting function of the zeros of $f - a$ with the multiplicities larger than p . And we use $\bar{N}_p\left(r, \frac{1}{f-a}\right)$ and $\bar{N}_{(p+1)}\left(r, \frac{1}{f-a}\right)$ to denote their corresponding reduced counting functions (ignoring multiplicities) respectively. We also use $N_p\left(r, \frac{1}{f-a}\right)$ to denote the counting function of the zeros of $f - a$ where a



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p -folds zero is counted m times if $m \leq p$ and p times if $m > p$. Define

$$\delta_p(a, f) = 1 - \limsup_{r \rightarrow \infty} \frac{N_p\left(r, \frac{1}{f-a}\right)}{T(r, f)}.$$

It is obvious that $\delta_p(a, f) \geq \delta(a, f)$ and

$$N_1\left(r, \frac{1}{f-a}\right) = \overline{N}\left(r, \frac{1}{f-a}\right).$$

Lahiri [6] improved Theorem C with weighted shared values and obtained the following theorem.

Theorem E. *Let f be a non-constant meromorphic function, k be a positive integer, and let $a \equiv a(z)$ be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. If*

(i) $a(z)$ has no zero (pole) which is also a zero (pole) of f or $f^{(k)}$ with the same multiplicity,

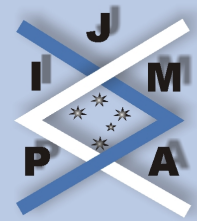
(ii) $f - a$ and $f^{(k)} - a$ share $(0, 2)$,

(iii) $2\delta_{2+k}(0, f) + (4 + k)\Theta(\infty, f) > 5 + k$,

then $f \equiv f^{(k)}$.

In 2005, Zhang [17] obtained the following result which is an improvement and complement of Theorem D.

Theorem F. *Let f be a non-constant meromorphic function, $k (\geq 1)$ and $l (\geq 0)$ be integers. Also, let $a \equiv a(z)$ be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. Suppose that $f - a$ and $f^{(k)} - a$ share $(0, l)$. Then $f \equiv f^{(k)}$ if one of the following conditions is satisfied,*



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(i) $l \geq 2$ and

$$(3 + k)\Theta(\infty, f) + 2\delta_{2+k}(0, f) > k + 4;$$

(ii) $l = 1$ and

$$(4 + k)\Theta(\infty, f) + 3\delta_{2+k}(0, f) > k + 6;$$

(iii) $l = 0$ (i.e. $f - a$ and $f^k - a$ share the value 0 IM) and

$$(6 + 2k)\Theta(\infty, f) + 5\delta_{2+k}(0, f) > 2k + 10.$$

It is natural to ask what happens if $f^{(k)}$ is replaced by a differential polynomial

$$(1.1) \quad L(f) = f^{(k)} + a_{k-1}f^{(k-1)} + \dots + a_0f$$

in Theorem E or F, where a_j ($j = 0, 1, \dots, k - 1$) are small meromorphic functions of f . Corresponding to this question, we obtain the following result which improves Theorems A ~ F and answers the four questions mentioned above.

Theorem 1.2. *Let f be a non-constant meromorphic function, $k(\geq 1)$ and $l(\geq 0)$ be integers. Also, let $a = a(z)$ be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. Suppose that $f - a$ and $L(f) - a$ share $(0, l)$. Then $f \equiv L(f)$ if one of the following assumptions holds,*

(i) $l \geq 2$ and

$$(1.2) \quad \delta_{2+k}(0, f) + \delta_2(0, f) + 3\Theta(\infty, f) + \delta(a, f) > 4;$$

(ii) $l = 1$ and

$$(1.3) \quad \delta_{2+k}(0, f) + \delta_2(0, f) + \frac{1}{2}\delta_{1+k}(0, f) + \frac{k+7}{2}\Theta(\infty, f) + \delta(a, f) > \frac{k}{2} + 5;$$



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(iii) $l = 0$ (i.e. $f - a$ and $L(f) - a$ share the value 0 IM) and

$$(1.4) \quad \delta_{2+k}(0, f) + 2\delta_{1+k}(0, f) + \delta_2(0, f) \\ + \Theta(0, f) + (6 + 2k)\Theta(\infty, f) + \delta(a, f) > 2k + 10.$$

Since $\delta_2(0, f) \geq \delta_{1+k}(0, f) \geq \delta_{2+k}(0, f) \geq \delta(0, f)$, we have the following corollary that improves Theorems A ~ F.

Corollary 1.3. *Let f be a non-constant meromorphic function, $k(\geq 1)$ and $l(\geq 0)$ be integers, and let $a \equiv a(z)$ be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. Suppose that $f - a$ and $f^{(k)} - a$ share $(0, l)$. Then $f \equiv f^{(k)}$ if one of the following three conditions holds,*

(i) $l \geq 2$ and

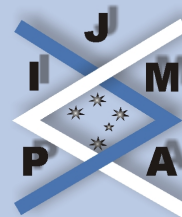
$$2\delta_{2+k}(0, f) + 3\Theta(\infty, f) + \delta(a, f) > 4;$$

(ii) $l = 1$ and

$$\frac{5}{2}\delta_{2+k}(0, f) + \frac{k+7}{2}\Theta(\infty, f) + \delta(a, f) > \frac{k}{2} + 5;$$

(iii) $l = 0$ (i.e. $f - a$ and $L(f) - a$ share the value 0 IM) and

$$5\delta_{2+k}(0, f) + (6 + 2k)\Theta(\infty, f) + \delta(a, f) > 2k + 10.$$



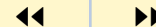
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2. Some Lemmas

Lemma 2.1 ([12]). *Let f be a non-constant meromorphic function. Then*

$$(2.1) \quad N\left(r, \frac{1}{f^{(n)}}\right) \leq T(r, f^{(n)}) - T(r, f) + N\left(r, \frac{1}{f}\right) + S(r, f),$$

$$(2.2) \quad N\left(r, \frac{1}{f^{(n)}}\right) \leq N\left(r, \frac{1}{f}\right) + n\bar{N}(r, f) + S(r, f).$$

Suppose that F and G are two non-constant meromorphic functions such that F and G share the value 1 IM. Let z_0 be a 1-point of F of order p , a 1-point of G of order q . We denote by $N_L\left(r, \frac{1}{F-1}\right)$ the counting function of those 1-points of F where $p > q$, by $N_E^{(1)}\left(r, \frac{1}{F-1}\right)$ the counting function of those 1-points of F where $p = q = 1$, by $N_E^{(2)}\left(r, \frac{1}{F-1}\right)$ the counting function of those 1-points of F where $p = q \geq 2$; each point in these counting functions is counted only once. In the same way, we can define $N_L\left(r, \frac{1}{G-1}\right)$, $N_E^{(1)}\left(r, \frac{1}{G-1}\right)$ and $N_E^{(2)}\left(r, \frac{1}{G-1}\right)$ (see [14]). In particular, if F and G share 1 CM, then

$$(2.3) \quad N_L\left(r, \frac{1}{F-1}\right) = N_L\left(r, \frac{1}{G-1}\right) = 0.$$

With these notations, if F and G share 1 IM, it is easy to see that

$$(2.4) \quad \begin{aligned} \bar{N}\left(r, \frac{1}{F-1}\right) &= N_E^{(1)}\left(r, \frac{1}{F-1}\right) + N_L\left(r, \frac{1}{F-1}\right) \\ &\quad + N_L\left(r, \frac{1}{G-1}\right) + N_E^{(2)}\left(r, \frac{1}{G-1}\right) \end{aligned}$$

$$= \bar{N} \left(r, \frac{1}{G-1} \right).$$

Lemma 2.2 ([13]). *Let*

$$(2.5) \quad H = \left(\frac{F''}{F'} - \frac{2F'}{F-1} \right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1} \right),$$

where F and G are two nonconstant meromorphic functions. If F and G share 1 IM and $H \not\equiv 0$, then

$$(2.6) \quad N_E^{(1)} \left(r, \frac{1}{F-1} \right) \leq N(r, H) + S(r, F) + S(r, G).$$

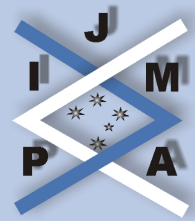
Lemma 2.3. *Let f be a transcendental meromorphic function, $L(f)$ be defined by (1.1). If $L(f) \not\equiv 0$, we have*

$$(2.7) \quad N \left(r, \frac{1}{L} \right) \leq T(r, L) - T(r, f) + N \left(r, \frac{1}{f} \right) + S(r, f),$$

$$(2.8) \quad N \left(r, \frac{1}{L} \right) \leq k\bar{N}(r, f) + N \left(r, \frac{1}{f} \right) + S(r, f).$$

Proof. By the first fundamental theorem and the lemma of logarithmic derivatives, we have

$$\begin{aligned} N \left(r, \frac{1}{L} \right) &= T(r, L) - m \left(r, \frac{1}{L} \right) + O(1) \\ &\leq T(r, L) - \left(m \left(r, \frac{1}{f} \right) - m \left(r, \frac{L(f)}{f} \right) \right) + O(1) \end{aligned}$$



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$$\begin{aligned} &\leq T(r, L) - \left(T(r, f) - N\left(r, \frac{1}{f}\right) \right) + S(r, f) \\ &\leq T(r, L) - T(r, f) + N\left(r, \frac{1}{f}\right) + S(r, f). \end{aligned}$$

This proves (2.7). Since

$$\begin{aligned} T(r, L) &= m(r, L) + N(r, L) \\ &\leq m(r, f) + m\left(r, \frac{L}{f}\right) + N(r, f) + k\bar{N}(r, f) \\ &= T(r, f) + k\bar{N}(r, f) + S(r, f), \end{aligned}$$

from this and (2.7), we obtain (2.8). Lemma 2.3 is thus proved. ■

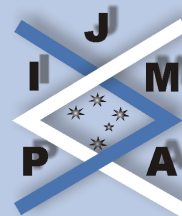
Lemma 2.4. *Let f be a non-constant meromorphic function, $L(f)$ be defined by (1.1), and let p be a positive integer. If $L(f) \not\equiv 0$, we have*

$$(2.9) \quad N_p\left(r, \frac{1}{L}\right) \leq T(r, L) - T(r, f) + N_{p+k}\left(r, \frac{1}{f}\right) + S(r, f),$$

$$(2.10) \quad N_p\left(r, \frac{1}{L}\right) \leq k\bar{N}(r, f) + N_{p+k}\left(r, \frac{1}{f}\right) + S(r, f).$$

Proof. From (2.8), we have

$$\begin{aligned} N_p\left(r, \frac{1}{L}\right) + \sum_{j=p+1}^{\infty} \bar{N}_{(j)}\left(r, \frac{1}{L}\right) \\ \leq N_{p+k}\left(r, \frac{1}{f}\right) + \sum_{j=p+k+1}^{\infty} \bar{N}_{(j)}\left(r, \frac{1}{f}\right) + k\bar{N}(r, f) + S(r, f), \end{aligned}$$



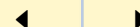
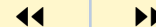
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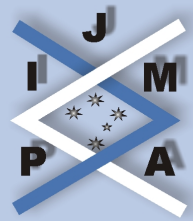
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then

$$\begin{aligned} N_p \left(r, \frac{1}{L} \right) &\leq N_{p+k} \left(r, \frac{1}{f} \right) + \sum_{j=p+k+1}^{\infty} \bar{N}_{(j)} \left(r, \frac{1}{f} \right) \\ &\quad - \sum_{j=p+1}^{\infty} \bar{N}_{(j)} \left(r, \frac{1}{L} \right) + k\bar{N}(r, f) + S(r, f) \\ &\leq N_{p+k} \left(r, \frac{1}{f} \right) + k\bar{N}(r, f) + S(r, f). \end{aligned}$$

Thus (2.10) holds. By the same arguments as above, we obtain (2.9) from (2.7). ■



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3. Proof of Theorem 1.2

Let

$$(3.1) \quad F = \frac{L(f)}{a}, \quad G = \frac{f}{a}.$$

From the conditions of Theorem 1.2, we know that F and G share $(1, l)$ except the zeros and poles of $a(z)$. From (3.1), we have

$$(3.2) \quad T(r, F) = O(T(r, f)) + S(r, f), \quad T(r, G) \leq T(r, f) + S(r, f),$$

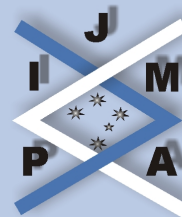
$$(3.3) \quad \bar{N}(r, F) = \bar{N}(r, G) + S(r, f).$$

It is obvious that f is a transcendental meromorphic function. Let H be defined by (2.5). We discuss the following two cases.

Case 1. $H \not\equiv 0$, by Lemma 2.2 we know that (2.6) holds. From (2.5) and (3.3), we have

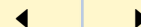
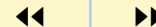
$$(3.4) \quad N(r, H) \leq \bar{N}_{(2)}\left(r, \frac{1}{F}\right) + \bar{N}_{(2)}\left(r, \frac{1}{G}\right) + \bar{N}(r, G) \\ + N_L\left(r, \frac{1}{F-1}\right) + N_L\left(r, \frac{1}{G-1}\right) + N_0\left(r, \frac{1}{F'}\right) + N_0\left(r, \frac{1}{G'}\right),$$

where $N_0\left(r, \frac{1}{F'}\right)$ denotes the counting function corresponding to the zeros of F' which are not the zeros of F and $F-1$, $N_0\left(r, \frac{1}{G'}\right)$ denotes the counting function corresponding to the zeros of G' which are not the zeros of G and $G-1$. From the



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second fundamental theorem in Nevanlinna's Theory, we have

$$(3.5) \quad T(r, F) + T(r, G) \leq \bar{N}\left(r, \frac{1}{F}\right) + \bar{N}(r, F) + \bar{N}\left(r, \frac{1}{F-1}\right) + \bar{N}\left(r, \frac{1}{G}\right) \\ + \bar{N}(r, G) + \bar{N}\left(r, \frac{1}{G-1}\right) - N_0\left(r, \frac{1}{F'}\right) - N_0\left(r, \frac{1}{G'}\right) + S(r, f).$$

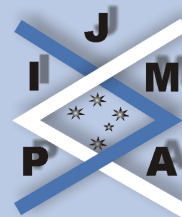
Noting that F and G share 1 IM except the zeros and poles of $a(z)$, we get from (2.4),

$$\bar{N}\left(r, \frac{1}{F-1}\right) + \bar{N}\left(r, \frac{1}{G-1}\right) \\ = 2N_E^{(1)}\left(r, \frac{1}{F-1}\right) + 2N_L\left(r, \frac{1}{F-1}\right) + 2N_L\left(r, \frac{1}{G-1}\right) \\ + 2N_E^{(2)}\left(r, \frac{1}{G-1}\right) + S(r, f).$$

Combining with (2.6) and (3.4), we obtain

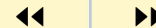
$$(3.6) \quad \bar{N}\left(r, \frac{1}{F-1}\right) + \bar{N}\left(r, \frac{1}{G-1}\right) \\ \leq N_{(2)}\left(r, \frac{1}{F}\right) + N_{(2)}\left(r, \frac{1}{G}\right) + \bar{N}(r, G) + 3N_L\left(r, \frac{1}{F-1}\right) + 3N_L\left(r, \frac{1}{G-1}\right) \\ + N_E^{(1)}\left(r, \frac{1}{F-1}\right) + 2N_E^{(2)}\left(r, \frac{1}{G-1}\right) + N_0\left(r, \frac{1}{F'}\right) + N_0\left(r, \frac{1}{G'}\right) + S(r, f).$$

We discuss the following three subcases.



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Subcase 1.1 $l \geq 2$. It is easy to see that

$$(3.7) \quad 3N_L \left(r, \frac{1}{F-1} \right) + 3N_L \left(r, \frac{1}{G-1} \right) \\ + 2N_E^{(2)} \left(r, \frac{1}{G-1} \right) + N_E^{(1)} \left(r, \frac{1}{F-1} \right) \\ \leq N \left(r, \frac{1}{G-1} \right) + S(r, f).$$

From (3.6) and (3.7), we have

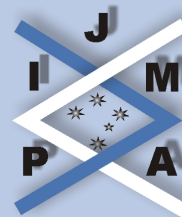
$$(3.8) \quad \bar{N} \left(r, \frac{1}{F-1} \right) + \bar{N} \left(r, \frac{1}{G-1} \right) \\ \leq N_{(2)} \left(r, \frac{1}{F} \right) + N_{(2)} \left(r, \frac{1}{G} \right) + \bar{N}(r, G) + N \left(r, \frac{1}{G-1} \right) \\ + N_0 \left(r, \frac{1}{F'} \right) + N_0 \left(r, \frac{1}{G'} \right) + S(r, f).$$

Substituting (3.8) into (3.5) and by using (3.3), we have

$$(3.9) \quad T(r, F) + T(r, G) \\ \leq 3\bar{N}(r, G) + N_2 \left(r, \frac{1}{F} \right) + N_2 \left(r, \frac{1}{G} \right) + N \left(r, \frac{1}{G-1} \right) + S(r, f).$$

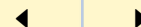
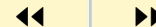
Noting that

$$N_2 \left(r, \frac{1}{F} \right) = N_2 \left(r, \frac{a}{L} \right) \leq N_2 \left(r, \frac{1}{L} \right) + S(r, f),$$



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we obtain from (2.9), (3.1) and (3.9) that

$$(3.10) \quad T(r, f) \leq 3\bar{N}(r, f) + N_{2+k} \left(r, \frac{1}{f} \right) + N_2 \left(r, \frac{1}{f} \right) - m \left(r, \frac{1}{G-1} \right) + S(r, f),$$

which contradicts the assumption (1.2) of Theorem 1.2.

Subcase 1.2 $l = 1$. Noting that

$$\begin{aligned} 2N_L \left(r, \frac{1}{F-1} \right) + 3N_L \left(r, \frac{1}{G-1} \right) + 2N_E^{(2)} \left(r, \frac{1}{G-1} \right) + N_E^{(1)} \left(r, \frac{1}{F-1} \right) \\ \leq N \left(r, \frac{1}{G-1} \right) + S(r, f), \end{aligned}$$

$$\begin{aligned} N_L \left(r, \frac{1}{F-1} \right) &\leq \frac{1}{2} N \left(r, \frac{F}{F'} \right) \\ &\leq \frac{1}{2} N \left(r, \frac{F'}{F} \right) + S(r, f) \\ &\leq \frac{1}{2} \left(\bar{N} \left(r, \frac{1}{F} \right) + \bar{N}(r, F) \right) + S(r, f) \\ &\leq \frac{1}{2} \left(N_1 \left(r, \frac{1}{F} \right) + \bar{N}(r, f) \right) + S(r, f) \\ &\leq \frac{1}{2} \left(N_{1+k} \left(r, \frac{1}{f} \right) + (k+1)\bar{N}(r, f) \right) + S(r, f), \end{aligned}$$

and by the same reasoning as in Subcase 1.1, we get

$$T(r, f) \leq \frac{k+7}{2} \bar{N}(r, f) + N_{2+k} \left(r, \frac{1}{f} \right) + N_2 \left(r, \frac{1}{f} \right)$$

$$+ \frac{1}{2}N_{1+k} \left(r, \frac{1}{f} \right) - m \left(r, \frac{1}{G-1} \right) + S(r, f),$$

which contradicts the assumption (1.3) of Theorem 1.2.

Subcase 1.3 $l = 0$. Noting that

$$\begin{aligned} N_L \left(r, \frac{1}{F-1} \right) + 2N_L \left(r, \frac{1}{G-1} \right) + 2N_E^{(2)} \left(r, \frac{1}{G-1} \right) + N_E^{(1)} \left(r, \frac{1}{F-1} \right) \\ \leq N \left(r, \frac{1}{G-1} \right) + S(r, f), \end{aligned}$$

$$2N_L \left(r, \frac{1}{F-1} \right) + N_L \left(r, \frac{1}{G-1} \right) \leq 2N \left(r, \frac{1}{F'} \right) + N \left(r, \frac{1}{G'} \right),$$

and by the same reasoning as in the Subcase 1.2, we get a contradiction.

Case 2. $H \equiv 0$. By integration, we get from (2.5) that

$$(3.11) \quad \frac{1}{G-1} = \frac{A}{F-1} + B,$$

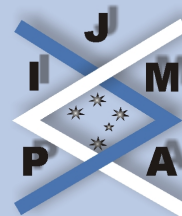
where $A (\neq 0)$ and B are constants. From (3.11) we have

$$(3.12) \quad N(r, F) = N(r, G) = N(r, f) = S(r, f), \quad \Theta(\infty, f) = 1,$$

and

$$(3.13) \quad G = \frac{(B+1)F + (A-B-1)}{BF + (A-B)}, \quad F = \frac{(B-A)G + (A-B-1)}{BG - (B+1)}.$$

We discuss the following three subcases.



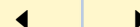
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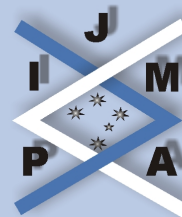
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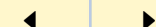
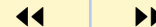
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Subcase 2.1 Suppose that $B \neq 0, -1$. From (3.13) we have $\bar{N}(r, 1/(G - \frac{B+1}{B})) = \bar{N}(r, F)$. From this and the second fundamental theorem, we have

$$\begin{aligned} T(r, f) &\leq T(r, G) + S(r, f) \\ &\leq \bar{N}(r, G) + \bar{N}\left(r, \frac{1}{G}\right) + \bar{N}\left(r, \frac{1}{G - \frac{B+1}{B}}\right) + S(r, f) \\ &\leq \bar{N}\left(r, \frac{1}{G}\right) + \bar{N}(r, F) + \bar{N}(r, G) + S(r, f) \\ &\leq \bar{N}\left(r, \frac{1}{f}\right) + S(r, f), \end{aligned}$$

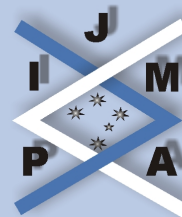
which contradicts the assumptions of Theorem 1.2.

Subcase 2.2 Suppose that $B = 0$. From (3.13) we have

$$(3.14) \quad G = \frac{F + (A - 1)}{A}, \quad F = AG - (A - 1).$$

If $A \neq 1$, from (3.14) we can obtain $\bar{N}(r, 1/(G - \frac{A-1}{A})) = \bar{N}(r, 1/F)$. From this and the second fundamental theorem, we have

$$\begin{aligned} 2T(r, f) &\leq 2T(r, G) + S(r, f) \\ &\leq \bar{N}(r, G) + \bar{N}\left(r, \frac{1}{G}\right) + \bar{N}\left(r, 1/\left(G - \frac{A-1}{A}\right)\right) \\ &\quad + \bar{N}\left(r, \frac{1}{G-1}\right) + S(r, f) \\ &\leq \bar{N}\left(r, \frac{1}{G}\right) + \bar{N}\left(r, \frac{1}{F}\right) + \bar{N}\left(r, \frac{1}{G-1}\right) + S(r, f), \end{aligned}$$



which contradicts the assumptions of Theorem 1.2. Thus $A = 1$. From (3.14) we have $F \equiv G$, then $f \equiv L$.

Subcase 2.3 Suppose that $B = -1$, from (3.13) we have

$$(3.15) \quad G = \frac{A}{-F + (A + 1)}, \quad F = \frac{(A + 1)G - A}{G}.$$

If $A \neq -1$, we obtain from (3.15) that $N\left(r, 1/\left(G - \frac{A}{A+1}\right)\right) = N(r, 1/F)$. By the same reasoning discussed in Subcase 2.2, we obtain a contradiction. Hence $A = -1$. From (3.15), we get $F \cdot G \equiv 1$, that is

$$(3.16) \quad f \cdot L \equiv a^2.$$

From (3.16), we have

$$(3.17) \quad N\left(r, \frac{1}{f}\right) + N(r, f) = S(r, f),$$

and so $T\left(r, \frac{f^{(k)}}{f}\right) = S(r, f)$. From (3.17), we obtain

$$\begin{aligned} 2T\left(r, \frac{f}{a}\right) &= T\left(r, \frac{f^2}{a^2}\right) \\ &= T\left(r, \frac{a^2}{f^2}\right) + O(1) \\ &= T\left(r, \frac{L}{f}\right) + O(1) = S(r, f), \end{aligned}$$

and so $T(r, f) = S(r, f)$, this is impossible. This completes the proof of Theorem 1.2.

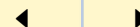
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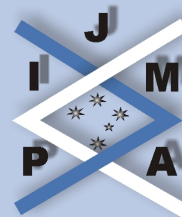
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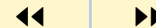
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4. Remarks

Let f and g be non-constant meromorphic functions, $a(z)$ be a small function of f and g , and k be a positive integer or ∞ . We denote by $\overline{N}_E^{(k)}(r, a)$ the counting function of common zeros of $f - a$ and $g - a$ with the same multiplicities $p \leq k$, by $\overline{N}_0^{(k+1)}(r, a)$ the counting function of common zeros of $f - a$ and $g - a$ with the multiplicities $p \geq k + 1$, and denote by $\overline{N}_0(r, a)$ the counting function of common zeros of $f - a$ and $g - a$; each point in these counting functions is counted only once.

Definition 4.1. Let f and g be non-constant meromorphic functions, a be a small function of f and g , and k be a positive integer or ∞ . We say that f and g share “ (a, k) ” if $k = 0$, and

$$\overline{N}\left(r, \frac{1}{f-a}\right) - \overline{N}_0(r, a) = S(r, f),$$

$$\overline{N}\left(r, \frac{1}{g-a}\right) - \overline{N}_0(r, a) = S(r, g);$$

or $k \neq 0$, and

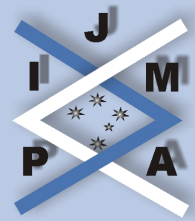
$$\overline{N}_{(k)}\left(r, \frac{1}{f-a}\right) - \overline{N}_E^{(k)}(r, a) = S(r, f),$$

$$\overline{N}_{(k)}\left(r, \frac{1}{g-a}\right) - \overline{N}_E^{(k)}(r, a) = S(r, g),$$

$$\overline{N}_{(k+1)}\left(r, \frac{1}{f-a}\right) - \overline{N}_0^{(k+1)}(r, a) = S(r, f),$$

$$\overline{N}_{(k+1)}\left(r, \frac{1}{g-a}\right) - \overline{N}_0^{(k+1)}(r, a) = S(r, g).$$

By the above definition and a similar argument to that used in the proof of Theorem 1.2, we conclude that Theorem 1.2 and Corollary 1.3 still hold if the condition that $f - a$ and $L(f) - a$ (or $f^{(k)} - a$) share $(0, l)$ is replaced by the assumption that $f - a$ and $L(f) - a$ (or $f^{(k)} - a$) share “ $(0, l)$ ”.



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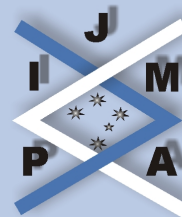
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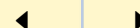
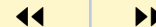
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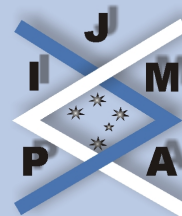
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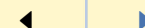
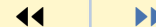
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