SOME RESULTS RELATED TO A CONJECTURE OF R. BRÜCK

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Abstract: In this paper, we investigate the uniqueness problems of meromorphic functions that share a small function with its differential polynomials, and give some results which are related to a conjecture of R. Brück and improve some results of Liu, Gu, Lahiri and Zhang, and also answer some questions of Kit-Wing Yu.

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1. Introduction and Results

In this paper a meromorphic function will mean meromorphic in the whole complex plane. We say that two meromorphic functions f and g share a finite value a IM (ignoring multiplicities) when $f - a$ and $g - a$ have the same zeros. If $f - a$ and $g-a$ have the same zeros with the same multiplicities, then we say that f and g share the value a CM (counting multiplicities). It is assumed that the reader is familiar with the standard symbols and fundamental results of Nevanlinna theory, as found in [5] and [15]. For any non-constant meromorphic function f, we denote by $S(r, f)$ any quantity satisfying

$$\lim_{r\to\infty} \frac{S(r,f)}{T(r,f)} = 0,$$

possibly outside of a set of finite linear measure in $\mathbb{R}$. Suppose that $a(z)$ is a meromorphic function, we say that $a(z)$ is a small function of f, if $T(r, a) = S(r, f)$.

Let l be a non-negative integer or infinite. For any $a \in \mathbb{C}\bigcup\{\infty\}$, we denote by $E_l(a, f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \le l$ and $l + 1$ times if $m > l$. If $E_l(a, f) = E_l(a, g)$, we say that f and g share the value a with weight l (see [6]).

We say that f and g share (a, l) if f and g share the value a with weight l. It is easy to see that f and g share (a, l) implies f and g share (a, p) for $0 \le p \le l$. Also we note that f and g share a value a IM or CM if and only if f and g share $(a, 0)$ or (a, ∞) respectively (see [6]).

L.A. Rubel and C.C. Yang [9], E. Mues and N. Steinmetz [8], G. Gundersen [3] and L.-Z. Yang [10], J.-H. Zheng and S.P. Wang [18], and many other authors have obtained elegant results on the uniqueness problems of entire functions that share values CM or IM with their first or k-th derivatives. In the aspect of only one CM value, R. Brück [1] posed the following conjecture.

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Conjecture 1.1. Let f be a non-constant entire function. Suppose that $\rho_1(f)$ is not a positive integer or infinite, if f and f' share one finite value a CM, then

$$\frac{f'-a}{f-a}=c$$

for some non-zero constant c, where $\rho_1(f)$ is the first iterated order of f which is defined by

$$\rho_1(f)=\limsup_{r\to\infty}\frac{\log\log T(r,f)}{\log r}.$$

R. Brück also showed in the same paper that the conjecture is true if $a=0$ or $N\left(r,\frac{1}{f'}\right)=S(r,f)$ (no growth condition in the later case). Furthermore in 1998, G.G. Gundersen and L.Z. Yang [4] proved that the conjecture is true if f is of finite order, and in 1999, L. Z. Yang [11] generalized their results to the k-th derivatives. In 2004, Z.-X. Chen and K. H. Shon [2] proved that the conjecture is true for entire functions of first iterated order $\rho_1<1/2$. In 2003, Kit-Wing Yu [16] considered the case that a is a small function, and obtained the following results.

Theorem A. *Let f be a non-constant entire function, let k be a positive integer, and let a be a small meromorphic function of f such that $a(z)\not\equiv 0,\infty$. If $f-a$ and $f^{(k)}-a$ share the value 0 CM and $\delta(0,f)>\frac{3}{4}$, then $f\equiv f^{(k)}$.*

Theorem B. *Let f be a non-constant, non-entire meromorphic function, let k be a positive integer, and let a be a small meromorphic function of f such that $a(z)\not\equiv 0,\infty$. If f and a do not have any common pole, and if $f-a$ and $f^{(k)}-a$ share the value 0 CM and $4\delta(0,f)+2(8+k)\Theta(\infty,f)>19+2k$, then $f\equiv f^{(k)}$.*

In the same paper, Kit-Wing Yu [16] posed the following questions.

Problem 1. *Can a CM shared value be replaced by an IM shared value in Theorem A?*

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Problem 2. *Is the condition $\delta(0, f) > \frac{3}{4}$ sharp in Theorem A?*

Problem 3. *Is the condition $4\delta(0, f)+2(8+k)\Theta(\infty, f) > 19+2k$ sharp in Theorem B?*

Problem 4. *Can the condition "f and a do not have any common pole" be deleted in Theorem B?*

In 2004, Liu and Gu [7] obtained the following results.

Theorem C. *Let $k \geq 1$ and let f be a non-constant meromorphic function, and let a be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. If $f - a$ and $f^{(k)} - a$ share the value 0 CM, $f^{(k)}$ and a do not have any common poles of the same multiplicities and*

$$2\delta(0, f) + 4\Theta(\infty, f) > 5,$$

then $f \equiv f^{(k)}$.

Theorem D. *Let $k \geq 1$ and let f be a non-constant entire function, and let a be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. If $f - a$ and $f^{(k)} - a$ share the value 0 CM and $\delta(0, f) > \frac{1}{2}$, then $f \equiv f^{(k)}$.*

Let p be a positive integer and $a \in \mathbb{C}\bigcup\{\infty\}$. We denote by $N_{p)}\left(r, \frac{1}{f-a}\right)$ the counting function of the zeros of $f - a$ with the multiplicities less than or equal to p, and by $N_{(p+1}\left(r, \frac{1}{f-a}\right)$ the counting function of the zeros of $f - a$ with the multiplicities larger than p. And we use $\overline{N}_{p)}\left(r, \frac{1}{f-a}\right)$ and $\overline{N}_{(p+1}\left(r, \frac{1}{f-a}\right)$ to denote their corresponding reduced counting functions (ignoring multiplicities) respectively. We also use $N_p\left(r, \frac{1}{f-a}\right)$ to denote the counting function of the zeros of $f - a$ where a

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p-folds zero is counted m times if $m \leq p$ and p times if $m > p$. Define

$$\delta_p(a, f) = 1 - \limsup_{r\to\infty} \frac{N_p\left(r, \frac{1}{f-a}\right)}{T(r, f)}.$$

It is obvious that $\delta_p(a, f) \geq \delta(a, f)$ and

$$N_1\left(r, \frac{1}{f-a}\right) = \overline{N}\left(r, \frac{1}{f-a}\right).$$

Lahiri [6] improved Theorem C with weighted shared values and obtained the following theorem.

Theorem E. *Let f be a non-constant meromorphic function, k be a positive integer, and let $a \equiv a(z)$ be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. If*

(i) $a(z)$ has no zero (pole) which is also a zero (pole) of f or $f^{(k)}$ with the same multiplicity,

(ii) $f - a$ and $f^{(k)} - a$ share $(0, 2)$,

(iii) $2\delta_{2+k}(0, f) + (4 + k)\Theta(\infty, f) > 5 + k$,

then $f \equiv f^{(k)}$.

In 2005, Zhang [17] obtained the following result which is an improvement and complement of Theorem D.

Theorem F. *Let f be a non-constant meromorphic function, $k\ (\geq 1)$ and $l\ (\geq 0)$ be integers. Also, let $a \equiv a(z)$ be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. Suppose that $f - a$ and $f^{(k)} - a$ share $(0, l)$. Then $f \equiv f^{(k)}$ if one of the following conditions is satisfied,*

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(i) $l \geq 2$ *and*

$$(3+k)\Theta(\infty, f) + 2\delta_{2+k}(0, f) > k + 4;$$

(ii) $l = 1$ *and*

$$(4+k)\Theta(\infty, f) + 3\delta_{2+k}(0, f) > k + 6;$$

(iii) $l = 0$ *(i.e.* $f - a$ *and* $f^k - a$ *share the value* 0 *IM) and*

$$(6+2k)\Theta(\infty, f) + 5\delta_{2+k}(0, f) > 2k + 10.$$

It is natural to ask what happens if $f^{(k)}$ is replaced by a differential polynomial

$$L(f) = f^{(k)} + a_{k-1} f^{(k-1)} + \cdots + a_0 f \tag{1.1}$$

in Theorem E or F, where a_j $(j = 0, 1, \ldots, k-1)$ are small meromorphic functions of f. Corresponding to this question, we obtain the following result which improves Theorems A $\sim$ F and answers the four questions mentioned above.

Theorem 1.2. *Let* f *be a non-constant meromorphic function,* $k(\geq 1)$ *and* $l(\geq 0)$ *be integers. Also, let* $a = a(z)$ *be a small meromorphic function of* f *such that* $a(z) \not\equiv 0, \infty$. *Suppose that* $f - a$ *and* $L(f) - a$ *share* $(0, l)$. *Then* $f \equiv L(f)$ *if one of the following assumptions holds,*

(i) $l \geq 2$ *and*

$$\delta_{2+k}(0, f) + \delta_2(0, f) + 3\Theta(\infty, f) + \delta(a, f) > 4; \tag{1.2}$$

(ii) $l = 1$ *and*

$$\delta_{2+k}(0, f) + \delta_2(0, f) + \frac{1}{2}\delta_{1+k}(0, f) + \frac{k+7}{2}\Theta(\infty, f) + \delta(a, f) > \frac{k}{2} + 5; \tag{1.3}$$

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(iii) *$l = 0$ (i.e. $f - a$ and $L(f) - a$ share the value 0 IM) and*

$$(1.4)\quad \delta_{2+k}(0, f) + 2\delta_{1+k}(0, f) + \delta_2(0, f) + \Theta(0, f) + (6 + 2k)\Theta(\infty, f) + \delta(a, f) > 2k + 10.$$

Since $\delta_2(0, f) \geq \delta_{1+k}(0, f) \geq \delta_{2+k}(0, f) \geq \delta(0, f)$, we have the following corollary that improves Theorems A $\sim$ F.

Corollary 1.3. *Let f be a non-constant meromorphic function, $k(\geq 1)$ and $l(\geq 0)$ be integers, and let $a \equiv a(z)$ be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. Suppose that $f - a$ and $f^{(k)} - a$ share $(0, l)$. Then $f \equiv f^{(k)}$ if one of the following three conditions holds,*

(i) *$l \geq 2$ and*

$$2\delta_{2+k}(0, f) + 3\Theta(\infty, f) + \delta(a, f) > 4;$$

(ii) *$l = 1$ and*

$$\frac{5}{2}\delta_{2+k}(0, f) + \frac{k+7}{2}\Theta(\infty, f) + \delta(a, f) > \frac{k}{2} + 5;$$

(iii) *$l = 0$ (i.e. $f - a$ and $L(f) - a$ share the value 0 IM) and*

$$5\delta_{2+k}(0, f) + (6 + 2k)\Theta(\infty, f) + \delta(a, f) > 2k + 10.$$

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2. Some Lemmas

Lemma 2.1 ([12]). *Let f be a non-constant meromorphic function. Then*

$$N\left(r,\frac{1}{f^{(n)}}\right)\le T(r,f^{(n)})-T(r,f)+N\left(r,\frac{1}{f}\right)+S(r,f), \tag{2.1}$$

$$N\left(r,\frac{1}{f^{(n)}}\right)\le N\left(r,\frac{1}{f}\right)+n\overline{N}(r,f)+S(r,f). \tag{2.2}$$

Suppose that F and G are two non-constant meromorphic functions such that F and G share the value 1 IM. Let z_0 be a 1-point of F of order p, a 1-point of G of order q. We denote by $N_L\left(r,\frac{1}{F-1}\right)$ the counting function of those 1-points of F where $p>q$, by $N_E^{1)}\left(r,\frac{1}{F-1}\right)$ the counting function of those 1-points of F where $p=q=1$, by $N_E^{(2}\left(r,\frac{1}{F-1}\right)$ the counting function of those 1-points of F where $p=q\ge 2$; each point in these counting functions is counted only once. In the same way, we can define $N_L\left(r,\frac{1}{G-1}\right)$, $N_E^{1)}\left(r,\frac{1}{G-1}\right)$ and $N_E^{(2}\left(r,\frac{1}{G-1}\right)$ (see [14]). In particular, if F and G share 1 CM, then

$$N_L\left(r,\frac{1}{F-1}\right)=N_L\left(r,\frac{1}{G-1}\right)=0. \tag{2.3}$$

With these notations, if F and G share 1 IM, it is easy to see that

$$\begin{aligned}\overline{N}\left(r,\frac{1}{F-1}\right)&=N_E^{1)}\left(r,\frac{1}{F-1}\right)+N_L\left(r,\frac{1}{F-1}\right)\\&\quad+N_L\left(r,\frac{1}{G-1}\right)+N_E^{(2}\left(r,\frac{1}{G-1}\right)\end{aligned} \tag{2.4}$$

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$$= \overline{N}\left(r, \frac{1}{G-1}\right).$$

Lemma 2.2 ([13]). *Let*

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right), \tag{2.5}$$

where F and G are two nonconstant meromorphic functions. If F and G share 1 IM and $H \not\equiv 0$, then

$$N_E^{1)}\left(r, \frac{1}{F-1}\right) \leq N(r,H) + S(r,F) + S(r,G). \tag{2.6}$$

Lemma 2.3. *Let f be a transcendental meromorphic function, $L(f)$ be defined by (1.1). If $L(f) \not\equiv 0$, we have*

$$N\left(r, \frac{1}{L}\right) \leq T(r,L) - T(r,f) + N\left(r, \frac{1}{f}\right) + S(r,f), \tag{2.7}$$

$$N\left(r, \frac{1}{L}\right) \leq k\overline{N}(r,f) + N\left(r, \frac{1}{f}\right) + S(r,f). \tag{2.8}$$

Proof. By the first fundamental theorem and the lemma of logarithmic derivatives, we have

$$\begin{aligned} N\left(r, \frac{1}{L}\right) &= T(r,L) - m\left(r, \frac{1}{L}\right) + O(1) \\ &\leq T(r,L) - \left(m\left(r, \frac{1}{f}\right) - m\left(r, \frac{L(f)}{f}\right)\right) + O(1) \end{aligned}$$

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$$\leq T(r,L) - \left(T(r,f) - N\left(r,\frac{1}{f}\right)\right) + S(r,f)$$
$$\leq T(r,L) - T(r,f) + N\left(r,\frac{1}{f}\right) + S(r,f).$$

This proves (2.7). Since

$$\begin{aligned} T(r,L) &= m(r,L) + N(r,L) \\ &\leq m(r,f) + m\left(r,\frac{L}{f}\right) + N(r,f) + k\overline{N}(r,f) \\ &= T(r,f) + k\overline{N}(r,f) + S(r,f), \end{aligned}$$

from this and (2.7), we obtain (2.8). Lemma 2.3 is thus proved. ∎

Lemma 2.4. *Let f be a non-constant meromorphic function, $L(f)$ be defined by (1.1), and let p be a positive integer. If $L(f) \not\equiv 0$, we have*

$$N_p\left(r,\frac{1}{L}\right) \leq T(r,L) - T(r,f) + N_{p+k}\left(r,\frac{1}{f}\right) + S(r,f), \tag{2.9}$$

$$N_p\left(r,\frac{1}{L}\right) \leq k\overline{N}(r,f) + N_{p+k}\left(r,\frac{1}{f}\right) + S(r,f). \tag{2.10}$$

Proof. From (2.8), we have

$$\begin{aligned} N_p\left(r,\frac{1}{L}\right) &+ \sum_{j=p+1}^{\infty} \overline{N}_{(j}\left(r,\frac{1}{L}\right) \\ &\leq N_{p+k}\left(r,\frac{1}{f}\right) + \sum_{j=p+k+1}^{\infty} \overline{N}_{(j}\left(r,\frac{1}{f}\right) + k\overline{N}(r,f) + S(r,f), \end{aligned}$$

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then

$$N_p\left(r,\frac{1}{L}\right) \le N_{p+k}\left(r,\frac{1}{f}\right) + \sum_{j=p+k+1}^{\infty} \overline{N}_{(j}\left(r,\frac{1}{f}\right) - \sum_{j=p+1}^{\infty} \overline{N}_{(j}\left(r,\frac{1}{L}\right) + k\overline{N}(r,f) + S(r,f)$$
$$\le N_{p+k}\left(r,\frac{1}{f}\right) + k\overline{N}(r,f) + S(r,f).$$

Thus (2.10) holds. By the same arguments as above, we obtain (2.9) from (2.7). ∎

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3. Proof of Theorem 1.2

Let

$$F=\frac{L(f)}{a},\qquad G=\frac{f}{a}. \tag{3.1}$$

From the conditions of Theorem 1.2, we know that F and G share $(1,l)$ except the zeros and poles of $a(z)$. From (3.1), we have

$$T(r,F)=O\big(T(r,f)\big)+S(r,f),\quad T(r,G)\le T(r,f)+S(r,f), \tag{3.2}$$

$$\overline{N}(r,F)=\overline{N}(r,G)+S(r,f). \tag{3.3}$$

It is obvious that f is a transcendental meromorphic function. Let H be defined by (2.5). We discuss the following two cases.

Case 1. *$H\not\equiv 0$, by Lemma 2.2 we know that (2.6) holds. From (2.5) and (3.3), we have*

$$\begin{aligned} N(r,H)\le{}&\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+\overline{N}(r,G)\\ &+N_L\left(r,\frac{1}{F-1}\right)+N_L\left(r,\frac{1}{G-1}\right)+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right), \end{aligned} \tag{3.4}$$

where $N_0\left(r,\frac{1}{F'}\right)$ denotes the counting function corresponding to the zeros of F' which are not the zeros of F and $F-1$, $N_0\left(r,\frac{1}{G'}\right)$ denotes the counting function corresponding to the zeros of G' which are not the zeros of G and $G-1$. From the

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second fundamental theorem in Nevanlinna's Theory, we have

$$(3.5)\quad T(r,F)+T(r,G)\le \overline{N}\left(r,\frac{1}{F}\right)+\overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G}\right)$$
$$+\overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G-1}\right)-N_0\left(r,\frac{1}{F'}\right)-N_0\left(r,\frac{1}{G'}\right)+S(r,f).$$

Noting that F and G share 1 IM except the zeros and poles of $a(z)$, we get from (2.4),

$$\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)$$
$$=2N_E^{1)}\left(r,\frac{1}{F-1}\right)+2N_L\left(r,\frac{1}{F-1}\right)+2N_L\left(r,\frac{1}{G-1}\right)$$
$$+2N_E^{(2}\left(r,\frac{1}{G-1}\right)+S(r,f).$$

Combining with (2.6) and (3.4), we obtain

$$(3.6)\quad \overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)$$
$$\le N_{(2}\left(r,\frac{1}{F}\right)+N_{(2}\left(r,\frac{1}{G}\right)+\overline{N}(r,G)+3N_L\left(r,\frac{1}{F-1}\right)+3N_L\left(r,\frac{1}{G-1}\right)$$
$$+N_E^{1)}\left(r,\frac{1}{F-1}\right)+2N_E^{(2}\left(r,\frac{1}{G-1}\right)+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)+S(r,f).$$

We discuss the following three subcases.

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Subcase 1.1 $l \geq 2$. It is easy to see that

$$(3.7)\quad 3N_L\left(r,\frac{1}{F-1}\right)+3N_L\left(r,\frac{1}{G-1}\right) + 2N_E^{(2}\left(r,\frac{1}{G-1}\right)+N_E^{1)}\left(r,\frac{1}{F-1}\right) \leq N\left(r,\frac{1}{G-1}\right)+S(r,f).$$

From (3.6) and (3.7), we have

$$(3.8)\quad \overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right) \leq N_{(2}\left(r,\frac{1}{F}\right)+N_{(2}\left(r,\frac{1}{G}\right)+\overline{N}(r,G)+N\left(r,\frac{1}{G-1}\right) + N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)+S(r,f).$$

Substituting (3.8) into (3.5) and by using (3.3), we have

$$(3.9)\quad T(r,F)+T(r,G) \leq 3\overline{N}(r,G)+N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+N\left(r,\frac{1}{G-1}\right)+S(r,f).$$

Noting that

$$N_2\left(r,\frac{1}{F}\right)=N_2\left(r,\frac{a}{L}\right)\leq N_2\left(r,\frac{1}{L}\right)+S(r,f),$$

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we obtain from (2.9), (3.1) and (3.9) that

$$(3.10)\quad T(r,f) \le 3\overline{N}(r,f)+N_{2+k}\left(r,\frac{1}{f}\right)+N_2\left(r,\frac{1}{f}\right)-m\left(r,\frac{1}{G-1}\right)+S(r,f),$$

which contradicts the assumption (1.2) of Theorem 1.2.

Subcase 1.2 $l=1$. Noting that

$$2N_L\left(r,\frac{1}{F-1}\right)+3N_L\left(r,\frac{1}{G-1}\right)+2N_E^{(2}\left(r,\frac{1}{G-1}\right)+N_E^{1)}\left(r,\frac{1}{F-1}\right) \le N\left(r,\frac{1}{G-1}\right)+S(r,f),$$

$$\begin{aligned}
N_L\left(r,\frac{1}{F-1}\right) &\le \frac{1}{2}N\left(r,\frac{F}{F'}\right)\\
&\le \frac{1}{2}N\left(r,\frac{F'}{F}\right)+S(r,f)\\
&\le \frac{1}{2}\left(\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}(r,F)\right)+S(r,f)\\
&\le \frac{1}{2}\left(N_1\left(r,\frac{1}{F}\right)+\overline{N}(r,f)\right)+S(r,f)\\
&\le \frac{1}{2}\left(N_{1+k}\left(r,\frac{1}{f}\right)+(k+1)\overline{N}(r,f)\right)+S(r,f),
\end{aligned}$$

and by the same reasoning as in Subcase 1.1, we get

$$T(r,f) \le \frac{k+7}{2}\overline{N}(r,f)+N_{2+k}\left(r,\frac{1}{f}\right)+N_2\left(r,\frac{1}{f}\right)$$

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$$+\frac{1}{2}N_{1+k}\left(r,\frac{1}{f}\right)-m\left(r,\frac{1}{G-1}\right)+S(r,f),$$

which contradicts the assumption (1.3) of Theorem 1.2.

Subcase 1.3 $l=0$. Noting that

$$N_L\left(r,\frac{1}{F-1}\right)+2N_L\left(r,\frac{1}{G-1}\right)+2N_E^{(2}\left(r,\frac{1}{G-1}\right)+N_E^{1)}\left(r,\frac{1}{F-1}\right)$$
$$\leq N\left(r,\frac{1}{G-1}\right)+S(r,f),$$

$$2N_L\left(r,\frac{1}{F-1}\right)+N_L\left(r,\frac{1}{G-1}\right)\leq 2N\left(r,\frac{1}{F'}\right)+N\left(r,\frac{1}{G'}\right),$$

and by the same reasoning as in the Subcase 1.2, we get a contradiction.

Case 2. $H\equiv 0$. *By integration, we get from (2.5) that*

$$\frac{1}{G-1}=\frac{A}{F-1}+B, \tag{3.11}$$

where $A\ (\neq 0)$ *and* B *are constants. From (3.11) we have*

$$N(r,F)=N(r,G)=N(r,f)=S(r,f),\quad \Theta(\infty,f)=1, \tag{3.12}$$

and

$$G=\frac{(B+1)F+(A-B-1)}{BF+(A-B)},\qquad F=\frac{(B-A)G+(A-B-1)}{BG-(B+1)}. \tag{3.13}$$

We discuss the following three subcases.

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Subcase 2.1 Suppose that $B \neq 0, -1$. From (3.13) we have $\overline{N}\left(r, 1/\left(G - \frac{B+1}{B}\right)\right) = \overline{N}(r, F)$. From this and the second fundamental theorem, we have

$$\begin{aligned} T(r,f) &\leq T(r,G) + S(r,f) \\ &\leq \overline{N}(r,G) + \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}\left(r, \frac{1}{G - \frac{B+1}{B}}\right) + S(r,f) \\ &\leq \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}(r,F) + \overline{N}(r,G) + S(r,f) \\ &\leq \overline{N}\left(r, \frac{1}{f}\right) + S(r,f), \end{aligned}$$

which contradicts the assumptions of Theorem 1.2.

Subcase 2.2 Suppose that $B = 0$. From (3.13) we have

$$G = \frac{F + (A-1)}{A}, \qquad F = AG - (A-1). \tag{3.14}$$

If $A \neq 1$, from (3.14) we can obtain $\overline{N}\left(r, 1/\left(G - \frac{A-1}{A}\right)\right) = \overline{N}(r, 1/F)$. From this and the second fundamental theorem, we have

$$\begin{aligned} 2T(r,f) &\leq 2T(r,G) + S(r,f) \\ &\leq \overline{N}(r,G) + \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}\left(r, 1/\left(G - \frac{A-1}{A}\right)\right) \\ &\quad + \overline{N}\left(r, \frac{1}{G-1}\right) + S(r,f) \\ &\leq \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{G-1}\right) + S(r,f), \end{aligned}$$

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which contradicts the assumptions of Theorem 1.2. Thus $A = 1$. From (3.14) we have $F \equiv G$, then $f \equiv L$.

Subcase 2.3 Suppose that $B = -1$, from (3.13) we have

$$G = \frac{A}{-F + (A+1)}, \qquad F = \frac{(A+1)G - A}{G}. \tag{3.15}$$

If $A \neq -1$, we obtain from (3.15) that $N\left(r, 1/\left(G - \frac{A}{A+1}\right)\right) = N(r, 1/F)$. By the same reasoning discussed in Subcase 2.2, we obtain a contradiction. Hence $A = -1$. From (3.15), we get $F \cdot G \equiv 1$, that is

$$f \cdot L \equiv a^2. \tag{3.16}$$

From (3.16), we have

$$N\left(r, \frac{1}{f}\right) + N(r, f) = S(r, f), \tag{3.17}$$

and so $T\left(r, \frac{f^{(k)}}{f}\right) = S(r, f)$. From (3.17), we obtain

$$\begin{aligned} 2T\left(r, \frac{f}{a}\right) &= T\left(r, \frac{f^2}{a^2}\right) \\ &= T\left(r, \frac{a^2}{f^2}\right) + O(1) \\ &= T\left(r, \frac{L}{f}\right) + O(1) = S(r, f), \end{aligned}$$

and so $T(r, f) = S(r, f)$, this is impossible. This completes the proof of Theorem 1.2.

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4. Remarks

Let f and g be non-constant meromorphic functions, $a(z)$ be a small function of f and g, and k be a positive integer or ∞. We denote by $\overline{N}_E^{k)}(r,a)$ the counting function of common zeros of $f-a$ and $g-a$ with the same multiplicities $p \leq k$, by $\overline{N}_0^{(k+1}(r,a)$ the counting function of common zeros of $f-a$ and $g-a$ with the multiplicities $p \geq k+1$, and denote by $\overline{N}_0(r,a)$ the counting function of common zeros of $f-a$ and $g-a$; each point in these counting functions is counted only once.

Definition 4.1. Let f and g be non-constant meromorphic functions, a be a small function of f and g, and k be a positive integer or ∞. We say that f and g share "(a,k)" if $k=0$, and

$$\overline{N}\left(r,\frac{1}{f-a}\right)-\overline{N}_0(r,a)=S(r,f),$$
$$\overline{N}\left(r,\frac{1}{g-a}\right)-\overline{N}_0(r,a)=S(r,g);$$

or $k\neq 0$, and

$$\overline{N}_{k)}\left(r,\frac{1}{f-a}\right)-\overline{N}_E^{k)}(r,a)=S(r,f),$$
$$\overline{N}_{k)}\left(r,\frac{1}{g-a}\right)-\overline{N}_E^{k)}(r,a)=S(r,g),$$

$$\overline{N}_{(k+1}\left(r,\frac{1}{f-a}\right)-\overline{N}_0^{(k+1}(r,a)=S(r,f),$$
$$\overline{N}_{(k+1}\left(r,\frac{1}{g-a}\right)-\overline{N}_0^{(k+1}(r,a)=S(r,g).$$

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By the above definition and a similar argument to that used in the proof of Theorem 1.2, we conclude that Theorem 1.2 and Corollary 1.3 still hold if the condition that $f - a$ and $L(f) - a$ (or $f^{(k)} - a$) share $(0, l)$ is replaced by the assumption that $f - a$ and $L(f) - a$ (or $f^{(k)} - a$) share "$(0, l)$".

A Conjecture of
R. Brück
Ji-Long Zhang and Lian-Zhong Yang
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