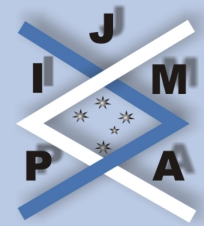


COEFFICIENT INEQUALITIES FOR CERTAIN CLASSES OF ANALYTIC AND UNIVALENT FUNCTIONS



Coefficient Inequalities

Toshio Hayami, Shigeyoshi Owa and

H.M. Srivastava

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Abstract: For functions $f(z)$ which are starlike of order α , convex of order α , and λ -spiral-like of order α in the open unit disk \mathbb{U} , some interesting sufficient conditions involving coefficient inequalities for $f(z)$ are discussed. Several (known or new) special cases and consequences of these coefficient inequalities are also considered.

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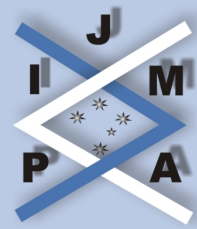
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1. Introduction, Definitions and Preliminaries

Let \mathcal{A}_0 be the class of functions $f(z)$ of the form:

$$(1.1) \quad f(z) = a_0 + a_1z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

If $f(z) \in \mathcal{A}_0$ is given by (1.1), together with the following normalization:

$$a_0 = 0 \quad \text{and} \quad a_1 = 1,$$

then we say that $f(z) \in \mathcal{A}$.

If $f(z) \in \mathcal{A}$ satisfies the following inequality:

$$(1.2) \quad \Re \left(\frac{z f'(z)}{f(z)} \right) > \alpha \quad (z \in \mathbb{U}; 0 \leq \alpha < 1),$$

then $f(z)$ is said to be starlike of order α in \mathbb{U} . We denote by $\mathcal{S}^*(\alpha)$ the subclass of \mathcal{A} consisting of functions $f(z)$ which are starlike of order α in \mathbb{U} . Similarly, we say that $f(z)$ is in the class $\mathcal{K}(\alpha)$ of convex functions of order α in \mathbb{U} if $f(z) \in \mathcal{A}$ satisfies the following inequality:

$$(1.3) \quad \Re \left(1 + \frac{z f''(z)}{f'(z)} \right) > \alpha \quad (z \in \mathbb{U}; 0 \leq \alpha < 1).$$

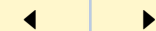
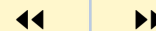
It is easily observed from (1.2) and (1.3) that (see, for details, [3])

$$f(z) \in \mathcal{K}(\alpha) \iff z f'(z) \in \mathcal{S}^*(\alpha) \quad (0 \leq \alpha < 1).$$



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As usual, in our present investigation, we write

$$\mathcal{S}^* := \mathcal{S}^*(0) \quad \text{and} \quad \mathcal{K} := \mathcal{K}(0).$$

Furthermore, we let \mathcal{B} denote the class of functions $p(z)$ of the form:

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n,$$

which are analytic in \mathbb{U} .

Each of the following lemmas will be needed in our present investigation.

Lemma 1. *A function $p(z) \in \mathcal{B}$ satisfies the following condition:*

$$\Re[p(z)] > 0 \quad (z \in \mathbb{U})$$

if and only if

$$p(z) \neq \frac{\zeta - 1}{\zeta + 1} \quad (z \in \mathbb{U}; \zeta \in \mathbb{C}; |\zeta| = 1).$$

Proof. For the sake of completeness, we choose to give a proof of Lemma 1, even though it is fairly obvious that the following bilinear (or Möbius) transformation:

$$w = \frac{z - 1}{z + 1}$$

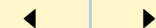
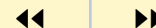
maps the unit circle $\partial\mathbb{U}$ onto the imaginary axis $\Re(w) = 0$. Indeed, for all ζ such that $|\zeta| = 1$ ($\zeta \in \mathbb{C}$), we set

$$w = \frac{\zeta - 1}{\zeta + 1} \quad (\zeta \in \mathbb{C}; |\zeta| = 1).$$



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Then

$$|\zeta| = \left| \frac{1+w}{1-w} \right| = 1,$$

which shows that

$$\Re(w) = \Re\left(\frac{\zeta-1}{\zeta+1}\right) = 0 \quad (\zeta \in \mathbb{C}; |\zeta| = 1).$$

Moreover, by noting that $p(0) = 1$ for $p(z) \in \mathcal{B}$, we know that

$$p(z) \neq \frac{\zeta-1}{\zeta+1} \quad (z \in \mathbb{U}; \zeta \in \mathbb{C}; |\zeta| = 1).$$

This evidently completes the proof of Lemma 1. □

Lemma 2. *A function $f(z) \in \mathcal{A}$ is in the class $\mathcal{S}^*(\alpha)$ if and only if*

$$(1.4) \quad 1 + \sum_{n=2}^{\infty} A_n z^{n-1} \neq 0,$$

where

$$A_n = \frac{n+1-2\alpha+(n-1)\zeta}{2-2\alpha} a_n.$$

Proof. Upon setting

$$p(z) = \frac{\frac{zf'(z)}{f(z)} - \alpha}{1-\alpha} \quad (f(z) \in \mathcal{S}^*(\alpha)),$$

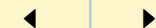
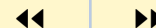
we find that

$$p(z) \in \mathcal{B} \quad \text{and} \quad \Re[p(z)] > 0 \quad (z \in \mathbb{U}).$$



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Using Lemma 1, we have

$$(1.5) \quad \frac{zf'(z) - \alpha}{1 - \alpha} \neq \frac{\zeta - 1}{\zeta + 1} \quad (z \in \mathbb{U}; \zeta \in \mathbb{C}; |\zeta| = 1),$$

which readily yields

$$(\zeta + 1)zf'(z) + (1 - 2\alpha - \zeta)f(z) \neq 0$$

$$(f(z) \in \mathcal{S}^*(\alpha); z \in \mathbb{U}; \zeta \in \mathbb{C}; |\zeta| = 1).$$

Thus we find that

$$(\zeta + 1)z + (\zeta + 1) \left(\sum_{n=2}^{\infty} na_n z^n \right) + (1 - 2\alpha - \zeta) \left(z + \sum_{n=2}^{\infty} a_n z^n \right) \neq 0$$
$$(z \in \mathbb{U}; \zeta \in \mathbb{C}; |\zeta| = 1),$$

that is, that

$$(1.6) \quad 2(1 - \alpha)z \left(1 + \sum_{n=2}^{\infty} \frac{n + 1 - 2\alpha + (n - 1)\zeta}{2(1 - \alpha)} a_n z^{n-1} \right) \neq 0$$
$$(z \in \mathbb{U}; \zeta \in \mathbb{C}; |\zeta| = 1).$$

Now, dividing both sides of (1.6) by $2(1 - \alpha)z$ ($z \neq 0$), we obtain

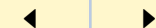
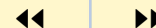
$$1 + \sum_{n=2}^{\infty} \frac{n + 1 - 2\alpha + (n - 1)\zeta}{2(1 - \alpha)} a_n z^{n-1} \neq 0$$
$$(z \in \mathbb{U}; \zeta \in \mathbb{C}; |\zeta| = 1),$$

which completes the proof of Lemma 2 (see also Remark 2 below). □



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Remark 1. It follows from the normalization conditions:

$$a_0 = 0 \quad \text{and} \quad a_1 = 1$$

that

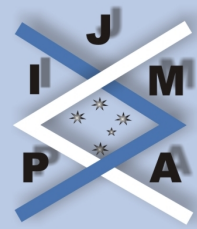
$$A_0 = \frac{1 - 2\alpha - x}{2 - 2\alpha} a_0 = 0 \quad \text{and} \quad A_1 = \frac{2 - 2\alpha}{2 - 2\alpha} a_1 = 1.$$

Remark 2. The assertion (1.4) of Lemma 2 is equivalent to

$$\frac{1}{z} \left(f(z) * \frac{z + \frac{\zeta + 2\alpha - 1}{2 - 2\alpha} z^2}{(1 - z)^2} \right) \neq 0 \quad (z \in \mathbb{U}),$$

which was given earlier by Silverman *et al.* [2]. Furthermore, in its special case when $\alpha = 0$, Lemma 2 yields a recent result of Nezhmetdinov and Ponnusamy [1] for the sufficient conditions involving the coefficients of $f(z)$ to be in the class \mathcal{S}^* .

The object of the present paper is to give some generalizations of the aforementioned result due to Nezhmetdinov and Ponnusamy [1]. We also briefly discuss several interesting corollaries and consequences of our main results.



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2. Coefficient Conditions for Functions in the Class $\mathcal{S}^*(\alpha)$

Our first result for functions $f(z)$ to be in the class $\mathcal{S}^*(\alpha)$ is contained in Theorem 1 below.

Theorem 1. *If $f(z) \in \mathcal{A}$ satisfies the following condition:*

$$(2.1) \quad \sum_{n=2}^{\infty} \left(\left| \sum_{k=1}^n \left[\sum_{j=1}^k (-1)^{k-j} (j+1-2\alpha) \binom{\beta}{k-j} a_j \right] \binom{\gamma}{n-k} \right| + \left| \sum_{k=1}^{\infty} \left[\sum_{j=1}^k (-1)^{k-j} (j-1) \binom{\beta}{k-j} a_j \right] \binom{\gamma}{n-k} \right| \right) \leq 2(1-\alpha)$$

$(0 \leq \alpha < 1; \beta \in \mathbb{R}; \gamma \in \mathbb{R}),$

then $f(z) \in \mathcal{S}^*(\alpha)$.

Proof. First of all, we note that

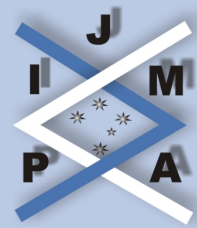
$$(1-z)^\beta \neq 0 \quad \text{and} \quad (1+z)^\gamma \neq 0 \quad (z \in \mathbb{U}; \beta \in \mathbb{R}; \gamma \in \mathbb{R}).$$

Hence, if the following inequality:

$$(2.2) \quad \left(1 + \sum_{n=2}^{\infty} A_n z^{n-1} \right) (1-z)^\beta (1+z)^\gamma \neq 0 \quad (z \in \mathbb{U}; \beta \in \mathbb{R}; \gamma \in \mathbb{R})$$

holds true, then we have

$$1 + \sum_{n=2}^{\infty} A_n z^{n-1} \neq 0,$$



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which is the relation (1.4) of Lemma 2. It is easily seen that (2.1) is equivalent to

$$(2.3) \quad \left(1 + \sum_{n=2}^{\infty} A_n z^{n-1}\right) \left(\sum_{n=0}^{\infty} (-1)^n b_n z^n\right) \left(\sum_{n=0}^{\infty} c_n z^n\right) \neq 0,$$

where, for convenience,

$$b_n := \binom{\beta}{n} \quad \text{and} \quad c_n := \binom{\gamma}{n}.$$

Considering the Cauchy product of the first two factors, (2.3) can be rewritten as follows:

$$(2.4) \quad \left(1 + \sum_{n=2}^{\infty} B_n z^{n-1}\right) \left(\sum_{n=0}^{\infty} c_n z^n\right) \neq 0,$$

where

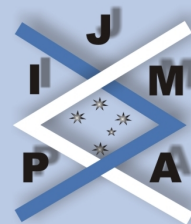
$$B_n := \sum_{j=1}^n (-1)^{n-j} A_j b_{n-j}.$$

Furthermore, by applying the same method for the Cauchy product in (2.4), we find that

$$1 + \sum_{n=2}^{\infty} \left(\sum_{k=1}^n B_k c_{n-k}\right) z^{n-1} \neq 0 \quad (z \in \mathbb{U})$$

or, equivalently, that

$$1 + \sum_{n=2}^{\infty} \left[\sum_{k=1}^n \left(\sum_{j=1}^k (-1)^{k-j} A_j b_{k-j} \right) c_{n-k} \right] z^{n-1} \neq 0 \quad (z \in \mathbb{U}).$$



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Thus, if $f(z) \in \mathcal{A}$ satisfies the following inequality:

$$\sum_{n=2}^{\infty} \left| \sum_{k=1}^n \left(\sum_{j=1}^k (-1)^{k-j} A_j b_{k-j} \right) c_{n-k} \right| \leq 1,$$

that is, if

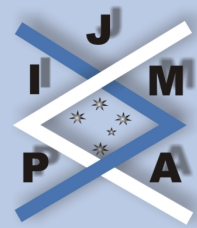
$$\begin{aligned} & \frac{1}{2(1-\alpha)} \sum_{n=2}^{\infty} \left| \sum_{k=1}^n \left(\sum_{j=1}^k (-1)^{k-j} [(j+1-2\alpha) + (j-1)\zeta] a_j b_{k-j} \right) c_{n-k} \right| \\ & \leq \frac{1}{2(1-\alpha)} \sum_{n=2}^{\infty} \left(\left| \sum_{k=1}^n \left[\sum_{j=1}^k (-1)^{k-j} (j+1-2\alpha) a_j b_{k-j} \right] c_{n-k} \right| \right. \\ & \quad \left. + |\zeta| \left| \sum_{k=1}^n \left[\sum_{j=1}^k (-1)^{k-j} (j-1) b_{k-j} a_j \right] c_{n-k} \right| \right) \\ & \leq 1 \quad (0 \leq \alpha < 1; \zeta \in \mathbb{C}; |\zeta| = 1), \end{aligned}$$

then $f(z) \in \mathcal{S}^*(\alpha)$. This completes the proof of Theorem 1. □

Setting $\alpha = 0$ in Theorem 1, we deduce the following corollary.

Corollary 1. *If $f(z) \in \mathcal{A}$ satisfies the following condition:*

$$\begin{aligned} (2.5) \quad & \sum_{n=2}^{\infty} \left(\left| \sum_{k=1}^n \left[\sum_{j=1}^k (-1)^{k-j} (j+1) \binom{\beta}{k-j} a_j \right] \binom{\gamma}{n-k} \right| \right. \\ & \left. + \left| \sum_{k=1}^n \left[\sum_{j=1}^k (-1)^{k-j} (j-1) \binom{\beta}{k-j} a_j \right] \binom{\gamma}{n-k} \right| \right) \leq 2 \\ & (\beta \in \mathbb{R}; \gamma \in \mathbb{R}), \end{aligned}$$



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then $f(z) \in \mathcal{S}^*$.

Remark 3. If, in the hypothesis (2.5) of Corollary 1, we set

$$\beta - 1 = \gamma = 0 \quad \text{or} \quad \beta = \gamma = 1 \quad \text{or} \quad \beta - 2 = \gamma = 0,$$

we arrive at the result given by Nezhmetdinov and Ponnusamy [1]. Moreover, for $\beta = \gamma = 0$ in Theorem 1, we obtain Corollary 2 below.

Corollary 2. If $f(z) \in \mathcal{A}$ satisfies the following coefficient inequality:

$$(2.6) \quad \sum_{n=2}^{\infty} (n - \alpha) |a_n| \leq 1 - \alpha \quad (0 \leq \alpha < 1),$$

then $f(z) \in \mathcal{S}^*(\alpha)$.

In particular, by putting $\alpha = 0$ in (2.6), we get the following well-known coefficient condition for the familiar class \mathcal{S}^* of starlike functions in \mathbb{U} .

Corollary 3. If $f(z) \in \mathcal{A}$ satisfies the following coefficient inequality:

$$(2.7) \quad \sum_{n=2}^{\infty} n |a_n| \leq 1,$$

then $f(z) \in \mathcal{S}^*$.

We next derive the coefficient condition for functions $f(z)$ to be in the class $\mathcal{K}(\alpha)$.



Theorem 2. If $f(z) \in \mathcal{A}$ satisfies the following condition:

$$(2.8) \quad \sum_{n=2}^{\infty} \left(\left| \sum_{k=1}^n \left[\sum_{j=1}^k (-1)^{k-j} j(j+1-2\alpha) \binom{\beta}{k-j} a_j \right] \binom{\gamma}{n-k} \right| \right. \\ \left. + \left| \sum_{k=1}^{\infty} \left[\sum_{j=1}^k (-1)^{k-j} j(j-1) \binom{\beta}{k-j} a_j \right] \binom{\gamma}{n-k} \right| \right) \leq 2(1-\alpha) \\ (0 \leq \alpha < 1; \beta \in \mathbb{R}; \gamma \in \mathbb{R}),$$

then $f(z) \in \mathcal{K}(\alpha)$.

Proof. Since $zf'(z)$ belongs to the class $\mathcal{S}^*(\alpha)$ if and only if $f(z)$ is in the class $\mathcal{K}(\alpha)$, and since

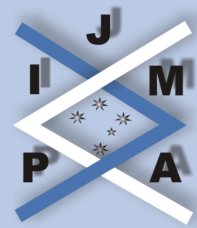
$$(2.9) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

and

$$(2.10) \quad zf'(z) = z + \sum_{n=2}^{\infty} na_n z^n,$$

upon replacing a_j in Theorem 1 by ja_j , we readily prove Theorem 2. \square

By considering some special values for the parameters α , β and γ , we can deduce the following corollaries.



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Corollary 4. If $f(z) \in \mathcal{A}$ satisfies the following condition:

$$(2.11) \quad \sum_{n=2}^{\infty} \left(\left| \sum_{k=1}^n \left[\sum_{j=1}^k (-1)^{k-j} j(j+1) (-1)^{k-j} \binom{\beta}{k-j} a_j \right] \binom{\gamma}{n-k} \right| \right. \\ \left. + \left| \sum_{k=1}^{\infty} \left[\sum_{j=1}^k (-1)^{k-j} j(j-1) \binom{\beta}{k-j} a_j \right] \binom{\gamma}{n-k} \right| \right) \\ \leq 2 \quad (\beta \in \mathbb{R}; \gamma \in \mathbb{R}),$$

then $f(z) \in \mathcal{K}$.

Corollary 5. If $f(z) \in \mathcal{A}$ satisfies the following coefficient inequality:

$$(2.12) \quad \sum_{n=2}^{\infty} n(n-\alpha)|a_n| \leq 1 - \alpha \quad (0 \leq \alpha < 1),$$

then $f(z) \in \mathcal{K}(\alpha)$.

Corollary 6. If $f(z) \in \mathcal{A}$ satisfies the following coefficient inequality:

$$(2.13) \quad \sum_{n=2}^{\infty} n^2 |a_n| \leq 1,$$

then $f(z) \in \mathcal{K}$.



3. Coefficient Conditions for Functions in the Class $\mathcal{SP}(\lambda, \alpha)$

In this section, we consider the subclass $\mathcal{SP}(\lambda, \alpha)$ of \mathcal{A} , which consists of functions $f(z) \in \mathcal{A}$ if and only if the following inequality holds true:

$$(3.1) \quad \Re \left[e^{i\lambda} \left(\frac{zf'(z)}{f(z)} - \alpha \right) \right] > 0 \quad \left(z \in \mathbb{U}; 0 \leq \alpha < 1; -\frac{\pi}{2} < \lambda < \frac{\pi}{2} \right).$$

For $f(z) \in \mathcal{SP}(\lambda, \alpha)$, we first derive Lemma 3 below.

Lemma 3. *A function $f(z) \in \mathcal{A}$ is in the class $\mathcal{SP}(\lambda, \alpha)$ if and only if*

$$(3.2) \quad 1 + \sum_{n=2}^{\infty} C_n z^{n-1} \neq 0,$$

where

$$C_n := \frac{n-1 + 2(1-\alpha)e^{-i\lambda} \cos \lambda + (n-1)\zeta}{2(1-\alpha)e^{-i\lambda} \cos \lambda} a_n.$$

Proof. Letting

$$p(z) = \frac{e^{i\lambda} \left(\frac{zf'(z)}{f(z)} - \alpha \right) - i(1-\alpha) \sin \lambda}{(1-\alpha) \cos \lambda},$$

we see that

$$p(z) \in \mathcal{B} \quad \text{and} \quad \Re[p(z)] > 0 \quad (z \in \mathbb{U}).$$

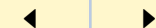
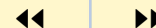
It follows from Lemma 1 that

$$(3.3) \quad \frac{e^{i\lambda} \left(\frac{zf'(z)}{f(z)} - \alpha \right) - i(1-\alpha) \sin \lambda}{(1-\alpha) \cos \lambda} \neq \frac{\zeta - 1}{\zeta + 1} \quad (z \in \mathbb{U}; \zeta \in \mathbb{C}; |\zeta| = 1).$$



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We need not consider Lemma 1 for the case when $z = 0$, because (3.3) implies that

$$p(0) \neq \frac{\zeta - 1}{\zeta + 1} \quad (\zeta \in \mathbb{C}; |\zeta| = 1).$$

It also follows from (3.3) that

$$\frac{e^{i\lambda} [zf'(z) - \alpha f(z)] - i(1 - \alpha)f(z) \sin \lambda}{(1 - \alpha) \cos \lambda} \neq \left(\frac{\zeta - 1}{\zeta + 1} \right) f(z) \\ (z \in \mathbb{U}; \zeta \in \mathbb{C}; |\zeta| = 1),$$

which readily yields

$$(\zeta + 1) \{e^{i\lambda} [zf'(z) - \alpha f(z)] - i(1 - \alpha)f(z) \sin \lambda\} \neq (\zeta - 1)(1 - \alpha)f(z) \cos \lambda \\ (z \in \mathbb{U}; \zeta \in \mathbb{C}; |\zeta| = 1)$$

or, equivalently,

$$(3.4) \quad (\zeta + 1)e^{i\lambda}zf'(z) - \alpha e^{i\lambda}f(z) - \zeta \alpha e^{i\lambda}f(z) \\ - i(1 - \alpha)f(z) \sin \lambda - i\zeta(1 - \alpha)f(z) \sin \lambda \\ \neq \zeta(1 - \alpha)f(z) \cos \lambda - (1 - \alpha)f(z) \cos \lambda \\ (z \in \mathbb{U}; \zeta \in \mathbb{C}; |\zeta| = 1).$$

We find from (3.4) that

$$(\zeta + 1)e^{i\lambda}zf'(z) - \alpha e^{i\lambda}f(z) - \zeta \alpha e^{i\lambda}f(z) - \zeta(1 - \alpha)e^{i\lambda}f(z) + (1 - \alpha)e^{-i\lambda}f(z) \neq 0 \\ (z \in \mathbb{U}; \zeta \in \mathbb{C}; |\zeta| = 1),$$

that is, that

$$(1 + \zeta)e^{i\lambda}zf'(z) + (e^{-i\lambda} - 2\alpha \cos \lambda - \zeta e^{i\lambda})f(z) \neq 0 \\ (z \in \mathbb{U}; \zeta \in \mathbb{C}; |\zeta| = 1),$$



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which, in light of (1.1) with $a_0 = a_1 - 1 = 0$, assumes the following form:

$$(\zeta + 1)e^{i\lambda} \left(z + \sum_{n=2}^{\infty} na_n z^n \right) + (e^{-i\lambda} - \zeta e^{i\lambda} - 2\alpha \cos \lambda) \left(z + \sum_{n=2}^{\infty} a_n z^n \right) \neq 0$$

$$(z \in \mathbb{U}; \zeta \in \mathbb{C}; |\zeta| = 1)$$

or, equivalently,

$$(3.5) \quad 2(1-\alpha)z \cos \lambda \left(1 + \sum_{n=2}^{\infty} \frac{n + e^{-2i\lambda} - 2\alpha e^{-i\lambda} \cos \lambda + (n-1)\zeta}{2(1-\alpha)e^{-i\lambda} \cos \lambda} a_n z^{n-1} \right) \neq 0$$

$$(z \in \mathbb{U}; \zeta \in \mathbb{C}; |\zeta| = 1).$$

Finally, upon dividing both sides of (3.5) by

$$2(1-\alpha)z \cos \lambda \neq 0$$

and noting that

$$e^{-2i\lambda} = -1 + 2e^{-i\lambda} \cos \lambda,$$

we obtain

$$1 + \sum_{n=2}^{\infty} \frac{n-1 + 2(1-\alpha)e^{-i\lambda} \cos \lambda + (n-1)\zeta}{2(1-\alpha)e^{-i\lambda} \cos \lambda} a_n \neq 0$$

$$\left(0 \leq \alpha < 1; -\frac{\pi}{2} < \lambda < \frac{\pi}{2}; \zeta \in \mathbb{C}; |\zeta| = 1 \right),$$

which completes the proof of Lemma 3 (see also the proof of a known result [1, Theorem 3.1]). \square

By applying Lemma 3, we now prove Theorem 3 below.



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Theorem 3. If $f(z) \in \mathcal{A}$ satisfies the following condition:

$$(3.6) \quad \sum_{n=2}^{\infty} \left(\left| \sum_{k=1}^n \left[\sum_{j=1}^k (-1)^{k-j} [j - \alpha + (1 - \alpha)e^{-2i\lambda}] \binom{\beta}{k-j} a_j \right] \binom{\gamma}{n-k} \right| \right. \\ \left. + \left| \sum_{k=1}^{\infty} \left[\sum_{j=1}^k (-1)^{k-j} (j-1) \binom{\beta}{k-j} a_j \right] \binom{\gamma}{n-k} \right| \right) \leq 2(1 - \alpha) \cos \lambda \\ \left(0 \leq \alpha < 1; -\frac{\pi}{2} < \lambda < \frac{\pi}{2}; \beta \in \mathbb{R}; \gamma \in \mathbb{R} \right),$$

then $f(z) \in \mathcal{SP}(\lambda, \alpha)$.

Proof. Applying the same method as in the proof of Theorem 1, we see that $f(z)$ is in the class $\mathcal{SP}(\lambda, \alpha)$ if

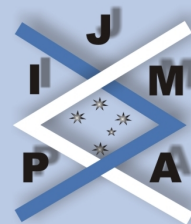
$$(3.7) \quad \sum_{n=2}^{\infty} \left| \sum_{k=1}^n \left(\sum_{j=1}^k (-1)^{k-j} C_j b_{k-j} \right) c_{n-k} \right| \leq 1$$

where, as before,

$$b_n := \binom{\beta}{n} \quad \text{and} \quad c_n := \binom{\gamma}{n},$$

the coefficients C_n being given as in Lemma 3. It follows from the inequality (3.7) that

$$\frac{1}{|2(1 - \alpha)e^{-i\lambda} \cos \lambda|} \cdot \sum_{n=2}^{\infty} \left| \sum_{k=1}^n \left[\sum_{j=1}^k \left((-1)^{k-j} (j-1 + 2(1 - \alpha)e^{-i\lambda} \cos \lambda) + \zeta(j-1) \right) a_j b_{k-j} \right] c_{n-k} \right|$$



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$$\leq \frac{1}{2(1-\alpha)\cos\lambda} \cdot \sum_{n=2}^{\infty} \left(\left| \sum_{k=1}^n \left[\sum_{j=1}^k (-1)^{k-j} (j-\alpha + (1-\alpha)(-1+2e^{-i\lambda}\cos\lambda)) b_{k-j} a_j \right] c_{n-k} \right| + |\zeta| \left| \sum_{k=1}^n \left[\sum_{j=1}^k (-1)^{k-j} (j-1) b_{k-j} a_j \right] c_{n-k} \right| \right)$$

$$(3.8) \leq 1 \quad \left(0 \leq \alpha < 1; -\frac{\pi}{2} < \lambda < \frac{\pi}{2}; \zeta \in \mathbb{C}; |\zeta| = 1 \right),$$

which implies that, if $f(z)$ satisfies the hypothesis (3.6) of Theorem 3, then $f(z) \in \mathcal{SP}(\lambda, \alpha)$. This completes the proof of Theorem 3. \square

In its special case when

$$\beta - 1 = \gamma = 0 \quad \text{or} \quad \beta = \gamma = 1 \quad \text{or} \quad \beta - 2 = \gamma = 0,$$

Theorem 3 would immediately yield the following corollary.

Corollary 7 (cf. [1]). *If $f(z) \in \mathcal{A}$ satisfies any one of the following conditions:*

$$(3.9) \quad \sum_{n=2}^{\infty} \left(\left| [n-\alpha + (1-\alpha)e^{-2i\lambda}](a_n - a_{n-1}) + a_{n-1} \right| + |(n-1)(a_n - a_{n-1}) + a_{n-1}| \right) \leq 2(1-\alpha)\cos\lambda \quad \left(0 \leq \alpha < 1; -\frac{\pi}{2} < \lambda < \frac{\pi}{2} \right)$$

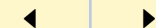
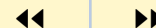
or

$$(3.10) \quad \sum_{n=2}^{\infty} \left(\left| [n-\alpha + (1-\alpha)e^{-2i\lambda}](a_n - a_{n-2}) + 2a_{n-2} \right| \right)$$



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$$\begin{aligned}
 &+ |(n-1)(a_n - a_{n-2}) + 2a_{n-2}| \\
 &\leq 2(1-\alpha) \cos \lambda \quad \left(0 \leq \alpha < 1; -\frac{\pi}{2} < \lambda < \frac{\pi}{2}\right)
 \end{aligned}$$

or

$$\begin{aligned}
 (3.11) \quad &\sum_{n=2}^{\infty} \left(|[n-1-\alpha+(1-\alpha)e^{-2i\lambda}](a_n - 2a_{n-1} + a_{n-2}) + a_n - a_{n-2}| \right. \\
 &\quad \left. + |(n-2)(a_n - 2a_{n-1} + a_{n-2}) + a_n - a_{n-2}| \right) \\
 &\leq 2(1-\alpha) \cos \lambda \quad \left(0 \leq \alpha < 1; -\frac{\pi}{2} < \lambda < \frac{\pi}{2}\right),
 \end{aligned}$$

then $f(z) \in \mathcal{SP}(\lambda, \alpha)$.

Remark 4. For $\lambda = 0$, Theorem 3 implies Theorem 1. Furthermore, by setting $\alpha = 0$ in Theorem 3, we arrive at the following sufficient condition for functions $f(z) \in \mathcal{A}$ to be in the class $\mathcal{SP}(\lambda)$.

Corollary 8. If $f(z) \in \mathcal{A}$ satisfies the following condition:

$$\begin{aligned}
 (3.12) \quad &\sum_{n=2}^{\infty} \left(\left| \sum_{k=1}^n \left[\sum_{j=1}^k (-1)^{k-j} (j + e^{-2i\lambda}) \binom{\beta}{k-j} a_j \right] \binom{\gamma}{n-k} \right| \right. \\
 &\quad \left. + \left| \sum_{k=1}^{\infty} \left[\sum_{j=1}^k (-1)^{k-j} (j-1) \binom{\beta}{k-j} a_j \right] \binom{\gamma}{n-k} \right| \right) \\
 &\leq 2 \cos \lambda \quad \left(0 \leq \alpha < 1; \beta \in \mathbb{R}; \gamma \in \mathbb{R}; -\frac{\pi}{2} < \lambda < \frac{\pi}{2}\right),
 \end{aligned}$$

then

$$f(z) \in \mathcal{SP}(\lambda) := \mathcal{SP}(\lambda, 0).$$

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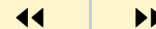
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