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ON STRONGLY GENERALIZED PREINVEX FUNCTIONS

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ABSTRACT. In this paper, we define and introduce some new concepts of strongly φ-preinvex (φ-invex) functions and strongly $\varphi\eta$-monotone operators. We establish some new relationships among various concepts of φ-preinvex (φ-invex) functions. As special cases, one can obtain various new and known results from our results. Results obtained in this paper can be viewed as refinement and improvement of previously known results.

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1. INTRODUCTION

In recent years, several extensions and generalizations have been considered for classical convexity. A significant generalization of convex functions is that of invex functions introduced by Hanson [1]. Hanson's initial result inspired a great deal of subsequent work which has greatly expanded the role and applications of invexity in nonlinear optimization and other branches of pure and applied sciences. Weir and Mond [9] have studied the basic properties of the preinvex functions and their role in optimization. It is well-known that the preinvex functions and invex sets may not be convex functions and convex sets. In recent years, these concepts and results have been investigated extensively in [2], [4], [6] – [9].

Equally important is another generalization of the convex function called the φ-convex function which was introduced and studied by Noor [3]. In particular, these generalizations of the convex functions are quite different and do not contain each other. In this paper, we introduce and consider another class of nonconvex functions, which include these generalizations as special cases. This class of nonconvex functions is called the strongly φ-preinvex (φ-invex) functions. Several new concepts of $\varphi\eta$-monotonicity are introduced. We establish the relationship between these classes and derive some new results. As special cases, one can obtain some

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new and correct versions of known results. Results obtained in this paper present a refinement and improvement of previously known results.

2. PRELIMINARIES

Let K be a nonempty closed set in a real Hilbert space H. We denote by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ the inner product and norm respectively. Let $F : K \to H$ and $\eta(\cdot,\cdot) : K \times K \to \mathbb{R}$ be continuous functions. Let $\varphi : K \longrightarrow \mathbb{R}$ be a continuous function.

Definition 2.1 ([5])**.** Let $u \in K$. Then the set K is said to be φ-invex at u with respect to $\eta(\cdot,\cdot)$ and $\varphi(\cdot)$, if

$$u + te^{i\varphi}\eta(v,u) \in K, \quad \forall u, v \in K, \quad t \in [0,1].$$

K is said to be an φ-invex set with respect to η and φ, if K is φ-invex at each $u \in K$. The φ-invex set K is also called a $\varphi\eta$-connected set. Note that the convex set with $\varphi = 0$ and $\eta(v,u) = v - u$ is an φ-invex set, but the converse is not true. For example, the set $K = R - \left(-\frac{1}{2}, \frac{1}{2}\right)$ is an φ-invex set with respect to η and $\varphi = 0$, where

$$\eta(v,u) = \begin{cases} v - u, & \text{for} \quad v > 0, u > 0 \quad \text{or} \quad v < 0, u < 0 \\ u - v, & \text{for} \quad v < 0, u > 0 \quad \text{or} \quad v < 0, u < 0. \end{cases}$$

It is clear that K is not a convex set.

Remark 2.1.
(i) If $\varphi = 0$, then the set K is called the invex (η-connected) set, see [2], [4], [9].
(ii) If $\eta(v,u) = v - u$, then the set K is called the φ-convex set, see Noor [3].
(iii) If $\varphi = 0$ and $\eta(v,u) = v - u$, then the set K is called the convex set.

From now onward K is a nonempty closed φ-invex set in H with respect to φ and $\eta(\cdot,\cdot)$, unless otherwise specified.

Definition 2.2. The function F on the φ-invex set K is said to be strongly φ-preinvex with respect to η and φ, if there exists a constant $\mu > 0$ such that

$$F(u + te^{i\varphi}\eta(v,u)) \le (1-t)F(u) + tF(v) - \mu t(1-t)\|\eta(v,u)\|^2, \quad \forall u, v \in K, \quad t \in [0,1].$$

The function F is said to be strongly φ-preconcave if and only if $-F$ is φ-preinvex. Note that every strongly convex function is a strongly φ-preinvex function, but the converse is not true.

Definition 2.3. The function F on the φ-invex set K is called strongly quasi φ-preinvex with respect to φ and η, if there exists a constant $\mu > 0$ such that

$$F(u + te^{i\varphi}\eta(v,u)) \le \max\{F(u), F(v)\} - \mu t(1-t)\|\eta(v,u)\|^2, \quad \forall u, v \in K, \quad t \in [0,1].$$

Definition 2.4. The function F on the φ-invex set K is said to be logarithmic φ-preinvex with respect to φ and η, if there exists a constant $\mu > 0$ such that

$$F(u + te^{i\varphi}\eta(v,u)) \le (F(u))^{1-t}(F(v))^t - \mu t(1-t)\|\eta(v,u)\|^2, \quad u, v \in K, \quad t \in [0,1],$$

where $F(\cdot) > 0$.

From the above definitions, we have

$$\begin{aligned} F(u + te^{i\varphi}\eta(v,u)) &\le (F(u))^{1-t}(F(v))^t - \mu t(1-t)\|\eta(v,u)\|^2 \\ &\le (1-t)F(u) + tF(v) - \mu t(1-t)\|\eta(v,u)\|^2 \\ &\le \max\{F(u), F(v)\} - \mu t(1-t)\|\eta(v,u)\|^2 \\ &< \max\{F(u), F(v)\} - \mu t(1-t)\|\eta(v,u)\|^2. \end{aligned}$$

For $t = 1$, Definitions 2.2 and 2.4 reduce to the following, which is mainly due to Noor and Noor [5].

Condition A.

$$F(u + e^{i\varphi}\eta(v,u)) \leq F(v), \quad \forall u, v \in K,$$

which plays an important part in studying the properties of the φ-preinvex (φ-invex) functions.

For $\varphi = 0$, Condition A reduces to the following for preinvex functions

Condition B.

$$F(u + \eta(v,u)) \leq F(v), \quad \forall u, v \in K.$$

For the applications of Condition B, see [2, 4, 7, 8].

Definition 2.5. The differentiable function F on the φ-invex set K is said to be a strongly φ-invex function with respect to φ and $\eta(\cdot,\cdot)$, if there exists a constant $\mu > 0$ such that

$$F(v) - F(u) \geq \langle F'_\varphi(u), \eta(v,u)\rangle + \mu\|\eta(v,u)\|^2, \quad \forall u, v \in K,$$

where $F'_\varphi(u)$ is the differential of F at u in the direction of $v - u \in K$. Note that for $\varphi = 0$, we obtain the original definition of strongly invexity.

It is well known that the concepts of preinvex and invex functions play a significant role in mathematical programming and optimization theory, see [1] – [9] and the references therein.

Remark 2.2. Note that for $\mu = 0$, Definitions 2.2 – 2.5 reduce to the ones in [5].

Definition 2.6. An operator $T : K \longrightarrow H$ is said to be:

(i) strongly η-monotone, iff there exists a constant $\alpha > 0$ such that

$$\langle Tu, \eta(v,u)\rangle + \langle Tv, \eta(u,v)\rangle \leq -\alpha\{\|\eta(v,u)\|^2 + \|\eta(u,v)\|^2\}, \quad \forall u, v \in K.$$

(ii) η-monotone, iff

$$\langle Tu, \eta(v,u)\rangle + \langle Tv, \eta(u,v)\rangle \leq 0, \quad \forall u, v \in K.$$

(iii) strongly η-pseudomonotone, iff there exists a constant $\nu > 0$ such that

$$\langle Tu, \eta(v,u)\rangle + \nu\|\eta(v,u)\|^2 \geq 0 \quad \Longrightarrow \quad -\langle Tv, \eta(u,v)\rangle \geq 0, \quad \forall u, v \in K.$$

(iv) strongly relaxed η-pseudomonotone, iff, there exists a constant $\mu > 0$ such that

$$\langle Tu, \eta(v,u)\rangle \geq 0 \quad \Longrightarrow \quad -\langle Tv, \eta(u,v)\rangle + \mu\|\eta(u,v)\|^2 \geq 0, \quad \forall u, v \in K.$$

(v) strictly η-monotone, iff,

$$\langle Tu, \eta(v,u)\rangle + \langle Tv, \eta(u,v)\rangle < 0, \quad \forall u, v \in K.$$

(vi) η-pseudomonotone, iff,

$$\langle Tu, \eta(v,u)\rangle \geq 0 \quad \Longrightarrow \quad \langle Tv, \eta(u,v)\rangle \leq 0, \quad \forall u, v \in K.$$

(vii) quasi η-monotone, iff,

$$\langle Tu, \eta(v,u)\rangle > 0 \quad \Longrightarrow \quad \langle Tv, \eta(u,v)\rangle \leq 0, \quad \forall u, v \in K.$$

(viii) strictly η-pseudomonotone, iff,

$$\langle Tu, \eta(v,u)\rangle \geq 0 \quad \Longrightarrow \quad \langle Tv, \eta(u,v)\rangle < 0, \quad \forall u, v \in K.$$

Note for $\varphi = 0, \quad \forall u, v \in K$, the φ-invex set K becomes an invex set. In this case, Definition 2.7 is exactly the same as in [4, 5, 6, 8]. In addition, if $\varphi = 0$ and $\eta(v,u) = v - u$, then the φ-invex set K is the convex set K. This clearly shows that Definition 2.7 is more general than and includes the ones in [4, 5, 6, 7, 8] as special cases.

Definition 2.7. A differentiable function F on an φ-invex set K is said to be strongly pseudo $\varphi\eta$-invex function, iff, there exists a constant $\mu > 0$ such that

$$\langle F'_\varphi(u), \eta(v,u)\rangle + \mu\|\eta(u,v)\|^2 \geq 0 \quad \Longrightarrow \quad F(v) - F(u) \geq 0, \quad \forall u, v \in K.$$

Definition 2.8. A differentiable function F on K is said to be strongly quasi φ-invex, if there exists a constant $\mu > 0$ such that

$$F(v) \leq F(u) \quad \Longrightarrow \quad \langle F'_\varphi(u), \eta(v,u)\rangle + \mu\|\eta(v,u)\|^2 \leq 0, \quad \forall u, v \in K.$$

Definition 2.9. The function F on the set K is said to be pseudo α-invex, if

$$\langle F'_\varphi(u), \eta(v,u)\rangle \geq 0, \quad \Longrightarrow \quad F(v) \geq F(u), \quad \forall u, v \in K.$$

Definition 2.10. A differentiable function F on the K is said to be quasi φ-invex, if such that

$$F(v) \leq F(u) \quad \Longrightarrow \quad \langle F'_\varphi(u), \eta(v,u)\rangle \leq 0, \quad \forall u, v \in K.$$

Note that if $\varphi = 0$, then the φ-invex set K is exactly the invex set K and consequently Definitions 2.8 – 2.10 are exactly the same as in [6, 7]. In particular, if $\varphi = 0$ and $\eta(v,u) = -\eta(v,u), \forall u, v \in K$, that is, the function $\eta(\cdot,\cdot)$ is skew-symmetric, then Definitions 2.7 – 2.10 reduce to the ones in [6, 7, 8]. This shows that the concepts introduced in this paper represent an improvement of the previously known ones. All the concepts defined above play important and fundamental parts in mathematical programming and optimization problems.

We also need the following assumption regarding the function $\eta(\cdot,\cdot)$, and φ, which is due to Noor and Noor [5].

Condition C. *Let* $\eta(\cdot,\cdot) : K \times K \longrightarrow H$ *and* φ *satisfy the assumptions*

$$\eta(u, u + te^{i\varphi}\eta(v,u)) = -t\eta(v,u)$$
$$\eta(v, u + te^{i\varphi}\eta(v,u)) = (1-t)\eta(v,u), \quad \forall u, v \in K, \quad t \in [0,1].$$

Clearly for $t = 0$, *we have* $\eta(u,v) = 0$, *if and only if* $u = v, \forall u, v \in K$. *One can easily show* [7, 8] *that* $\eta(u + te^{i\varphi}\eta(v,u), u) = t\eta(v,u), \quad \forall u, v \in K$.

Note that for $\varphi = 0$, Condition C collapses to the following condition, which is due to Mohan and Neogy [2].

Condition D. *Let* $\eta(\cdot,\cdot) : K \times K \longrightarrow H$ *satisfy the assumptions*

$$\eta(u, u + t\eta(v,u)) = -t\eta(v,u),$$
$$\eta(v, u + t\eta(v,u)) = (1-t)\eta(v,u), \quad \forall u, v \in K, \quad t \in [0,1].$$

For applications of Condition D, see [2], [4] – [8].

3. MAIN RESULTS

In this section, we consider some basic properties of strong φ-preinvex functions and strongly φ-invex functions on the invex set K.

Theorem 3.1. *Let* F *be a differentiable function on the* φ*-invex set* K *in* H *and let Condition C hold. Then the function* F *is a strongly* φ*-preinvex function if and only if* F *is a strongly* φ*-invex function.*

Proof. Let F be a strongly φ-preinvex function on the invex set K. Then there exists a function $\eta(\cdot,\cdot) : K \times K \longrightarrow \mathbb{R}$ and a constant $\mu > 0$ such that

$$F(u + te^{i\varphi}\eta(v,u)) \leq (1-t)F(u) + tF(v) - t(1-t)\mu\|\eta(v,u)\|^2, \quad \forall u, v \in K,$$

which can be written as

$$F(v) - F(u) \geq \frac{F(u + te^{i\varphi}\eta(v,u)) - F(u)}{t} + (1-t)\mu\|\eta(v,u)\|^2.$$

Letting $t \longrightarrow 0$ in the above inequality, we have

$$F(v) - F(u) \geq \langle F'_\varphi(u), \eta(v,u)\rangle + \mu\|\eta(v,u)\|^2,$$

which implies that F is a strongly φ-invex function.

Conversely, let F be a strongly φ-invex function on the φ-invex function K. Then $\forall u, v \in K, t \in [0,1], \quad v_t = u + te^{i\varphi}\eta(v,u) \in K$ and using Condition C, we have

$$\begin{aligned} F(v) - F(u + te^{i\varphi}\eta(v,u)) &\geq \langle F'_\varphi(u + te^{i\varphi}\eta(v,u)), \eta(v, u + te^{i\varphi}\eta(v,u))\rangle \\ &\qquad + \mu\|\eta(v, u + te^{i\varphi}\eta(v,u))\|^2 \\ &= (1-t)\langle F'_\varphi(u + te^{i\varphi}\eta(v,u)), \eta(v,u)\rangle \\ &\qquad + \mu(1-t)^2\|\eta(v,u)\|^2. \end{aligned} \tag{3.1}$$

In a similar way, we have

$$\begin{aligned} F(u) - F(u + te^{i\varphi}\eta(v,u)) &\geq \langle F'_\varphi(u + te^{i\varphi}\eta(v,u)), \eta(u, u + te^{i\varphi}\eta(v,u)) \\ &\qquad + \mu\|\eta(u, u + te^{i\varphi}\eta(v,u))\| \\ &= -t\langle F'_\varphi(u + te^{i\varphi}\eta(v,u)), \eta(v,u))\rangle + t^2\|\eta(v,u)\|^2. \end{aligned} \tag{3.2}$$

Multiplying (3.1) by t and (3.2) by $(1-t)$ and adding the resultant, we have

$$F(u + te^{i\varphi}\eta(v,u)) \leq (1-t)F(u) + tF(v) - \mu t(1-t)\|\eta(v,u)\|^2,$$

showing that F is a strongly φ-preinvex function. □

Theorem 3.2. *Let F be differntiable on the φ-invex set K. Let Condition A and Condition C hold. Then F is a strongly φ-invex function if and only if its differential F'_φ is strongly $\varphi\eta$-monotone.*

Proof. Let F be a strongly φ-invex function on the φ-invex set K. Then

$$F(v) - F(u) \geq \langle F'_\varphi(u), \eta(v,u)\rangle + \mu\|\eta(v,u)\|^2, \quad \forall u, v \in K. \tag{3.3}$$

Changing the role of u and v in (3.3), we have

$$F(u) - F(v) \geq \langle F'_\varphi(v), \eta(u,v)\rangle + \mu\|\eta(u,v)\|^2, \quad \forall u, v \in K. \tag{3.4}$$

Adding (3.3) and (3.4), we have

$$\langle F'_\varphi(u), \eta(v,u)\rangle + \langle F'_\varphi(v), \eta(u,v)\rangle \leq -\mu\{\|\eta(v,u)\|^2 + \|\eta(u,v)\|^2\}, \tag{3.5}$$

which shows that F'_φ is strongly $\varphi\eta$-monotone.

Conversely, let F'_φ be strongly $\varphi\eta$-monotone. From (3.5), we have

$$\langle F'_\varphi(v), \eta(u,v)\rangle \leq \langle F'_\varphi(u), \eta(v,u)\rangle - \mu\{\|\eta(v,u)\|^2 + \|\eta(u,v)\|^2\}. \tag{3.6}$$

Since K is an φ-invex set, $\forall u, v \in K, t \in [0,1] \quad v_t = u + te^{i\varphi}\eta(v,u) \in K$. Taking $v = v_t$ in (3.6) and using Condition C, we have

$$\begin{aligned} \langle F'_\varphi(v_t), \eta(u, u + te^{i\varphi}\eta(v,u)\rangle &\leq \langle F'_\varphi(u), \eta(u + te^{i\varphi}\eta(v,u), u)\rangle - \mu\{\|\eta(u + te^{i\varphi}\eta(v,u), u)\|^2 \\ &\qquad + \|\eta(u, u + te^{i\varphi}\eta(v,u)\|^2\} \\ &= -t\langle F'_\varphi(u), \eta(v,u)\rangle - 2t^2\mu\|\eta(v,u)\|^2, \end{aligned}$$

which implies that

$$\langle F'_\varphi(v_t), \eta(v,u)\rangle \geq \langle F'_\varphi(u), \eta(v,u)\rangle + 2\mu t\|\eta(v,u)\|^2. \tag{3.7}$$

Let $g(t) = F(u + te^{i\varphi}\eta(v,u))$. Then from (3.7), we have

$$\begin{aligned} g'(t) &= \langle F'_\varphi(u + te^{i\varphi}\eta(v,u)), \eta(v,u)\rangle \\ &\geq \langle F'_\varphi(u), \eta(v,u)\rangle + 2\mu t\|\eta(v,u)\|^2. \end{aligned} \tag{3.8}$$

Integrating (3.8) between 0 and 1, we have

$$g(1) - g(0) \geq \langle F'_\varphi(u), \eta(v,u)\rangle + \mu\|\eta(v,u)\|^2,$$

that is,

$$F(u + e^{i\varphi}\eta(v,u)) - F(u) \geq \langle F'(u), \eta(v,u)\rangle + \mu\|\eta(v,u)\|^2.$$

By using Condition A, we have

$$F(v) - F(u) \geq \langle F'_\varphi(u), \eta(v,u)\rangle + \mu\|\eta(v,u)\|^2,$$

which shows that F is a strongly φ-invex function on the invex set K. □

From Theorem 3.1 and Theorem 3.2, we have:
strongly φ-preinvex functions $F \implies$ strongly φ-invex functions $F \implies$ strongly $\varphi\eta$-monotonicity of the differential F'_φ and conversely if Conditions A and C hold.

For $\mu = 0$, Theorems 3.1 and 3.2 reduce to the following results, which are mainly due to Noor and Noor [5].

Theorem 3.3. *Let F be a differentiable function on the φ-invex set K in H and let Condition C hold. Then the function F is a φ-preinvex function if and only if F is a φ-invex function.*

Theorem 3.4. *Let F be differentiable function and let Condition C hold. Then the function F is φ-preinvex (invex) function if and only if its differential F'_φ is $\varphi\eta$-monotone.*

We now give a necessary condition for strongly $\varphi\eta$-pseudo-invex function.

Theorem 3.5. *Let F'_φ be strongly relaxed $\varphi\eta$-pseudomonotone and Conditions A and C hold. Then F is strongly $\varphi\eta$-pseudo-invex function.*

Proof. Let F'_φ be strongly relaxed $\varphi\eta$-pseudomonotone. Then, $\forall u, v \in K$,

$$\langle F'_\varphi(u), \eta(v,u)\rangle \geq 0,$$

implies that

$$-\langle F'_\varphi(v), \eta(u,v)\rangle \geq \alpha\|\eta(u,v)\|^2. \tag{3.9}$$

Since K is an φ-invex set, $\forall u, v \in K, t \in [0,1], v_t = u + te^{i\varphi}\eta(v,u) \in K$. Taking $v = v_t$ in (3.9) and using Condition C, we have

$$\langle F'_\varphi(u + te^{i\varphi}\eta(v,u)), \eta(v,u)\rangle \geq t\alpha\|\eta(v,u)\|^2. \tag{3.10}$$

Let

$$g(t) = F(u + te^{i\varphi}\eta(v,u)), \quad \forall u, v \in K, t \in [0,1].$$

Then, using (3.10), we have

$$g'(t) = \langle F'_\varphi(u + te^{i\varphi}\eta(v,u)), \eta(v,u)\rangle \geq t\alpha\|\eta(v,u)\|^2.$$

Integrating the above relation between 0 and 1, we have

$$g(1) - g(0) \geq \frac{\alpha}{2}\|\eta(v,u)\|^2,$$

that is,

$$F(u + e^{i\varphi}\eta(v,u)) - F(u) \geq \frac{\alpha}{2}\|\eta(v,u)\|^2,$$

which implies, using Condition A,

$$F(v) - F(u) \geq \frac{\alpha}{2}\|\eta(v,u)\|^2,$$

showing that F is strongly $\varphi\eta$-pseudo-invex function. □

As special cases of Theorem 3.5, we have the following:

Theorem 3.6. *Let the differential $F'_\varphi(u)$ of a function $F(u)$ on the φ-invex set K be $\varphi\eta$-pseudomonotone. If Conditions A and C hold, then F is a pseudo $\varphi\eta$-invex function.*

Theorem 3.7. *Let the differential $F'_\varphi(u)$ of a function $F(u)$ on the invex set K be strongly η-pseudomonotone. If Conditions A and C hold, then F is a strongly pseudo η-invex function.*

Theorem 3.8. *Let the differential $F'_\varphi(u)$ of a function $F(u)$ on the invex set K be strongly η-pseudomonotone. If Conditions B and D hold, then F is a strongly pseudo η-invex function.*

Theorem 3.9. *Let the differential $F'_\varphi(u)$ of a function $F(u)$ on the invex set K be η-pseudomonotone. If Conditions B and D hold, then F is a pseudo invex function.*

Theorem 3.10. *Let the differential $F'_\varphi(u)$ of a differentiable φ-preinvex function $F(u)$ be Lipschitz continuous on the φ-invex set K with a constant $\beta > 0$. If Condition A holds, then*

$$F(v) - F(u) \leq \langle F'_\varphi(u), \eta(v,u)\rangle + \frac{\beta}{2}\|\eta(v,u)\|^2, \quad \forall u, v \in K.$$

Proof. $\forall u, v \in K, t \in [0,1], \quad u + te^{i\varphi}\eta(v,u) \in K,$ since K is an φ-invex set. Now we consider the function

$$\varphi(t) = F(u + te^{i\varphi}\eta(v,u)) - F(u) - t\langle F'_\varphi(u), \eta(v,u)\rangle.$$

from which it follows that $\varphi(0) = 0$ and

$$\varphi'(t) = \langle F'_\varphi(u + te^{i\varphi}\eta(v,u)), \eta(v,u)\rangle - \langle F'_\varphi(u), \eta(v,u)\rangle. \tag{3.11}$$

Integrating (3.10) between 0 and 1, we have

$$\begin{aligned}
\varphi(1) &= F(u + e^{i\varphi}\eta(v,u)) - F(u) - \langle F'_\varphi(u), \eta(v,u)\rangle \\
&\leq \int_0^1 |\varphi'(t)| dt \\
&= \int_0^1 \left|\langle F'_\varphi(u + te^{i\varphi}\eta(v,u)), \eta(v,u)\rangle - \langle F'_\varphi(u), \eta(v,u)\rangle\right| dt \\
&\leq \beta \int_0^1 t\|\eta(v,u)\|^2 dt = \frac{\beta}{2}\|\eta(v,u)\|^2,
\end{aligned}$$

which implies that

$$F(u + e^{i\varphi}\eta(v,u)) - F(u) \leq \langle F'_\varphi(u), \eta(v,u)\rangle + \frac{\beta}{2}\|\eta(v,u)\|^2. \tag{3.12}$$

from which, using Condition A, we obtain

$$F(v) - F(u) \leq \langle F'_\varphi(u), \eta(v,u)\rangle + \frac{\beta}{2}\|\eta(v,u)\|^2.$$

□

Remark 3.11. For $\eta(v,u) = v - u$ and $\alpha(v,u) = 1$, the α-invex set K becomes a convex set and consequently Theorem 3.10 reduces to the well known result in convexity.

Definition 3.1. The function F is said to be sharply strongly pseudo φ-preinvex, if there exists a constant $\mu > 0$ such that

$$\langle F'_\varphi(u), \eta(v,u)\rangle \geq 0 \\ \implies F(v) \geq F(v + te^{i\varphi}\eta(v,u)) + \mu t(1-t)\|\eta(v,u)\|^2, \quad \forall u, v \in K, \quad t \in [0,1].$$

Theorem 3.12. *Let F be a sharply strong pseudo φ-preinvex function on K with a constant $\mu > 0$. Then*

$$-\langle F'_\varphi(v), \eta(v,u)\rangle \geq \mu\|\eta(v,u)\|^2, \quad \forall u, v \in K.$$

Proof. Let F be a sharply strongly pseudo φ-preinvex function on K. Then

$$F(v) \geq F(v + te^{i\varphi}\eta(v,u)) + \mu t(1-t)\|\eta(v,u)\|^2, \forall u, v \in K, \quad t \in [0,1].$$

from which we have

$$\frac{F(v + te^{i\varphi}\eta(v,u)) - F(v)}{t} + \mu(1-t)\|\eta(v,u)\|^2 \leq 0.$$

Taking the limit in the above inequality, as $t \longrightarrow 0,$ we have

$$-\langle F'_\varphi(v), \eta(v,u)\rangle \geq \mu\|\eta(v,u)\|^2,$$

the required result. □

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