journal of inequalities in pure and applied mathematics
http://jipam.vu.edu.au
issn: 1443-5756
Volume 8 (2007), Issue 2, Article 35, 9 pp.

A SIMULTANEOUS SYSTEM OF FUNCTIONAL INEQUALITIES AND MAPPINGS WHICH ARE WEAKLY OF A CONSTANT SIGN

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Received 17 October, 2006; accepted 07 June, 2007
Communicated by Zs. Pales

ABSTRACT. It is shown that, under some algebraic conditions on fixed reals $\alpha_1, \alpha_2, \dots, \alpha_{n+1}$ and vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{n+1} \in \mathbb{R}^n$, every continuous at a point function $f : \mathbb{R}^n \to \mathbb{R}$ satisfying the simultaneous system of inequalities

$$f(\mathbf{x} + \mathbf{a}_i) \leq \alpha_i + f(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n, i = 1, 2, \dots, n+1,$$

has to be of the form $f(\mathbf{x}) = \mathbf{p} \cdot \mathbf{x} + f(\mathbf{0})$, $\mathbf{x} \in \mathbb{R}^n$, with uniquely determined $\mathbf{p} \in \mathbb{R}^n$. For mappings with values in a Banach space which are weakly of a constant sign, a counterpart of this result is given.

Key words and phrases: Linear functional inequalities, Kronecker's theorem, Mappings of weakly constant sign .

2000 *Mathematics Subject Classification.* Primary 39B72; Secondary 26D05.

1. INTRODUCTION

In this paper we consider the simultaneous system of functional inequalities

$$f(\mathbf{x} + \mathbf{a}_i) \leq \alpha_i + f(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^n, \quad i = 1, 2, \dots, n+1, \tag{1.1}$$

where $n \in \mathbb{N}$, $\alpha_1, \alpha_2, \dots, \alpha_{n+1} \in \mathbb{R}$, $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{n+1} \in \mathbb{R}^n$ are fixed and $f : \mathbb{R}^n \to \mathbb{R}$ is an unknown function.

In Section 3, assuming Kronecker's type conditions on the vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{n+1}$ and a inequality involving some determinants depending on these vectors and scalars $\alpha_1, \alpha_2, \dots, \alpha_{n+1}$, we show (Theorem 3.1) that for every continuous at least at one point function f satisfying (1.1) there exists a unique vector $\mathbf{p} \in \mathbb{R}^n$ such that

$$f(\mathbf{x}) = \mathbf{p} \cdot \mathbf{x} + f(\mathbf{0}), \qquad \mathbf{x} \in \mathbb{R}^n.$$

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This result seems to be a little surprising in the context of an obvious fact that, in general, the accompanying simultaneous system of functional equations has a lot of very regular but nonlinear solutions, even "depending on an arbitrary function" (cf. M. Kuczma [4], [2] or [3], where the full construction of the solution in a special one dimensional case is given). Theorem 3.1 generalizes a suitable result of [2], where the one dimensional case

$$f(x+a) \le \alpha + f(x), \qquad f(x+b) \le \beta + f(x), \qquad x \in \mathbb{R},$$

is considered. Let us mention that in the case when $\alpha = \beta = 0$, Montel [5] considered the accompanying simultaneous system of equations.

In Section 4 we define a mapping to be weakly of a constant sign. Using this notion and a total system of linear functionals, we present a counterpart of Theorem 3.1 for functions with values in a Banach space.

2. KRONECKER'S THEOREM AND A LEMMA

The symbols $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are reserved, respectively, for the set of natural, integer, rational and real numbers.

We begin this section by recalling the following:

Theorem 2.1 (Kronecker (cf. [1, p. 382]))**.** *If the reals $\upsilon_1, \upsilon_2, \dots, \upsilon_n, 1$ are linearly independent over the field $\mathbb{Q}$, the numbers $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}$ are arbitrary, and N and ϵ are positive, then there are the integers $p_1, p_2, \dots, p_n$, and $m > N$, such that $|m\upsilon_i - p_i - \alpha_i| < \epsilon$ for each $i = 1, 2, \dots, n$.*

As an immediate consequence of this theorem we obtain the following

Corollary 2.2. *Let $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ be the standard base of the real linear space $\mathbb{R}^n$. If reals $\upsilon_1, \upsilon_2, \dots, \upsilon_n, 1$ are linearly independent over the field $\mathbb{Q}$, and $\mathbf{v} := (\upsilon_1, \upsilon_2, \dots, \upsilon_n)$, then the set*

$$\{m\mathbf{v} + p_1\mathbf{e}_1 + p_2\mathbf{e}_2 + \dots + p_n\mathbf{e}_n : m \in \mathbb{N},\ p_1, p_2, \dots, p_n \in \mathbb{Z}\}$$

is dense in $\mathbb{R}^n$.

In sequel we need a more special result which guarantees that the set of all linear combinations of the elements of the set $B = \{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{n+1}\}$ with natural coefficients is dense in $\mathbb{R}^n$.

Lemma 2.3. *Let $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$ be an arbitrary base of linear space $\mathbb{R}^n$ over $\mathbb{R}$. Suppose that $\upsilon_1, \upsilon_2, \dots, \upsilon_n \in \mathbb{R}$ are negative and the system of numbers $\upsilon_1, \upsilon_2, \dots, \upsilon_n, 1$ is linearly independent over the field $\mathbb{Q}$. If*

$$\mathbf{a}_{n+1} = \upsilon_1\mathbf{a}_1 + \upsilon_2\mathbf{a}_2 + \dots + \upsilon_n\mathbf{a}_n,$$

then the set

$$A := \left\{ \sum_{i=1}^{n+1} m^{(i)}\mathbf{a}_i : m^{(i)} \in \mathbb{N}, i = 1, 2, \dots, n+1 \right\}$$

is dense in $\mathbb{R}^n$.

Proof. Let $N > 0$ be fixed. Take $\mathbf{x} \in \mathbb{R}^n$. Then there exists a unique system of numbers $x_1, x_2, \dots, x_n \in \mathbb{R}$ such that

$$\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_n\mathbf{a}_n.$$

By Kronecker's theorem, for every $k \in \mathbb{N}$ there exist $m_k > N$ and $p_{ik} \in \mathbb{Z}$ such that

$$|m_k\upsilon_i + p_{ik} - x_i| = |m_k\upsilon_i - (-p_{ik}) - x_i| < \frac{1}{k}, \qquad i = 1, 2, \dots, n,$$

whence

$$\lim_{k\to\infty}(m_k\upsilon_i + p_{ik}) = x_i, \qquad i = 1, 2, \ldots, n.$$

Since $m_k \in \mathbb{N}$, $\upsilon_i < 0$ for $i = 1, 2, \ldots, n$, it follows that $p_{ik} \in \mathbb{N}$ for k large enough (in the opposite case we would have $\lim_{k\to\infty}(m_k\upsilon_i + p_{ik}) = -\infty$). It follows that, for k large enough, setting $m_k^{(n+1)} := m_k$ and $m_k^{(i)} := p_{ik}$, we get

$$m_k^{(i)} \in \mathbb{N}, \qquad i = 1, 2, \ldots, n+1.$$

Put

$$\mathbf{t}_k := \sum_{i=1}^{n}\left(m_k^{(n+1)}\upsilon_i + m_k^{(i)}\right)\mathbf{a}_i, \qquad k \in \mathbb{N}.$$

By the definition of $\mathbf{a}_{n+1}$ we hence get

$$\mathbf{t}_k = \sum_{i=1}^{n+1} m_k^{(i)}\mathbf{a}_i, \qquad k \in \mathbb{N},$$

whence $\mathbf{t}_k \in A$ for k sufficiently large. Moreover

$$\begin{aligned}\lim_{k\to\infty}\mathbf{t}_k &= \lim_{k\to\infty}\sum_{i=1}^{n}\left(m_k^{(n+1)}\upsilon_i + m_k^{(i)}\right)\mathbf{a}_i\\ &= \sum_{i=1}^{n}\lim_{k\to\infty}\left(m_k^{(n+1)}\upsilon_i + m_k^{(i)}\right)\mathbf{a}_i\\ &= \sum_{i=1}^{n} x_i\mathbf{a}_i = \mathbf{x}.\end{aligned}$$

This completes the proof of the density of A in $\mathbb{R}^n$. □

3. MAIN RESULT

Theorem 3.1. *Let $n \in \mathbb{N}$, $\mathbf{a}_i = (a_{i,1}, a_{i,2}, \ldots, a_{i,n}) \in \mathbb{R}^n$, $i = 1, 2, \ldots, n$, negative reals $\upsilon_1, \upsilon_2, \ldots, \upsilon_n$ and $\alpha_1, \alpha_2, \ldots, \alpha_{n+1} \in \mathbb{R}$ be fixed and such that:*

(i) *$\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ form a base of the linear space $\mathbb{R}^n$ over $\mathbb{R}$;*
(ii) *$\upsilon_1, \upsilon_2, \ldots, \upsilon_n, 1$ are linearly independent over $\mathbb{Q}$,*
(iii) *$\mathbf{a}_{n+1} = \upsilon_1\mathbf{a}_1 + \upsilon_2\mathbf{a}_2 + \cdots + \upsilon_n\mathbf{a}_n$,*
(iv)

$$(-1)^n\left(\operatorname{sgn}\mathbf{W}^{(n)}\right)\begin{vmatrix}\alpha_1 & \alpha_2 & \cdots & \alpha_n & \alpha_{n+1}\\ a_{1,1} & a_{2,1} & \cdots & a_{n,1} & a_{n+1,1}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ a_{1,n} & a_{2,n} & \cdots & a_{n,n} & a_{n+1,n}\end{vmatrix} \le 0,$$

where

$$\mathbf{W}^{(n)} := \begin{vmatrix} a_{1,1} & a_{2,1} & \cdots & a_{n,1}\\ a_{1,2} & a_{2,2} & \cdots & a_{n,2}\\ \cdots & \cdots & \cdots & \cdots\\ a_{1,n} & a_{2,n} & \cdots & a_{n,n}\end{vmatrix}.$$

If a continuous at least at one point function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfies the simultaneous system of functional inequalities

$$f(\mathbf{x} + \mathbf{a}_i) \leq \alpha_i + f(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^n,\ i = 1, 2, \dots, n+1, \tag{3.1}$$

then there exists a unique $\mathbf{p} \in \mathbb{R}^n$ such that

$$f(\mathbf{x}) = \mathbf{p} \cdot \mathbf{x} + f(\mathbf{0}), \qquad \mathbf{x} \in \mathbb{R}^n.$$

Moreover,

$$\mathbf{p} = \frac{1}{\mathbf{W}^{(n)}} \left[p_1^{(n)}, p_2^{(n)}, \dots, p_n^{(n)}\right],$$

where

$$p_i^{(n)} := \begin{vmatrix} a_{1,1} & a_{2,1} & \cdots & a_{n,1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1,i-1} & a_{2,i-1} & \cdots & a_{n,i-1} \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ a_{1,i+1} & a_{2,i+1} & \cdots & a_{n,i+1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1,n} & a_{2,n} & \cdots & a_{n,n} \end{vmatrix}, \qquad i = 1, 2, \dots, n.$$

Proof. Suppose that the function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfies system (3.1). Using an easy induction argument, one can show that f satisfies the inequality

$$f\left(\sum_{i=1}^{n+1} m^{(i)} \mathbf{a}_i + \mathbf{x}\right) \leq \sum_{i=1}^{n+1} m^{(i)} \alpha_i + f(\mathbf{x}), \tag{3.2}$$

for all $m^{(i)} \in \mathbb{N}$, $i = 1, 2, \dots, n+1$ and all $\mathbf{x} \in \mathbb{R}^n$. By Lemma 2.3, the set

$$A := \left\{\sum_{i=1}^{n+1} m^{(i)} \mathbf{a}_i : m^{(i)} \in \mathbb{N}, \quad i = 1, 2, \dots, n+1\right\}$$

is dense in $\mathbb{R}^n$. Thus there exist the sequences of positive integers $(m_k^{(i)})_{k \in \mathbb{N}}$ for $i = 1, 2, \dots, n+1$ such that

$$\begin{aligned} \lim_{k\to\infty} \sum_{i=1}^{n+1} m_k^{(i)} \mathbf{a}_i &= \lim_{k\to\infty} \left(\sum_{i=1}^{n+1} m_k^{(i)} a_{i,1}, \sum_{i=1}^{n+1} m_k^{(i)} a_{i,2}, \dots, \sum_{i=1}^{n+1} m_k^{(i)} a_{i,n}\right) \\ &= (0, 0, \dots, 0), \end{aligned}$$

and, obviously,

$$\lim_{k\to\infty} m_k^{(i)} = \infty, \qquad i = 1, 2, \dots, n+1.$$

It follows that, for each $i = 1, 2, \dots, n$, the limit $\lim_{k\to\infty} \frac{m_k^{(i)}}{m_k^{(n+1)}}$ exists, and

$$\sum_{i=1}^{n+1} \left(\lim_{k\to\infty} \frac{m_k^{(i)}}{m_k^{(n+1)}} a_{i,j}\right) = \lim_{k\to\infty} \frac{\sum\limits_{i=1}^{n+1} m_k^{(i)} a_{i,j}}{m_k^{(n+1)}} = 0, \qquad j = 1, 2, \dots, n.$$

Consequently,

$$\lim_{k\to\infty} \frac{m_k^{(i)}}{m_k^{(n+1)}} = \frac{\mathbf{W}_i^{(n)}}{\mathbf{W}^{(n)}}, \qquad i = 1, 2, \dots, n, \tag{3.3}$$

where

$$\mathbf{W}_i^{(n)} := \begin{vmatrix} a_{1,1} & \cdots & a_{i-1,1} & -a_{n+1,1} & a_{i+1,1} & \cdots & a_{n,1} \\ a_{1,2} & \cdots & a_{i-1,2} & -a_{n+1,2} & a_{i+1,2} & \cdots & a_{n,2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{1,n} & \cdots & a_{i-1,n} & -a_{n+1,n} & a_{i+1,n} & \cdots & a_{n,n} \end{vmatrix}, \qquad i = 1, 2, \dots, n.$$

Let $\mathbf{x}_0$ be a point of the continuity of f. From inequality (3.2), we get

$$\frac{f\left(\sum_{i=1}^{n+1} m_k^{(i)} \mathbf{a}_i + \mathbf{x}_0\right)}{m_k^{(n+1)}} \leq \sum_{i=1}^{n+1} \frac{m_k^{(i)}}{m_k^{(n+1)}} \alpha_i + \frac{f(\mathbf{x}_0)}{m_k^{(n+1)}}.$$

Letting here $k \to \infty$ and applying (3.3), we obtain

$$0 \leq \sum_{i=1}^{n} \frac{\mathbf{W}_i^{(n)}}{\mathbf{W}^{(n)}} \alpha_i + \alpha_{n+1}. \tag{3.4}$$

Setting

$$\overline{\mathbf{W}_i^{(n)}} := - \begin{vmatrix} a_{1,1} & \cdots & a_{i-1,1} & a_{i+1,1} & \cdots & a_{n,1} & a_{n+1,1} \\ a_{1,2} & \cdots & a_{i-1,2} & a_{i+1,2} & \cdots & a_{n,2} & a_{n+1,2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{1,n} & \cdots & a_{i-1,n} & a_{i+1,n} & \cdots & a_{n,n} & a_{n+1,n} \end{vmatrix}, \qquad i = 1, 2, \dots, n,$$

we can write inequality (3.4) in the form

$$0 \leq \operatorname{sgn} \mathbf{W}^{(n)} \left(\sum_{i=1}^{n} (-1)^{n-i+1} \overline{\mathbf{W}_i^{(n)}} \alpha_i + \mathbf{W}^{(n)} \alpha_{n+1} \right),$$

whence, by the Laplace expansion of a determinant,

$$0 \leq (-1)^n \operatorname{sgn} \mathbf{W}^{(n)} \begin{vmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n & \alpha_{n+1} \\ a_{1,1} & a_{2,1} & \cdots & a_{n,1} & a_{n+1,1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{1,n} & a_{2,n} & \cdots & a_{n,n} & a_{n+1,n} \end{vmatrix}.$$

Hence, taking into account condition (iv), we infer that

$$\begin{vmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n & \alpha_{n+1} \\ a_{1,1} & a_{2,1} & \cdots & a_{n,1} & a_{n+1,1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{1,n} & a_{2,n} & \cdots & a_{n,n} & a_{n+1,n} \end{vmatrix} = 0.$$

Now, by Laplace's expansion theorem,

$$\alpha_{n+1} = (-1)^{n+1} \frac{\sum_{i=1}^{n} (-1)^{1-i} \overline{\mathbf{W}_i^{(n)}} \alpha_i}{\mathbf{W}^{(n)}},$$

whence by (3.2),

$$f\left(\sum_{i=1}^{n+1} m^{(i)}\mathbf{a}_i+\mathbf{x}\right)$$
$$\leq \sum_{i=1}^{n} m^{(i)}\alpha_i + m^{(n+1)}(-1)^{n+1}\frac{\sum_{i=1}^{n}(-1)^{1-i}\overline{\mathbf{W}_i^{(n)}}\alpha_i}{\mathbf{W}^{(n)}}+f(\mathbf{x})$$
$$=\frac{1}{\mathbf{W}^{(n)}}\left[\sum_{i=1}^{n} m^{(i)}\mathbf{W}^{(n)}\alpha_i+m^{(n+1)}\sum_{i=1}^{n}(-1)^{n+2-i}\overline{\mathbf{W}_i^{(n)}}\alpha_i\right]+f(\mathbf{x}),$$

for all $m^{(i)}\in\mathbb{N}$, and $x\in\mathbb{R}^n$.

Applying Laplace's theorem for the ith columne of $\mathbf{W}^{(n)}$ and for the last columne of $\overline{\mathbf{W}_i^{(n)}}$, we have

$$\sum_{i=1}^{n} m^{(i)}\mathbf{W}^{(n)}\alpha_i+m^{(n+1)}\sum_{i=1}^{n}(-1)^{n+2-i}\overline{\mathbf{W}_i^{(n)}}\alpha_i$$
$$=\sum_{i=1}^{n} m^{(i)}\alpha_i\sum_{j=1}^{n}(-1)^{i+j}a_{i,j}\mathbf{W}_{ij}^{(n)}$$
$$+m^{(n+1)}\left(\sum_{i=1}^{n}(-1)^{n-i}a_{i,j}\alpha_i\left(\sum_{j=1}^{n}(-1)^{n+j}a_{n+1,j}\overline{\mathbf{W}_{ij}^{(n)}}\right)\right),$$

where $\mathbf{W}_{ij}^{(n)}$ is obtained from $\mathbf{W}_i^{(n)}$ by deleting the jth row and ith column, and $\overline{\mathbf{W}_{ij}^{(n)}}$ is obtained from $\overline{\mathbf{W}_i^{(n)}}$ by deleting the jth row and the last column. Since

$$\mathbf{W}_{ij}^{(n)}=\overline{\mathbf{W}_{ij}^{(n)}},\qquad i,j=1,2,\dots,n,$$

applying Fubini's theorem for sums, we have

$$\sum_{i=1}^{n} m^{(i)}\mathbf{W}^{(n)}\alpha_i+m^{(n+1)}\sum_{i=1}^{n}(-1)^{n+2-i}\overline{\mathbf{W}_i^{(n)}}\alpha_i$$
$$=\sum_{j=1}^{n}\sum_{k=1}^{n} m^{(k)}a_{k,j}\left((-1)^{k+j}\mathbf{W}_{kj}^{(n)}\alpha_k\right)+m^{(n+1)}a_{n+1,j}\sum_{i=1}^{n}(-1)^{j-i}\mathbf{W}_{ij}^{(n)}\alpha_i.$$

Adding and subtracting the term $\sum_{i=1}^{n}(-1)^{j-i}\mathbf{W}_{ij}^{(n)}\alpha_i$ in the sum over k gives

$$\sum_{i=1}^{n} m^{(i)}\mathbf{W}^{(n)}\alpha_i+m^{(n+1)}\sum_{i=1}^{n}(-1)^{n+2-i}\overline{\mathbf{W}_i^{(n)}}\alpha_i$$
$$=\sum_{j=1}^{n}\left(\sum_{k=1}^{n} m^{(k)}a_{k,j}+m^{(n+1)}a_{n+1,j}\right)\left(\sum_{i=1}^{n}(-1)^{j-i}\mathbf{W}_{ij}^{(n)}\alpha_i\right)$$
$$+\sum_{j=1}^{n}\left(\sum_{k=1}^{n} m^{(k)}a_{k,j}\left(\sum_{i=1}^{n}(-1)^{j-i+1}\mathbf{W}_{ij}^{(n)}\alpha_i+(-1)^{k+j}\mathbf{W}_{kj}^{(n)}\alpha_k\right)\right).$$

Let us note that

$$\sum_{j=1}^{n}\left(\sum_{k=1}^{n} m^{(k)} a_{k,j}\left(\sum_{i=1}^{n}(-1)^{j-i+1}\mathbf{W}_{ij}^{(n)}\alpha_i+(-1)^{k+j}\mathbf{W}_{kj}^{(n)}\alpha_k\right)\right)=0.$$

Indeed, we have

$$\begin{aligned}\sum_{j=1}^{n}&\left(\sum_{k=1}^{n} m^{(k)} a_{k,j}\left(\sum_{i=1}^{n}(-1)^{j-i+1}\mathbf{W}_{ij}^{(n)}\alpha_i+(-1)^{k+j}\mathbf{W}_{kj}^{(n)}\alpha_k\right)\right)\\ &=\sum_{j=1}^{n}\sum_{k=1}^{n} m^{(k)} a_{k,j}\left(\sum_{i\in\{1,2,\dots,n\}-k}(-1)^{j-i+1}\mathbf{W}_{ij}^{(n)}\alpha_i\right)\\ &=\sum_{k=1}^{n} m^{(k)}\left(\sum_{i\in\{1,2,\dots,n\}-k}\alpha_i\left(\sum_{j=1}^{n}(-1)^{j-i+1}a_{k,j}\mathbf{W}_{ij}^{(n)}\right)\right)\end{aligned}$$

and

$$\sum_{j=1}^{n}(-1)^{j-i+1}a_{k,j}\mathbf{W}_{ij}^{(n)}$$

is a determinant with two equal columns. Thus we have shown that f satisfies the inequality

$$f\left(\sum_{i=1}^{n+1} m^{(i)}\mathbf{a}_i+\mathbf{x}\right)\le\sum_{j=1}^{n}\left(\left(\frac{1}{\mathbf{W}^n}\sum_{i=1}^{n}(-1)^{j-i}\mathbf{W}_{ij}^{(n)}\alpha_i\right)\left(\sum_{i=1}^{n+1} m^{(i)}a_{ij}\right)\right)+f(\mathbf{x}),$$

for all $m^{(i)}\in\mathbb{N}$, and $x\in\mathbb{R}^n$, which can be written in the form

$$f(\mathbf{t}+\mathbf{x})\le\mathbf{p}\cdot\mathbf{t}+f(\mathbf{x}),\qquad \mathbf{t}\in A,\mathbf{x}\in\mathbb{R}^n. \tag{3.5}$$

Now take an arbitrary $\mathbf{x}\in\mathbb{R}^n$. By the density of A there is a sequence $(\mathbf{t}_n)$ such that

$$\mathbf{t}_n\in A\ (n\in\mathbf{N}),\qquad \lim_{n\to\infty}\mathbf{t}_n=\mathbf{x}_0-\mathbf{x}.$$

From (3.5) we have

$$f(\mathbf{t}_n+\mathbf{x})\le\mathbf{p}\cdot\mathbf{t}_n+f(\mathbf{x}),\qquad n\in\mathbf{N}.$$

Letting here $n\to\infty$, and making use of the continuity of f at $\mathbf{x}_0$, we obtain

$$f(\mathbf{x}_0)\le\mathbf{p}\cdot(\mathbf{x}_0-\mathbf{x})+f(\mathbf{x}),\qquad \mathbf{x}\in\mathbb{R}^n.$$

To prove the converse inequality, note that replacing $\mathbf{x}$ by $\mathbf{x}-\mathbf{t}$ in (3.5) we get

$$f(\mathbf{x})\le\mathbf{p}\cdot\mathbf{t}+f(\mathbf{x}-\mathbf{t}),\qquad \mathbf{t}\in A,\mathbf{x}\in\mathbb{R}^n.$$

Taking a sequence $(\mathbf{t}_n)$ such that

$$\mathbf{t}_n\in A\ (n\in\mathbf{N}),\qquad \lim_{n\to\infty}\mathbf{t}_n=\mathbf{x}-\mathbf{x}_0,$$

(which, by the density of A, exists) we hence get

$$f(\mathbf{x})\le\mathbf{p}\cdot\mathbf{t}_n+f(\mathbf{x}-\mathbf{t}_n),\qquad n\in\mathbf{N}.$$

Letting here $n\to\infty$, and again making use of the continuity of f at $\mathbf{x}_0$, we obtain the inequality

$$f(\mathbf{x})\le\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_0)+f(\mathbf{x}_0),\qquad \mathbf{x}\in\mathbb{R}^n.$$

Thus

$$f(\mathbf{x})=\mathbf{p}\cdot\mathbf{x}+(f(\mathbf{x}_0)-\mathbf{p}\cdot\mathbf{x}_0),\qquad \mathbf{x}\in\mathbb{R}^n.$$

Since $f(\mathbf{0})=f(\mathbf{x}_0)-\mathbf{p}\cdot\mathbf{x}_0$, the proof is completed. □

4. APPLICATION FOR MAPPINGS WHICH ARE WEAKLY OF A CONSTANT SIGN

We start this section with the following.

Definition 4.1. Let X be an arbitrary nonempty set, Y - an arbitrary real linear topological space, Y^*- the conjugate space of Y, and $T \subset Y^*$.

(i) We say that a mapping $G : X \to Y$ is T-*weakly of a constant sign* if for each functional $\phi \in T$ either $\phi \circ G$ is nonpositive or $\phi \circ G$ is nonnegative.

(ii) The mappings $G_1 : X \to Y$ and $G_2 : X \to Y$ are said to be T-*weakly of the same sign* if $(\phi \circ G_1) \cdot (\phi \circ G_2) \geq 0$ for every functional $\phi \in T$.

Now applying Theorem 3.1 we prove:

Theorem 4.1. *Let Y be a real Banach space and $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\} \subset \mathbb{R}^n$, $\mathbf{a}_i = (a_{i,1}, a_{i,2}, \dots, a_{i,n})$, $i = 1, 2, \dots, n$ be a base of the real linear space $\mathbb{R}^n$. Suppose that reals $\upsilon_1, \upsilon_2, \dots, \upsilon_n$ are negative and $\upsilon_1, \upsilon_2, \dots, \upsilon_n, 1$ are linearly independent over the field $\mathbb{Q}$. Let $\mathbf{a}_{n+1} = \upsilon_1\mathbf{a}_1 + \upsilon_2\mathbf{a}_2 + \cdots + \upsilon_n\mathbf{a}_n$.*

If a mapping $F : \mathbb{R}^n \to Y$ is continuous at least at one point and there exist a total system $T \subset Y^*$ and the vectors $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_{n+1} \in Y$ such that

(i) *the mappings*

$$\mathbb{R}^n \ni \mathbf{x} \to F(\mathbf{x} + \mathbf{a}_i) - F(\mathbf{x}) - \mathbf{y}_i, \qquad i \in \{1, 2, \dots, n+1\}, \tag{*}$$

are T-weakly of a constant and the same sign;

(ii) *for every $\phi \in T$,*

$$(-1)^n \left(\operatorname{sgn} \mathbf{W}^{(n)}\right) \begin{vmatrix} \phi(\mathbf{y}_1) & \phi(\mathbf{y}_2) & \cdots & \phi(\mathbf{y}_n) & \phi(\mathbf{y}_{n+1}) \\ a_{1,1} & a_{2,1} & \cdots & a_{n,1} & a_{n+1,1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{1,n} & a_{2,n} & \cdots & a_{n,n} & a_{n+1,n} \end{vmatrix} \leq 0,$$

then there exists a unique linear continuous mapping $L : \mathbb{R}^n \to Y$ such that

$$F(\mathbf{x}) = L(\mathbf{x}) + F(\mathbf{0}), \quad \mathbf{x} \in \mathbb{R}^n. \tag{4.1}$$

Proof. Assume that $F : \mathbb{R}^n \to Y$ satisfies assumptions (i) and (ii). By Definition 4.1, we have either, for all $\phi \in T$ and $i = 1, 2, \dots, n+1$,

$$\phi\left(F(\mathbf{x} + \mathbf{a}_i) - F(\mathbf{x}) - \mathbf{y}_i\right) \leq 0, \quad \mathbf{x} \in \mathbb{R}^n,$$

or, for all $\phi \in T$ and $i = 1, 2, \dots, n+1$,

$$\phi\left(F(\mathbf{x} + \mathbf{a}_i) - F(\mathbf{x}) - \mathbf{y}_i\right) \geq 0, \quad \mathbf{x} \in \mathbb{R}^n.$$

Without any loss of generality we may confine the consideration to the first case. From the linearity of ϕ we obtain the simultaneous system of inequalities

$$\phi\left(F(\mathbf{x} + \mathbf{a}_i)\right) \leq \phi\left(F(\mathbf{x})\right) + \phi\left(\mathbf{y}_i\right), \quad \mathbf{x} \in \mathbb{R}^n; \quad \mathbf{y}_i \in Y, i = 1, 2, \dots, n+1; \quad \phi \in T.$$

Let us fix $\phi \in T$. Setting $f := \phi \circ F$ and $\alpha_i := \phi\left(\mathbf{y}_i\right)$, we hence get

$$f(\mathbf{x} + \mathbf{a}_i) \leq \alpha_i + f(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^n, \quad i = 1, 2, \dots, n+1,$$

i.e. f satisfies system (3.1). Since all the required assumptions of Theorem 3.1 are satisfied, there exists a vector $\mathbf{p}$, such that

$$f(\mathbf{x}) = \mathbf{p} \cdot \mathbf{x} + f(\mathbf{0}), \qquad \mathbf{x} \in \mathbb{R}^n.$$

Hence, by the definition of f,

$$\phi\left(F(\mathbf{x}) - F(\mathbf{0})\right) = \mathbf{p} \cdot \mathbf{x}, \qquad \mathbf{x} \in \mathbb{R}^n.$$

It follows that for every $\phi \in T$, there exists $\mathbf{p} := \mathbf{p}_\phi$, such that

$$\phi\left(F(\mathbf{x}) - F(\mathbf{0})\right) = \mathbf{p}_\phi \cdot \mathbf{x}, \qquad \mathbf{x} \in \mathbb{R}^n.$$

Setting $L := F - F(\mathbf{0})$, we hence get

$$\begin{aligned}\phi\left(L(\mathbf{x}+\mathbf{y})\right) &= \phi\left(F(\mathbf{x}+\mathbf{y}) - F(\mathbf{0})\right)\\ &= \mathbf{p}_\phi \cdot (\mathbf{x}+\mathbf{y})\\ &= \mathbf{p}_\phi \cdot \mathbf{x} + \mathbf{p}_\phi \cdot \mathbf{y}\\ &= \phi\left(F(\mathbf{x}) - F(\mathbf{0})\right) + \phi\left(F(\mathbf{y}) - F(\mathbf{0})\right)\\ &= \phi\left(L(\mathbf{x})\right) + \phi\left(L(\mathbf{y})\right),\end{aligned}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, whence

$$\phi\left(L(\mathbf{x}+\mathbf{y}) - L(\mathbf{x}) - L(\mathbf{y})\right) = 0, \qquad \mathbf{x}, \mathbf{y} \in \mathbb{R}^n,$$

for every functional $\phi \in T$. Since T is total set of functionals (cf. [6]),

$$L(\mathbf{x}+\mathbf{y}) - L(\mathbf{x}) - L(\mathbf{y}) = 0, \qquad \mathbf{x}, \mathbf{y} \in \mathbb{R}^n,$$

that is, L is an additive mapping. Now the continuity of F at least at one point implies that $L := F - F(\mathbf{0})$ is continuous and, consequently, a linear map. The proof is completed. □

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