

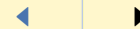
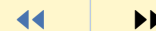
# PYTHAGOREAN PARAMETERS AND NORMAL STRUCTURE IN BANACH SPACES



**Pythagorean Parameters**  
Hongwei Jiao and Bijun Pang  
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*Key words:* Uniform non-squareness; Normal structure.

*Abstract:* Recently, Gao introduced some quadratic parameters, such as  $E_\epsilon(X)$  and  $f_\epsilon(X)$ . In this paper, we obtain some sufficient conditions for normal structure in terms of Gao's parameters, improving some known results.

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## 1. Introduction

There are several parameters and constants which are defined on the unit sphere or the unit ball of a Banach space. These parameters and constants, such as the James and von Neumann-Jordan constants, have been proved to be very useful in the descriptions of the geometric structure of Banach spaces.

Based on a Pythagorean theorem, Gao introduced some quadratic parameters recently [1, 2]. Using these parameters, one can easily distinguish several important classes of spaces such as uniform non-squareness or spaces having normal structure.

In this paper, we are going to continue the study in Gao's parameters. Moreover, we obtain some sufficient conditions for a Banach space to have normal structure.

Let  $X$  be a Banach space and  $X^*$  its dual. We shall assume throughout this paper that  $B_X$  and  $S_X$  denote the unit ball and unit sphere of  $X$ , respectively.

One of Gao's parameters  $E_\epsilon(X)$  is defined by the formula

$$E_\epsilon(X) = \sup\{\|x + \epsilon y\|^2 + \|x - \epsilon y\|^2 : x, y \in S_X\},$$

where  $\epsilon$  is a nonnegative number. It is worth noting that  $E_\epsilon(X)$  was also introduced by Saejung [3] and Yang-Wang [5] recently. Let us now collect some properties related to this parameter (see [1, 4, 5]).

- (1)  $X$  is uniformly non-square if and only if  $E_\epsilon(X) < 2(1 + \epsilon)^2$  for some  $\epsilon \in (0, 1]$ .
- (2)  $X$  has uniform normal structure if  $E_\epsilon(X) < 1 + (1 + \epsilon)^2$  for some  $\epsilon \in (0, 1]$ .
- (3)  $E_\epsilon(X) = E_\epsilon(\tilde{X})$ , where  $\tilde{X}$  is the ultrapower of  $X$ .
- (4)  $E_\epsilon(X) = \sup\{\|x + \epsilon y\|^2 + \|x - \epsilon y\|^2 : x, y \in B_X\}$ .

It follows from the property (4) that

$$E_\epsilon(X) = \inf \left\{ \frac{\|x + \epsilon y\|^2 + \|x - \epsilon y\|^2}{\max(\|x\|^2, \|y\|^2)} : x, y \in X, \|x\| + \|y\| \neq 0 \right\}.$$

Now let us pay attention to another Gao's parameter  $f_\epsilon(X)$ , which is defined by the formula

$$f_\epsilon(X) = \inf\{\|x + \epsilon y\|^2 + \|x - \epsilon y\|^2 : x, y \in S_X\},$$

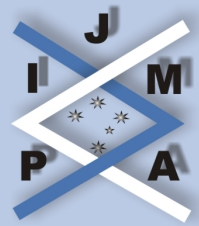
where  $\epsilon$  is a nonnegative number.

We quote some properties related to this parameter (see [1, 2]).

(1) If  $f_\epsilon(X) > 2$  for some  $\epsilon \in (0, 1]$ , then  $X$  is uniformly non-square.

(2)  $X$  has uniform normal structure if  $f_1(X) > 32/9$ .

Using a similar method to [4, Theorem 3], we can also deduce that  $f_\epsilon(X) = f_\epsilon(\tilde{X})$ , where  $\tilde{X}$  is the ultrapower of  $X$ .



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## 2. Main Results

We start this section with some definitions. Recall that  $X$  is called *uniformly non-square* if there exists  $\delta > 0$ , such that if  $x, y \in S_X$  then  $\|x + y\|/2 \leq 1 - \delta$  or  $\|x - y\|/2 \leq 1 - \delta$ . In what follows, we shall show that  $f_\epsilon(X)$  also provides a characterization of the uniformly non-square spaces, namely  $f_1(X) > 2$ .

**Theorem 2.1.**  $X$  is uniformly non-square if and only if  $f_1(X) > 2$ .

*Proof.* It is convenient for us to assume in this proof that  $\dim X < \infty$ . The extension of the results to the general case is immediate, depending only on the formula

$$f_\epsilon(X) = \inf\{f_\epsilon(Y) : Y \text{ subspace of } X \text{ and } \dim Y = 2\}.$$

We are going to prove that uniform non-squareness implies  $f_1(X) > 2$ . Assume on the contrary that  $f_1(X) = 2$ . It follows from the definition of  $f_\epsilon(X)$  that there exist  $x, y \in S_X$  so that

$$\|x + y\|^2 + \|x - y\|^2 = 2.$$

Then, since  $\|x + y\| + \|x - y\| \geq 2$ , we have

$$\|x \pm y\|^2 = 2 - \|x \mp y\|^2 \leq 2 - (2 - \|x \pm y\|)^2,$$

which implies that  $\|x \pm y\| = 1$ . Now let us put  $u = x + y$ ,  $v = x - y$ , then  $u, v \in S_X$  and  $\|u \pm v\| = 2$ . This is a contradiction. The converse of this assertion was proved by Gao [2, Theorem 2.8], and thus the proof is complete.  $\square$

Consider now the definitions of normal structure. A Banach space  $X$  is said to have (*weak*) *normal structure* provided that every (weakly compact) closed bounded convex subset  $C$  of  $X$  with  $\text{diam}(C) > 0$ , contains a non-diametral point, i.e., there exists  $x_0 \in C$  such that  $\sup\{\|x - x_0\| : x \in C\} < \text{diam}(C)$ . It is clear that normal



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structure and weak normal structure coincides when  $X$  is reflexive. A Banach space  $X$  is said to have *uniform normal structure* if  $\inf \{\text{diam}(C)/\text{rad}(C)\} > 1$ , where the infimum is taken over all bounded closed convex subsets  $C$  of  $X$  with  $\text{diam}(C) > 0$ .

To study the relation between normal structure and Gao's parameter, we need a sufficient condition for normal structure, which was posed by Saejung [4, Lemma 2] recently.

**Theorem 2.2.** *Let  $X$  be a Banach space with*

$$E_\epsilon(X) < 2 + \epsilon^2 + \epsilon\sqrt{4 + \epsilon^2}$$

for some  $\epsilon \in (0, 1]$ , then  $X$  has uniform normal structure.

*Proof.* By our hypothesis it is enough to show that  $X$  has normal structure. Suppose that  $X$  lacks normal structure, then by [4, Lemma 2], there exist  $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \in S_{\tilde{X}}$  and  $\tilde{f}_1, \tilde{f}_2, \tilde{f}_3 \in S_{\tilde{X}^*}$  satisfying:

- (a)  $\|\tilde{x}_i - \tilde{x}_j\| = 1$  and  $\tilde{f}_i(\tilde{x}_j) = 0$  for all  $i \neq j$ .
- (b)  $\tilde{f}_i(\tilde{x}_i) = 1$  for  $i = 1, 2, 3$  and
- (c)  $\|\tilde{x}_3 - (\tilde{x}_2 + \tilde{x}_1)\| \geq \|\tilde{x}_2 + \tilde{x}_1\|$ .

Let  $2\alpha(\epsilon) = \sqrt{4 + \epsilon^2} + 2 - \epsilon$  and consider three possible cases.

**CASE 1.**  $\|\tilde{x}_1 + \tilde{x}_2\| \leq \alpha(\epsilon)$ . In this case, let us put  $\tilde{x} = \tilde{x}_1 - \tilde{x}_2$  and  $\tilde{y} = (\tilde{x}_1 + \tilde{x}_2)/\alpha(\epsilon)$ . It follows that  $\tilde{x}, \tilde{y} \in B_{\tilde{X}}$ , and

$$\begin{aligned} \|\tilde{x} + \epsilon\tilde{y}\| &= \|(1 + (\epsilon/\alpha(\epsilon)))\tilde{x}_1 - (1 - (\epsilon/\alpha(\epsilon)))\tilde{x}_2\| \\ &\geq (1 + (\epsilon/\alpha(\epsilon)))\tilde{f}_1(\tilde{x}_1) - (1 - (\epsilon/\alpha(\epsilon)))\tilde{f}_1(\tilde{x}_2) \\ &= 1 + (\epsilon/\alpha(\epsilon)), \end{aligned}$$



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$$\begin{aligned}\|\tilde{x} - \epsilon\tilde{y}\| &= \|(1 + (\epsilon/\alpha(\epsilon)))\tilde{x}_2 - (1 - (\epsilon/\alpha(\epsilon)))\tilde{x}_1\| \\ &\geq (1 + (\epsilon/\alpha(\epsilon)))\tilde{f}_2(\tilde{x}_2) - (1 - (\epsilon/\alpha(\epsilon)))\tilde{f}_2(\tilde{x}_1) \\ &= 1 + (\epsilon/\alpha(\epsilon)).\end{aligned}$$

**CASE 2.**  $\|\tilde{x}_1 + \tilde{x}_2\| \geq \alpha(\epsilon)$  and  $\|\tilde{x}_3 + \tilde{x}_2 - \tilde{x}_1\| \leq \alpha(\epsilon)$ . In this case, let us put  $\tilde{x} = \tilde{x}_2 - \tilde{x}_3$  and  $\tilde{y} = (\tilde{x}_3 + \tilde{x}_2 - \tilde{x}_1)/\alpha(\epsilon)$ . It follows that  $\tilde{x}, \tilde{y} \in B_{\tilde{X}}$ , and

$$\begin{aligned}\|\tilde{x} + \epsilon\tilde{y}\| &= \|(1 + (\epsilon/\alpha(\epsilon)))\tilde{x}_2 - (1 - (\epsilon/\alpha(\epsilon)))\tilde{x}_3 - (\epsilon/\alpha(\epsilon))\tilde{x}_1\| \\ &\geq (1 + (\epsilon/\alpha(\epsilon)))\tilde{f}_2(\tilde{x}_2) - (1 - (\epsilon/\alpha(\epsilon)))\tilde{f}_2(\tilde{x}_3) - (\epsilon/\alpha(\epsilon))\tilde{f}_2(\tilde{x}_1) \\ &= 1 + (\epsilon/\alpha(\epsilon)),\end{aligned}$$

$$\begin{aligned}\|\tilde{x} - \epsilon\tilde{y}\| &= \|(1 + (\epsilon/\alpha(\epsilon)))\tilde{x}_3 - (1 - (\epsilon/\alpha(\epsilon)))\tilde{x}_2 - (\epsilon/\alpha(\epsilon))\tilde{x}_1\| \\ &\geq (1 + (\epsilon/\alpha(\epsilon)))\tilde{f}_3(\tilde{x}_3) - (1 - (\epsilon/\alpha(\epsilon)))\tilde{f}_3(\tilde{x}_2) - (\epsilon/\alpha(\epsilon))\tilde{f}_3(\tilde{x}_1) \\ &= 1 + (\epsilon/\alpha(\epsilon)).\end{aligned}$$

**CASE 3.**  $\|\tilde{x}_1 + \tilde{x}_2\| \geq \alpha(\epsilon)$  and  $\|\tilde{x}_3 + \tilde{x}_2 - \tilde{x}_1\| \geq \alpha(\epsilon)$ . In this case, let us put  $\tilde{x} = \tilde{x}_3 - \tilde{x}_1$  and  $\tilde{y} = \tilde{x}_2$ . It follows that  $\tilde{x}, \tilde{y} \in S_{\tilde{X}}$ , and

$$\begin{aligned}\|\tilde{x} + \epsilon\tilde{y}\| &= \|\tilde{x}_3 + \epsilon\tilde{x}_2 - \tilde{x}_1\| \\ &\geq \|\tilde{x}_3 + \tilde{x}_2 - \tilde{x}_1\| - (1 - \epsilon) \\ &\geq \alpha(\epsilon) + \epsilon - 1,\end{aligned}$$

$$\begin{aligned}\|\tilde{x} - \epsilon\tilde{y}\| &= \|\tilde{x}_3 - (\epsilon\tilde{x}_2 + \tilde{x}_1)\| \\ &\geq \|\tilde{x}_3 - (\tilde{x}_2 + \tilde{x}_1)\| - (1 - \epsilon) \\ &\geq \alpha(\epsilon) + \epsilon - 1.\end{aligned}$$



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Then, by definition of  $E_\epsilon(X)$  and the fact  $E_\epsilon(X) = E_\epsilon(\tilde{X})$ ,

$$\begin{aligned} E_\epsilon(X) &\geq 2 \min \{1 + (\epsilon/\alpha(\epsilon)), \alpha(\epsilon) + \epsilon - 1\}^2 \\ &= 2 + \epsilon^2 + \epsilon\sqrt{4 + \epsilon^2}. \end{aligned}$$

This is a contradiction and thus the proof is complete.  $\square$

*Remark 1.* It is proved that  $E_\epsilon(X) < 1 + (1 + \epsilon)^2$  for some  $\epsilon \in (0, 1]$  implies that  $X$  has uniform normal structure. So Theorem 2.2 is an improvement of such a result.

**Theorem 2.3.** *Let  $X$  be a Banach space with*

$$f_\epsilon(X) > ((1 + \epsilon^2)^2 + 2\epsilon(1 - \epsilon^2))(2 + \epsilon^2 - \epsilon\sqrt{4 + \epsilon^2})$$

*for some  $\epsilon \in (0, 1]$ , then  $X$  has uniform normal structure.*

*Proof.* By our hypothesis it is enough to show that  $X$  has normal structure. Assume that  $X$  lacks normal structure, then from the proof of Theorem 2.2 we can find  $\tilde{x}, \tilde{y} \in B_{\tilde{X}}$  such that

$$\|\tilde{x} \pm \epsilon\tilde{y}\| \geq 1 + (\epsilon/\alpha(\epsilon)) = \alpha(\epsilon) + \epsilon - 1 =: \beta(\epsilon).$$

Put  $\tilde{u} = (\tilde{x} + \epsilon\tilde{y})/\beta(\epsilon)$  and  $\tilde{v} = (\tilde{x} - \epsilon\tilde{y})/\beta(\epsilon)$ . It follows that  $\|\tilde{u}\|, \|\tilde{v}\| \geq 1$ , and

$$\begin{aligned} \|\tilde{u} + \epsilon\tilde{v}\| &= \left\| \frac{1}{\beta(\epsilon)} ((1 + \epsilon)\tilde{x} + \epsilon(1 - \epsilon)\tilde{y}) \right\| \\ &\leq \frac{(1 + \epsilon) + \epsilon(1 - \epsilon)}{\beta(\epsilon)}, \\ \|\tilde{u} - \epsilon\tilde{v}\| &= \frac{1}{\beta(\epsilon)} ((1 - \epsilon)\tilde{x} + \epsilon(1 + \epsilon)\tilde{y}) \\ &\leq \frac{(1 - \epsilon) + \epsilon(1 + \epsilon)}{\beta(\epsilon)}. \end{aligned}$$





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Hence, by the definition of  $f_\epsilon(X)$  and the fact  $f_\epsilon(X) = f_\epsilon(\tilde{X})$ , we have

$$\begin{aligned} f_\epsilon(X) &\leq \frac{((1 + \epsilon) + \epsilon(1 - \epsilon))^2 + ((1 - \epsilon) + \epsilon(1 + \epsilon))^2}{\beta^2(\epsilon)} \\ &= ((1 + \epsilon^2)^2 + 2\epsilon(1 - \epsilon^2))(2 + \epsilon^2 - \epsilon\sqrt{4 + \epsilon^2}), \end{aligned}$$

which contradicts our hypothesis. □

*Remark 2.* Letting  $\epsilon = 1$ , one can easily get that if  $f_1(X) > 4(3 - \sqrt{5})$ , then  $X$  has uniform normal structure. So this is an extension and an improvement of [2, Theorem 5.3].

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