Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 7, Issue 2, Article 63, 2006

EXTENSIONS AND SHARPENINGS OF JORDAN'S AND KOBER'S INEQUALITIES

XIAOHUI ZHANG, GENDI WANG, AND YUMING CHU

DEPARTMENT OF MATHEMATICS
HUZHOU UNIVERSITY
HUZHOU 313000, P.R. CHINA.
xhzhang@hutc.zj.cn

Received 15 September, 2005; accepted 24 February, 2006
Communicated by S.S. Dragomir

ABSTRACT. In this paper the authors discuss some monotonicity properties of functions involving sine and cosine, and obtain some sharp inequalities for them. These inequalities are extensions and sharpenings of the well-known Jordan's and Kober's inequalities.

Key words and phrases: Monotonicity; Jordan's inequality; Kober's inequality; Extension and sharpening.

2000 *Mathematics Subject Classification.* Primary 26D05.

1. INTRODUCTION

The well-known inequalities

$$\frac{2}{\pi}x \le \sin x \le x, \qquad x \in \left[0, \frac{\pi}{2}\right] \tag{1.1}$$

and

$$\cos x \ge 1 - \frac{2}{\pi}x, \qquad x \in \left[0, \frac{\pi}{2}\right] \tag{1.2}$$

are called Jordan's and Kober's inequality, respectively. In fact, Jordan's and Kober's inequalities are dual in the sense that they follow from each other via the transformation $T:\ x \to \pi/2 - x$. Some different extensions and sharpenings of these inequalities have been obtained by many authors (see [1] – [4]).

In this note, we will extend and sharpen Jordan's and Kober's inequalities by using the monotone form of l'Hôpital's Rule (cf. [5, Theorem 1.25]) and obtain the following results:

ISSN (electronic): 1443-5756

The research is partly supported by N.S.Foundation of China under grant 10471039 and N.S.Foundation of Zhejiang Province under grant M103087.

274-05

Theorem 1.1. *For $x \in [0, \pi/2]$,*

$$\frac{2}{\pi}x + \frac{\pi-2}{\pi^2}x(\pi-2x) \le \sin x \le \frac{2}{\pi}x + \frac{2}{\pi^2}x(\pi-2x), \tag{1.3}$$

$$\frac{2}{\pi}x + \frac{1}{\pi^3}x(\pi^2-4x^2) \le \sin x \le \frac{2}{\pi}x + \frac{\pi-2}{\pi^3}x(\pi^2-4x^2), \tag{1.4}$$

and

$$1-\frac{2}{\pi}x + \frac{\pi-2}{\pi^2}x(\pi-2x) \le \cos x \le 1-\frac{2}{\pi}x + \frac{2}{\pi^2}x(\pi-2x), \tag{1.5}$$

where the coefficients are all best possible.

2. PROOF OF THEOREM 1.1

The following monotone form of l'Hôpital's Rule, which is put forward in [5, Theorem 1.25], is extremely useful in our proof.

Lemma 2.1 (The Monotone Form of l'Hôpital's Rule)**.** *For $-\infty < a < b < \infty$, let f, $g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$, and differentiable on (a,b), let $g'(x) \neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are*

$$\frac{f(x)-f(a)}{g(x)-g(a)} \qquad \text{and} \qquad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

We next prove the inequalities (1.3) – (1.5) by making use of the monotone form of l'Hôpital's Rule.

Proof of Inequality (1.3). Let $f(x) = \left(\frac{\sin x}{x} - \frac{2}{\pi}\right) / \left(\frac{\pi}{2} - x\right)$. Write $f_1(x) = \frac{\sin x}{x} - \frac{2}{\pi}$, and $f_2(x) = \frac{\pi}{2} - x$. Then $f_1(\pi/2) = f_2(\pi/2) = 0$ and

$$\frac{f_1'(x)}{f_2'(x)} = \frac{\sin x - x\cos x}{x^2} = \frac{f_3(x)}{f_4(x)}, \tag{2.1}$$

where $f_3(x) = \sin x - x\cos x$ and $f_4(x) = x^2$. Then $f_3(0) = f_4(0) = 0$ and

$$\frac{f_3'(x)}{f_4'(x)} = \frac{\sin x}{2}, \tag{2.2}$$

which is strictly increasing on $[0, \pi/2]$. By (2.1), (2.2) and the monotone form of l'Hôpital's rule, $f(x)$ is strictly increasing on $[0, \pi/2]$.

The limiting value $f(0) = \frac{2}{\pi}(1-\frac{2}{\pi})$ is clear. By (2.1) and l'Hôpital's Rule, we have $f(\pi/2) = \frac{4}{\pi^2}$.

The inequality (1.3) follows from the monotonicity and the limiting values of $f(x)$. □

Proof of Inequality (1.4). Let $g(x) = g_1(x)/g_2(x)$, where $g_1(x) = \frac{\sin x}{x} - \frac{2}{\pi}$ and $g_2(x) = \frac{\pi^2}{4} - x^2$. Then $g_1(\pi/2) = g_2(\pi/2) = 0$. By differentiation, we have

$$\frac{g_1'(x)}{g_2'(x)} = \frac{\sin x - x\cos x}{2x^3} = \frac{g_3(x)}{g_4(x)}, \tag{2.3}$$

where $g_3(x) = \sin x - x\cos x$ and $g_4(x) = 2x^3$. Then $g_3(0) = g_4(0) = 0$ and

$$\frac{g_3'(x)}{g_4'(x)} = \frac{\sin x}{6x}, \tag{2.4}$$

which is strictly decreasing on $[0, \pi/2]$. Hence, by the monotone form of l'Hôpital's rule, $g(x)$ is also strictly decreasing on $[0, \pi/2]$.

The limiting value $g(0) = \frac{4}{\pi^2}(1 - \frac{2}{\pi})$ is clear. By (2.3) and l'Hôpital's Rule, $g(\pi/2) = \frac{4}{\pi^3}$.

The inequality (1.4) follows from the monotonicity and the limiting values of $g(x)$. □

Proof of Inequality (1.5). Let $h(x) = \left(\frac{1-\cos x}{x} - \frac{2}{\pi}\right) / \left(\frac{\pi}{2} - x\right)$. Simple calculating similar to proofs of inequalities (1.3) and (1.4) will yield the monotonicity and limiting values of $h(x)$, and the inequality (1.5) follow. □

Remark 2.2.

(1) The inequalities (1.3) and (1.5) are $T-$dual to each other.

(2) Like the proof of inequality (1.4), we can construct a function

$$m(x) = \left(\frac{1 - \cos x}{x} - \frac{2}{\pi}\right) \Big/ \left(\frac{\pi^2}{4} - x^2\right)$$

and obtain the following inequality:

$$1 - \frac{2}{\pi}x + \frac{\pi - 2}{2\pi^3}x(\pi^2 - 4x^2) \le \cos x \le 1 - \frac{2}{\pi}x + \frac{2}{\pi^3}x(\pi^2 - 4x^2). \tag{2.5}$$

But the inequalities (1.4) and (2.5) are not $T-$dual. Comparing the inequality (1.5) with (2.5), we can find the inequality (1.5) is stronger than (2.5). Whereas the inequalities (1.3) and (1.4) cannot be compared on the whole interval $[0, \pi/2]$.

(3) Straightforward simplifications of the inequalities (1.3) – (1.5) yield that for $x \in [0, \pi/2]$,

$$x - \frac{2(\pi - 2)}{\pi^2}x^2 \le \sin x \le \frac{4x}{\pi} - \frac{4}{\pi^2}x^2, \tag{2.6}$$

$$\frac{3}{\pi}x - \frac{4}{\pi^3}x^3 \le \sin x \le x - \frac{4(\pi - 2)}{\pi^3}x^3, \tag{2.7}$$

and

$$1 - \frac{4 - \pi}{\pi}x - \frac{2(\pi - 2)}{\pi^2}x^2 \le \cos x \le 1 - \frac{4}{\pi^2}x^2. \tag{2.8}$$

REFERENCES

[1] G.H. HARDY, J.E. LITTLEWOOD AND G. PÓLYA, *Inequalities*, Second Edition, Cambridge, 1952.

[2] D.S. MITRINOVIC, *Analytic Inequalities*, Springer-Verlag, 1970.

[3] G. KLAMBAUER, *Problems and Properties in Analysis*, Marcel Dekker, Inc., New York and Basel, 1979.

[4] U. ABEL AND D. CACCIA, A sharpening of Jordan's inequality, *Amer. Math. Monthly*, **93** (1986), 568.

[5] G.D. ANDERSON, M.K. VAMANAMURTHY AND M. VUORINEN, *Conformal Invariants, Inequalities, and Quasiconformal Maps*, John Wiley & Sons, New York, 1997.