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A GENERALIZED OSTROWSKI-GRÜSS TYPE INEQUALITY FOR TWICE DIFFERENTIABLE MAPPINGS AND APPLICATIONS

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ABSTRACT. A generalized Ostrowski type inequality for twice differentiable mappings in terms of the upper and lower bounds of the second derivative is established. The inequality is applied to numerical integration.

Key words and phrases: Ostrowski inequality, Grüss inequality.

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1. INTRODUCTION

The integral inequality that establishes a connection between the integral of the product of two functions and the product of the integrals is known in the literature as the Grüss inequality. The inequality is as follows:

Theorem 1.1. *Let $f, g : [a, b] \longrightarrow \mathbb{R}$ be integrable functions such that $\Psi \le f(x) \le \varphi$ and $\gamma \le g(x) \le \Gamma$ for all $x \in [a, b]$, where $\Psi,\ \varphi,\ \gamma$ and Γ are constants. It follows that,*

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx\frac{1}{b-a}\int_a^b g(x)dx\right| \le \frac{1}{4}(\varphi - \Psi)\left(\Gamma - \gamma\right), \tag{1.1}$$

where the constant $\frac{1}{4}$ is sharp.

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In [2], S.S. Dragomir and S. Wang proved the following Ostrowski type inequality in terms of lower and upper bounds of the first derivative.

Theorem 1.2. *Let $f : [a,b] \longrightarrow \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) and where the first derivative satisfies the condition,*

$$\gamma \leq f'(x) \leq \Gamma \qquad \text{for all } x \in [a,b],$$

then,

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right)\right| \leq \frac{1}{4}(b-a)(\Gamma - \gamma) \tag{1.2}$$

for all $x \in [a,b]$.

In [1], S.S. Dragomir and N.S. Barnett, proved the following inequality.

Theorem 1.3. *Let $f : [a,b] \longrightarrow \mathbb{R}$ be continuous on $[a,b]$ and twice differentiable on (a,b), where the second derivative $f'' : (a,b) \longrightarrow \mathbb{R}$ satisfies the condition,*

$$\varphi \leq f''(x) \leq \Phi \quad \text{for all } x \in (a,b),$$

then,

$$\begin{aligned} &\left| f(x) + \left[\frac{(b-a)^2}{24} + \frac{1}{2}\left(x - \frac{a+b}{2}\right)^2\right]\frac{f'(b)-f'(a)}{b-a} \right. \\ &\qquad \left. - \left(x - \frac{a+b}{2}\right) f'(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \\ &\qquad\qquad \leq \frac{1}{8}(\Phi - \varphi)\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^2 \end{aligned} \tag{1.3}$$

for all $x \in [a,b]$.

In this paper we establish a more general form of (1.3) and apply the result to numerical integration.

2. MAIN RESULTS

Theorem 2.1. *Let $f : [a,b] \longrightarrow \mathbb{R}$ be a continuous mapping on $[a,b]$, and twice differentiable on (a,b) with second derivative $f'' : (a,b) \longrightarrow \mathbb{R}$ satisfying the condition:*

$$\varphi \leq f''(x) \leq \Phi, \qquad \text{for all } x \in \left[a + h\frac{b-a}{2}, b - h\frac{b-a}{2}\right].$$

It follows that,

$$\begin{aligned} &\left| (1-h)\left[f(x) - \left(x - \frac{a+b}{2}\right)f'(x)\right] + h\frac{f(a)+f(b)}{2} \right. \\ &\qquad + \left[\frac{1}{2}(1-h)\left(x - \frac{a+b}{2}\right)^2 - \frac{(3h-1)(b-a)^2}{24}\right]\left(\frac{f'(b)-f'(a)}{b-a}\right) \\ &\qquad \left. - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \frac{1}{8}(\Phi - \varphi)\left[\frac{1}{2}(b-a)(1-h) + \left|x - \frac{a+b}{2}\right|\right]^2, \end{aligned} \tag{2.1}$$

for all $x \in [a + h\frac{b-a}{2}, b - h\frac{b-a}{2}]$ and $h \in [0,1]$.

Proof. The proof uses the following identity,

$$\begin{aligned}\int_a^b f(t)dt &= (b-a)(1-h)f(x) \\ &\quad - (b-a)(1-h)\left(x-\frac{a+b}{2}\right)f'(x) + h\frac{b-a}{2}\left(f(a)+f(b)\right) \\ &\quad - \frac{h^2(b-a)^2}{8}\left(f'(b)-f'(a)\right) + \int_a^b K(x,t)f''(t)\,dt. \end{aligned} \tag{2.2}$$

for all $x \in [a + h\frac{b-a}{2}, b - h\frac{b-a}{2}]$, where the kernel $K : [a,b]^2 \to \mathbb{R}$ is defined by

$$K(x,t) = \begin{cases} \frac{1}{2}\left[t - \left(a + h\frac{b-a}{2}\right)\right]^2 & \text{if } t \in [a,x] \\ \frac{1}{2}\left[t - \left(b - h\frac{b-a}{2}\right)\right]^2 & \text{if } t \in (x,b]. \end{cases} \tag{2.3}$$

This is a particular form of the identity given in [3, p. 59; Corollary 2.3].

Observe that the kernel K satisfies the estimation

$$0 \le K(x,t) \le \begin{cases} \frac{1}{2}\left[\left(b - h\frac{b-a}{2}\right) - x\right]^2, & x \in \left[a + h\frac{b-a}{2}, \frac{a+b}{2}\right) \\ \frac{1}{2}\left[x - \left(a + h\frac{b-a}{2}\right)\right]^2, & x \in \left[\frac{a+b}{2}, b - h\frac{b-a}{2}\right]. \end{cases} \tag{2.4}$$

Applying the Grüss inequality for the mappings $f''(\cdot)$ and $K(x,\cdot)$ we get,

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b K(x,t)f''(t)\,dt - \frac{1}{b-a}\int_a^b K(x,t)\,dt\frac{1}{b-a}\int_a^b f''(t)\,dt\right| \\ &\qquad \le \frac{1}{4}(\Phi-\varphi) \times \begin{cases} \frac{1}{2}\left[\left(b - h\frac{b-a}{2}\right) - x\right]^2, & x \in \left[a + h\frac{b-a}{2}, \frac{a+b}{2}\right) \\ \frac{1}{2}\left[x - \left(a + h\frac{b-a}{2}\right)\right]^2, & x \in \left[\frac{a+b}{2}, b - h\frac{b-a}{2}\right]. \end{cases} \end{aligned} \tag{2.5}$$

Observe that,

$$\begin{aligned} \int_a^b K(x,t)dt &= \int_a^x \frac{\left[t - \left(a + h\frac{b-a}{2}\right)\right]^2}{2}dt + \int_x^b \frac{\left[t - \left(b - h\frac{b-a}{2}\right)\right]^2}{2}dt \\ &= \frac{1}{6}\left[\left(x - \left(a + h\frac{b-a}{2}\right)\right)^3 + \left(\left(b - h\frac{b-a}{2}\right) - x\right)^3 + \frac{h^3(b-a)^3}{4}\right] \\ &= (b-a)(1-h)\left[\frac{(b-a)^2(1-h)^2}{24} + \frac{1}{2}\left(x - \frac{a+b}{2}\right)^2\right] \\ &\qquad + \frac{h^3(b-a)^3}{24}. \end{aligned} \tag{2.6}$$

Using (2.6) in (2.5) , we get

$$\left|\frac{1}{b-a}\int_a^b K(x,t)\,f''(t)\,dt - \left[\frac{(b-a)^2(1-h)^3}{24}+\frac{1}{2}(1-h)\left(x-\frac{a+b}{2}\right)^2 + \frac{h^3(b-a)^2}{24}\right]\left(\frac{f'(b)-f'(a)}{b-a}\right)\right| \le \frac{1}{4}(\Phi-\varphi)\times\begin{cases}\frac{1}{2}\left[\left(b-h\frac{b-a}{2}\right)-x\right]^2, & x\in\left[a+h\frac{b-a}{2},\frac{a+b}{2}\right)\\ \frac{1}{2}\left[x-\left(a+h\frac{b-a}{2}\right)\right]^2, & x\in\left[\frac{a+b}{2},b-h\frac{b-a}{2}\right].\end{cases}$$

Also, by using identity (2.2) , the above inequality reduces to,

$$\left|(1-h)\left[f(x)-\left(x-\frac{a+b}{2}\right)f'(x)\right]+h\frac{f(a)+f(b)}{2} + \left[\frac{1}{2}(1-h)\left(x-\frac{a+b}{2}\right)^2-\frac{(3h-1)(b-a)^2}{24}\right]\left(\frac{f'(b)-f'(a)}{b-a}\right) - \frac{1}{b-a}\int_a^b f(t)dt\right| \le \frac{1}{4}(\Phi-\varphi)\times\begin{cases}\frac{1}{2}\left[\left(b-h\frac{b-a}{2}\right)-x\right]^2, & x\in\left[a+h\frac{b-a}{2},\frac{a+b}{2}\right);\\ \frac{1}{2}\left[x-\left(a+h\frac{b-a}{2}\right)\right]^2, & x\in\left[\frac{a+b}{2},b-h\frac{b-a}{2}\right].\end{cases}$$

Since,

$$\max\left\{\frac{\left[\left(b-h\frac{b-a}{2}\right)-x\right]^2}{2},\frac{\left[x-\left(a+h\frac{b-a}{2}\right)\right]^2}{2}\right\} = \begin{cases}\frac{1}{2}\left[\left(b-h\frac{b-a}{2}\right)-x\right]^2, & x\in\left[a+h\frac{b-a}{2},\frac{a+b}{2}\right)\\ \frac{1}{2}\left[x-\left(a+h\frac{b-a}{2}\right)\right]^2, & x\in\left[\frac{a+b}{2},b-h\frac{b-a}{2}\right],\end{cases}$$

but on the other hand,

$$\max\left\{\frac{\left[\left(b-h\frac{b-a}{2}\right)-x\right]^2}{2},\frac{\left[x-\left(a+h\frac{b-a}{2}\right)\right]^2}{2}\right\} = \frac{1}{2}\left[\frac{1}{2}(b-a)(1-h)+\left|x-\frac{a+b}{2}\right|\right]^2,$$

inequality (2.1) is proved. □

Remark 2.2. For $h=0$ in (2.1), we obtain (1.3).

Corollary 2.3. *If f is as in Theorem 2.1, then we have the following perturbed midpoint inequality:*

$$\left|(1-h)f\left(\frac{a+b}{2}\right)+h\frac{f(a)+f(b)}{2} - \frac{(3h-1)(b-a)}{24}(f'(b)-f'(a)) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{1}{32}(\Phi-\varphi)(b-a)^2(1-h)^2, \tag{2.7}$$

giving,

$$\left|f\left(\frac{a+b}{2}\right)+\frac{(b-a)}{24}(f'(b)-f'(a))-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{1}{32}(\Phi-\varphi)(b-a)^2, \tag{2.8}$$

for $h=0$.

Remark 2.4. The classical midpoint inequality states that

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{24}(b-a)^2\|f''\|_\infty. \tag{2.9}$$

If $\Phi-\varphi\leq\frac{4}{3}\|f''\|_\infty$, then the estimation provided by (2.7) is better than the estimation in the classical midpoint inequality (2.9). A sufficient condition for $\Phi-\varphi\leq\frac{4}{3}\|f''\|_\infty$ to be true is $0\leq\varphi\leq\Phi$.Indeed, if $0\leq\varphi\leq\Phi$, then $\Phi-\varphi\leq\|f''\|_\infty<\frac{4}{3}\|f''\|_\infty$.

Corollary 2.5. *Let f be as in Theorem 2.1, then,*

$$\left|\frac{f(a)+f(b)}{2}-\frac{(b-a)}{12}(f'(b)-f'(a))-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{1}{32}(\Phi-\varphi)(2-h)^2(b-a)^2. \tag{2.10}$$

Proof. Put $x=a$ and $x=b$ in turn in (2.1) and use the triangle inequality. □

Corollary 2.6. *Let f be as in Theorem 2.1, then we have the following perturbed Trapezoid inequality:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{(b-a)}{12}(f'(b)-f'(a))-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{1}{32}(\Phi-\varphi)(b-a)^2. \tag{2.11}$$

Proof. Put $h=1$ in (2.10). □

Remark 2.7. The classical Trapezoid inequality states that

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{12}(b-a)^2\|f''\|_\infty. \tag{2.12}$$

If we assume that $\Phi-\varphi\leq\frac{2}{3}\|f''\|_\infty$, then the estimation provided by (2.10) is better than the estimation in the classical Trapezoid inequality (2.12).

3. APPLICATIONS IN NUMERICAL INTEGRATION

Let $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ be a division of the interval $[a,b]$, $\xi_i \in [x_i, x_{i+1}]$, $(i = 0,1,\ldots,n-1)$ a sequence of intermediate points and $h_i := x_{i+1} - x_i$ $(i = 0,1,\ldots,n-1)$. Following the approach taken in [1] we have the following:

Theorem 3.1. *Let $f : [a,b] \longrightarrow \mathbb{R}$ be continuous on $[a,b]$ and a twice differentiable function on (a,b), whose second derivative, $f'' : (a,b) \longrightarrow \mathbb{R}$ satisfies:*

$$\varphi \le f''(x) \le \Phi, \qquad \text{for all } x \in (a,b),$$

then,

$$\int_a^b f(t)dt = A(f,f',I_n,\xi,\delta) + R(f,f',I_n,\xi,\delta), \tag{3.1}$$

where

$$\begin{aligned} A(f,f',I_n,\xi,\delta) &= (1-\delta)\sum_{i=0}^{n-1} h_i f(\xi_i) \\ &\quad - (1-\delta)\sum_{i=0}^{n-1} h_i\left(\xi_i - \frac{x_i + x_{i-1}}{2}\right) f'(\xi_i) + \delta\sum_{i=0}^{n-1} h_i\left(\frac{f(x_i)+f(x_{i+1})}{2}\right) \\ &\quad + \sum_{i=0}^{n-1}\left[\frac{1}{2}(1-\delta)\left(\xi_i - \frac{x_i+x_{i+1}}{2}\right)^2 \right. \\ &\quad \left. - \frac{(3\delta-1)h_i^2}{24}\right](f'(x_{i+1}) - f'(x_i)) \end{aligned} \tag{3.2}$$

and the remainder $R(f,f',I_n,\xi,\delta)$ satisfies the estimation:

$$\begin{aligned} |R(f,f',I_n,\xi,\delta)| &\le \frac{1}{8}(\Phi-\varphi)\sum_{i=0}^{n-1} h_i\left[\frac{(1-\delta)}{2}h_i + \left|\xi_i - \frac{x_i+x_{i+1}}{2}\right|\right]^2 \\ &\le \frac{1}{32}(\Phi-\varphi)(1-\delta)^2\sum_{i=0}^{n-1} h_i^3, \end{aligned} \tag{3.3}$$

where $\delta \in [0,1]$ and $x_i + \delta\frac{h_i}{2} \le \xi_i \le x_{i+1} - \delta\frac{h_i}{2}$.

Proof. Applying Theorem 2.1 on the interval $[x_i, x_{i+1}]$ $(i = 0,\ldots,n-1)$ gives:

$$\begin{aligned} &\left|(1-\delta)\left[h_i f(\xi_i) - h_i\left(\xi_i - \frac{x_i+x_{i+1}}{2}\right)f'(\xi_i)\right] + \delta h_i\left(\frac{f(x_i)+f(x_{i+1})}{2}\right)\right. \\ &\left. + \left[\frac{1}{2}(1-\delta)\left(\xi_i - \frac{x_i+x_{i+1}}{2}\right)^2 - \frac{(3\delta-1)h_i^2}{24}\right](f'(x_{i+1}) - f'(x_i)) - \int_{x_i}^{x_{i+1}} f(t)\,dt\right| \\ &\le \frac{1}{8}(\Phi-\varphi)h_i\left[\frac{1}{2}(1-\delta)h_i + \left|\xi_i - \frac{x_i+x_{i+1}}{2}\right|\right]^2, \\ &\le \frac{1}{8}(\Phi-\varphi)(1-\delta)^2 h_i^3 \end{aligned}$$

as

$$\left|\xi_i - \frac{x_i+x_{i+1}}{2}\right| \le (1-\delta)\frac{h_i}{2} \quad \text{for all } i \in \{0,1,\ldots,n-1\}$$

for any choice ξ_i of the intermediate points.

Summing the above inequalities over i from 0 to $n-1$, and using the generalized triangle inequality, we get the desired estimation (3.3). □

Corollary 3.2. *The following perturbed midpoint rule holds:*

$$\int_a^b f(x)\,dx = M(f,f',I_n) + R_M(f,f',I_n), \tag{3.4}$$

where

$$M(f,f',I_n) = \sum_{i=0}^{n-1} h_i f\left(\frac{x_i + x_{i+1}}{2}\right) + \frac{1}{24}\sum_{i=0}^{n-1} h_i^2 \left(f'(x_{i+1}) - f'(x_i)\right) \tag{3.5}$$

and the remainder term $R_M(f,f',I_n)$ satisfies the estimation:

$$|R_M(f,f',I_n)| \le \frac{1}{32}(\Phi - \varphi)\sum_{i=0}^{n-1} h_i^3. \tag{3.6}$$

Corollary 3.3. *The following perturbed trapezoid rule holds*

$$\int_a^b f(x)\,dx = T(f,f',I_n) + R_T(f,f',I_n), \tag{3.7}$$

where

$$T(f,f',I_n) = \sum_{i=0}^{n-1} h_i \frac{f(x_i) + f(x_{i+1})}{2} - \frac{1}{12}\sum_{i=0}^{n-1} h_i^2 \left(f'(x_{i+1}) - f'(x_i)\right) \tag{3.8}$$

and the remainder term $R_T(f,f',I_n)$ satisfies the estimation:

$$|R_T(f,f',I_n)| \le \frac{1}{32}(\Phi - \varphi)\sum_{i=0}^{n-1} h_i^3. \tag{3.9}$$

Remark 3.4. Note that the above mentioned perturbed midpoint formula (3.5) and perturbed trapezoid formula (3.8) can offer better approximations of the integral $\int_a^b f(x)\,dx$ for general classes of mappings as discussed in Remarks 2.2 and 2.4.

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