

AN EQUIVALENT FORM OF THE FUNDAMENTAL TRIANGLE INEQUALITY AND ITS APPLICATIONS



Fundamental Triangle Inequality
Shan-He Wu and Mihály Bencze
vol. 10, iss. 1, art. 16, 2009

[Title Page](#)

[Contents](#)

[◀◀](#) [▶▶](#)

[◀](#) [▶](#)

Page 1 of 15

[Go Back](#)

[Full Screen](#)

[Close](#)

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756

© 2007 Victoria University. All rights reserved.

SHAN-HE WU

Dept. of Mathematics and Computer Science
Longyan University
Longyan Fujian 364012, China
EMail: wushanhe@yahoo.com.cn

MIHÁLY BENCZE

Str. Harmanului 6
505600 Sacele-Négyfalu
Jud. Brasov, Romania
EMail: benczemihaly@yahoo.com

Received: 12 March, 2008

Accepted: 20 January, 2009

Communicated by: **S.S. Dragomir**

2000 AMS Sub. Class.: 26D05, 26D15, 51M16.

Key words: Fundamental triangle inequality, Equivalent form, Garfunkel-Bankoff inequality, Leuenberger's inequality.

Abstract: An equivalent form of the fundamental triangle inequality is given. The result is then used to obtain an improvement of the Leuenberger's inequality and a new proof of the Garfunkel-Bankoff inequality.

Acknowledgement: The present investigation was supported, in part, by the Natural Science Foundation of Fujian Province of China under Grant S0850023, and, in part, by the Science Foundation of Project of Fujian Province Education Department of China under Grant JA08231.

Both of the authors are grateful to the referees for their helpful and constructive comments which enhanced this paper.

Contents

1	Introduction and Main Results	3
2	Proof of Theorem 1.1	5
3	Application to a New Proof of the Garfunkel-Bankoff Inequality	7
4	Application to an Improvement of Leuenberger's Inequality	9



Fundamental Triangle Inequality

Shan-He Wu and Mihály Bencze

vol. 10, iss. 1, art. 16, 2009

[Title Page](#)

[Contents](#)



Page 2 of 15

[Go Back](#)

[Full Screen](#)

[Close](#)

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756



1. Introduction and Main Results

In what follows, we denote by A, B, C the angles of triangle ABC , let a, b, c denote the lengths of its corresponding sides, and let s, R and r denote respectively the semi-perimeter, circumradius and inradius of a triangle. We will customarily use the symbol of cyclic sums:

$$\sum f(a) = f(a) + f(b) + f(c), \quad \sum f(a, b) = f(a, b) + f(b, c) + f(c, a).$$

The fundamental triangle inequality is one of the cornerstones of geometric inequalities for triangles. It reads as follows:

$$(1.1) \quad 2R^2 + 10Rr - r^2 - 2(R - 2r)\sqrt{R^2 - 2Rr} \\ \leq s^2 \leq 2R^2 + 10Rr - r^2 + 2(R - 2r)\sqrt{R^2 - 2Rr}.$$

The equality holds in the left (or right) inequality of (1.1) if and only if the triangle is isosceles.

As is well known, the inequality (1.1) is a necessary and sufficient condition for the existence of a triangle with elements R, r and s . This classical inequality has many important applications in the theory of geometric inequalities and has received much attention. There exist a large number of papers that have been written about applying the inequality (1.1) to establish and prove the geometric inequalities for triangles, e.g., see [1] to [10] and the references cited therein.

In a recent paper [11], we presented a sharpened version of the fundamental triangle, as follows:

$$(1.2) \quad 2R^2 + 10Rr - r^2 - 2(R - 2r)\sqrt{R^2 - 2Rr} \cos \phi \\ \leq s^2 \leq 2R^2 + 10Rr - r^2 + 2(R - 2r)\sqrt{R^2 - 2Rr} \cos \phi,$$

Title Page

Contents

◀▶

◀▶

Page 3 of 15

Go Back

Full Screen

Close



where $\phi = \min\{|A - B|, |B - C|, |C - A|\}$.

The objective of this paper is to give an equivalent form of the fundamental triangle inequality. As applications, we shall apply our results to a new proof of the Garfunkel-Bankoff inequality and an improvement of the Leuenberger inequality. It will be shown that our new inequality can efficaciously reduce the computational complexity in the proof of certain inequalities for triangles. We state the main result in the following theorem:

Theorem 1.1. *For any triangle ABC the following inequalities hold true:*

$$(1.3) \quad \frac{1}{4}\delta(4 - \delta)^3 \leq \frac{s^2}{R^2} \leq \frac{1}{4}(2 - \delta)(2 + \delta)^3,$$

where $\delta = 1 - \sqrt{1 - (2r/R)}$. Furthermore, the equality holds in the left (or right) inequality of (1.3) if and only if the triangle is isosceles.

Fundamental Triangle Inequality

Shan-He Wu and Mihály Bencze

vol. 10, iss. 1, art. 16, 2009

Title Page

Contents



Page 4 of 15

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756



2. Proof of Theorem 1.1

We rewrite the fundamental triangle inequality (1.1) as:

$$(2.1) \quad 2 + \frac{10r}{R} - \frac{r^2}{R^2} - 2 \left(1 - \frac{2r}{R}\right) \sqrt{1 - \frac{2r}{R}} \\ \leq \frac{s^2}{R^2} \leq 2 + \frac{10r}{R} - \frac{r^2}{R^2} + 2 \left(1 - \frac{2r}{R}\right) \sqrt{1 - \frac{2r}{R}}.$$

By the Euler's inequality $R \geq 2r$ (see [1]), we observe that

$$0 \leq 1 - \frac{2r}{R} < 1.$$

Let

$$(2.2) \quad \sqrt{1 - \frac{2r}{R}} = 1 - \delta, \quad 0 < \delta \leq 1.$$

Also, the identity (2.2) is equivalent to the following identity:

$$(2.3) \quad \frac{r}{R} = \delta - \frac{1}{2}\delta^2.$$

By applying the identities (2.2) and (2.3) to the inequality (2.1), we obtain that

$$2 + 10 \left(\delta - \frac{1}{2}\delta^2\right) - \left(\delta - \frac{1}{2}\delta^2\right)^2 - 2 \left[1 - 2 \left(\delta - \frac{1}{2}\delta^2\right)\right] (1 - \delta) \\ \leq \frac{s^2}{R^2} \leq 2 + 10 \left(\delta - \frac{1}{2}\delta^2\right) - \left(\delta - \frac{1}{2}\delta^2\right)^2 + 2 \left[1 - 2 \left(\delta - \frac{1}{2}\delta^2\right)\right] (1 - \delta),$$

Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 5 of 15

Go Back

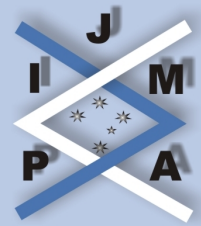
Full Screen

Close

that is

$$16\delta - 12\delta^2 + 3\delta^3 - \frac{1}{4}\delta^4 \leq \frac{s^2}{R^2} \leq 4 + 4\delta - \delta^3 - \frac{1}{4}\delta^4.$$

After factoring out common factors, the above inequality can be transformed into the desired inequality (1.3). This completes the proof of Theorem 1.1. \square



Fundamental Triangle Inequality

Shan-He Wu and Mihály Bencze

vol. 10, iss. 1, art. 16, 2009

[Title Page](#)

[Contents](#)



Page 6 of 15

[Go Back](#)

[Full Screen](#)

[Close](#)

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756



3. Application to a New Proof of the Garfunkel-Bankoff Inequality

Theorem 3.1. *If A , B , C are angles of an arbitrary triangle, then we have the inequality*

$$(3.1) \quad \sum \tan^2 \frac{A}{2} \geq 2 - 8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$$

The equality holds in (3.1) if and only if the triangle ABC is equilateral.

Inequality (3.1) was proposed by Garfunkel as a conjecture in [12], and it was first proved by Bankoff in [13]. In this section, we give a simplified proof of this Garfunkel-Bankoff inequality by means of the equivalent form of the fundamental triangle inequality.

Proof. From the identities for an arbitrary triangle (see [2]):

$$(3.2) \quad \sum \tan^2 \frac{A}{2} = \frac{(4R + r)^2}{s^2} - 2,$$

$$(3.3) \quad \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = \frac{r}{4R},$$

it is easy to see that the Garfunkel-Bankoff inequality is equivalent to the following inequality:

$$(3.4) \quad \left(4 - \frac{2r}{R}\right) \frac{s^2}{R^2} - \left(4 + \frac{r}{R}\right)^2 \leq 0.$$

Title Page

Contents

◀ ▶

◀ ▶

Page 7 of 15

Go Back

Full Screen

Close



Using the inequality (1.3) and the identity (2.3), we have

$$\begin{aligned}\left(4 - \frac{2r}{R}\right) \frac{s^2}{R^2} - \left(4 + \frac{r}{R}\right)^2 &\leq \frac{1}{4}(4 - 2\delta + \delta^2)(2 - \delta)(2 + \delta)^3 - \frac{1}{4}(4 - \delta)^2(2 + \delta)^2 \\ &= -\frac{1}{4}\delta^2(2 + \delta)^2(1 - \delta)^2 \\ &\leq 0,\end{aligned}$$

which implies the required inequality (3.4), hence the Garfunkel-Bankoff inequality is proved. \square

Fundamental Triangle Inequality

Shan-He Wu and Mihály Bencze

vol. 10, iss. 1, art. 16, 2009

Title Page

Contents



Page 8 of 15

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756



4. Application to an Improvement of Leuenberger's Inequality

In 1960, Leuenberger presented the following inequality concerning the sides and the circumradius of a triangle (see [1])

$$(4.1) \quad \sum \frac{1}{a} \geq \frac{\sqrt{3}}{R}.$$

Three years later, Steinig sharpened the inequality (4.1) to the following form ([14], see also [1])

$$(4.2) \quad \sum \frac{1}{a} \geq \frac{3\sqrt{3}}{2(R+r)}.$$

Mitrinović et al. [2, p. 173] showed another sharpened form of (4.1), as follows:

$$(4.3) \quad \sum \frac{1}{a} \geq \frac{5R-r}{2R^2 + (3\sqrt{3}-4)Rr}.$$

Recently, a unified improvement of the inequalities (4.2) and (4.3) was given by Wu [15], that is,

$$(4.4) \quad \sum \frac{1}{a} \geq \frac{11\sqrt{3}}{5R + 12r + k_0(2r - R)},$$

where $k_0 = 0.02206078402\dots$. It is the root on the interval $(0, 1/15)$ of the following equation

$$405k^5 + 6705k^4 + 129586k^3 + 1050976k^2 + 2795373k - 62181 = 0.$$

We show here a new improvement of the inequalities (4.2) and (4.3), which is stated in Theorem 4.1 below.

Fundamental Triangle Inequality

Shan-He Wu and Mihály Bencze

vol. 10, iss. 1, art. 16, 2009

[Title Page](#)

[Contents](#)



Page 9 of 15

[Go Back](#)

[Full Screen](#)

[Close](#)

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756

© 2007 Victoria University. All rights reserved.



Title Page

Contents



Page 10 of 15

Go Back

Full Screen

Close

Theorem 4.1. For any triangle ABC the following inequality holds true:

$$(4.5) \quad \sum \frac{1}{a} \geq \frac{\sqrt{25Rr - 2r^2}}{4Rr},$$

with equality holding if and only if the triangle ABC is equilateral.

Proof. By using the identity (2.3) and the identities for an arbitrary triangle (see [2]):

$$(4.6) \quad \sum ab = s^2 + 4Rr + r^2, \quad abc = 4sRr,$$

we have

$$(4.7) \quad \begin{aligned} & \left(\sum \frac{1}{a} \right)^2 - \frac{25Rr - 2r^2}{16R^2r^2} \\ &= \frac{(s^2 + 4Rr + r^2)^2}{16s^2R^2r^2} - \frac{25Rr - 2r^2}{16R^2r^2} \\ &= \frac{R^2}{16s^2r^2} \left[\frac{s^4}{R^4} - \left(\frac{17r}{R} - \frac{4r^2}{R^2} \right) \frac{s^2}{R^2} + \left(\frac{4r}{R} + \frac{r^2}{R^2} \right)^2 \right] \\ &= \frac{R^2}{16s^2r^2} \left[\frac{s^4}{R^4} - \left(-\delta^4 + 4\delta^3 - \frac{25}{2}\delta^2 + 17\delta \right) \frac{s^2}{R^2} \right. \\ & \quad \left. + \frac{1}{16}(4 - \delta)^2(4 - \delta^2)^2\delta^2 \right]. \end{aligned}$$

Let

$$f \left(\frac{s^2}{R^2} \right) = \left(\frac{s^2}{R^2} \right)^2 - \left(-\delta^4 + 4\delta^3 - \frac{25}{2}\delta^2 + 17\delta \right) \left(\frac{s^2}{R^2} \right) + \frac{1}{16}(4 - \delta)^2(4 - \delta^2)^2\delta^2.$$



[Title Page](#)

[Contents](#)



Page 11 of 15

[Go Back](#)

[Full Screen](#)

[Close](#)

It is easy to see that the quadratic function

$$f(x) = x^2 - \left(-\delta^4 + 4\delta^3 - \frac{25}{2}\delta^2 + 17\delta\right)x + \frac{1}{16}(4 - \delta)^2(4 - \delta^2)^2\delta^2$$

is increasing on the interval $\left[\frac{1}{2}(-\delta^4 + 4\delta^3 - \frac{25}{2}\delta^2 + 17\delta), +\infty\right)$.

Now, from the inequalities

$$\frac{s^2}{R^2} \geq \frac{1}{4}\delta(4 - \delta)^3$$

and

$$\frac{1}{4}\delta(4 - \delta)^3 - \frac{1}{2}\left(-\delta^4 + 4\delta^3 - \frac{25}{2}\delta^2 + 17\delta\right) = \delta\left(\frac{15}{2} - \frac{23}{4}\delta\right) + \delta^3 + \frac{1}{4}\delta^4 > 0,$$

we deduce that

$$\begin{aligned} f\left(\frac{s^2}{R^2}\right) &\geq f\left(\frac{\delta(4 - \delta)^3}{4}\right) \\ &= \frac{1}{16}\delta^2(4 - \delta)^6 - \frac{1}{4}\delta(4 - \delta)^3\left(-\delta^4 + 4\delta^3 - \frac{25}{2}\delta^2 + 17\delta\right) \\ &\quad + \frac{1}{16}(4 - \delta)^2(4 - \delta^2)^2\delta^2 \\ &= \frac{1}{8}(4 - \delta)^2(1 - \delta)^2(6 - \delta)\delta^3 \\ &\geq 0. \end{aligned}$$

The above inequality with the identity (4.7) lead us to

$$\left(\sum \frac{1}{a}\right)^2 - \frac{25Rr - 2r^2}{16R^2r^2} \geq 0.$$



Title Page

Contents



Page 12 of 15

Go Back

Full Screen

Close

Theorem 4.1 is thus proved. □

Remark 1. The inequality (4.5) is stronger than the inequalities (4.1), (4.2) and (4.3) because from the Euler inequality $R \geq 2r$ it is easy to verify that the following inequalities hold for any triangle.

$$(4.8) \quad \sum \frac{1}{a} \geq \frac{\sqrt{25Rr - 2r^2}}{4Rr} \geq \frac{5R - r}{2R^2 + (3\sqrt{3} - 4)Rr} \geq \frac{\sqrt{3}}{R},$$

$$(4.9) \quad \sum \frac{1}{a} \geq \frac{\sqrt{25Rr - 2r^2}}{4Rr} \geq \frac{3\sqrt{3}}{2(R + r)} \geq \frac{\sqrt{3}}{R}.$$

In addition, it is worth noticing that inequalities (4.4) and (4.5) are incomparable in general, which can be observed from the following fact.

Letting $a = \sqrt{3}$, $b = 1$, $c = 1$, then $R = 1$, $r = \sqrt{3} - \frac{3}{2}$, direct calculation gives

$$\begin{aligned} & \frac{\sqrt{25Rr - 2r^2}}{4Rr} - \frac{11\sqrt{3}}{5R + 12r + k_0(2r - R)} \\ & > \frac{\sqrt{25Rr - 2r^2}}{4Rr} - \frac{11\sqrt{3}}{5R + 12r + 0.023(2r - R)} \\ & = 0.11934 \dots > 0. \end{aligned}$$

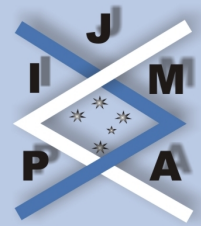
Letting $a = 2$, $b = \sqrt{2}$, $c = \sqrt{2}$, then $R = 1$, $r = \sqrt{2} - 1$, direct calculation gives

$$\begin{aligned} & \frac{\sqrt{25Rr - 2r^2}}{4Rr} - \frac{11\sqrt{3}}{5R + 12r + k_0(2r - R)} \\ & < \frac{\sqrt{25Rr - 2r^2}}{4Rr} - \frac{11\sqrt{3}}{5R + 12r + 0.022(2r - R)} \\ & = -0.00183 \dots < 0. \end{aligned}$$

As a further improvement of the inequality (4.5), we propose the following interesting problem:

Open Problem 4.3. *Determine the best constant k for which the inequality below holds*

$$(4.10) \quad \sum \frac{1}{a} \geq \frac{1}{4Rr} \sqrt{25Rr - 2r^2 + k \left(1 - \frac{2r}{R}\right) \frac{r^3}{R}}.$$



Fundamental Triangle Inequality

Shan-He Wu and Mihály Bencze

vol. 10, iss. 1, art. 16, 2009

[Title Page](#)

[Contents](#)



Page 13 of 15

[Go Back](#)

[Full Screen](#)

[Close](#)

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756

References

- [1] O. BOTTEMA, R.Z. DJORDJEVIĆ, R.R. JANIĆ, D.S. MITRINOVIĆ AND P.M. VASIĆ, *Geometric Inequalities*, Wolters-Noordhoff, Groningen, 1969.
- [2] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND V. VOLENEC, *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, Dordrecht, Netherlands, 1989.
- [3] D.S. MITRINOVIĆ, J.E. PEČARIĆ, V. VOLENEC AND J. CHEN, Addenda to the Monograph: Recent Advances in Geometric Inequalities, I, *J. Ningbo Univ.*, **4**(2) (1991), 1–145.
- [4] J.C. KUANG, *Applied Inequalities*, Third Ed., Shandong Science and Technology Press, Jinan, China, 2004 (in Chinese).
- [5] B.Q. LIU, *BOTTEMA, What We See*, Tibet People's Publishing House, Lasha, China, 2003 (in Chinese).
- [6] SH.-H. WU AND ZH.-H. ZHANG, A class of inequalities related to the angle bisectors and the sides of a triangle, *J. Inequal. Pure Appl. Math.*, **7**(3) (2006), Art. 108. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=698>].
- [7] SH.-H. WU AND ZH.-H. ZHANG, Some strengthened results on Gerretsen's inequality, *RGMA Res. Rep. Coll.*, **6**(3) (2003), Art. 16. [ONLINE: <http://rgmia.vu.edu.au/v6n3.html>].
- [8] SH.-L. CHEN, A simplified method to prove inequalities for triangle, *Research in Inequality*, Tibet People's Publishing House, Lasha, China, 2000 (in Chinese).



Fundamental Triangle Inequality

Shan-He Wu and Mihály Bencze

vol. 10, iss. 1, art. 16, 2009

Title Page

Contents



Page 14 of 15

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5755



- [9] SH.-L. CHEN, Certain inequalities for the elements R , r and s of an acute triangle, *Geometric Inequalities in China*, Jiangsu Educational Publishing House, Jiangsu, China, 1996 (in Chinese).
- [10] R.A. SATNOIAU, General power inequalities between the sides and the circumscribed and inscribed radii related to the fundamental triangle inequality, *Math. Inequal. Appl.*, **5**(4) (2002), 745–751.
- [11] SH.-H. WU, A sharpened version of the fundamental triangle inequality, *Math. Inequal. Appl.*, **11**(3) (2008), 477–482.
- [12] J. GARFUNKEL, Problem 825, *Crux Math.*, **9** (1983), 79.
- [13] L. BANKOFF, Solution of Problem 825, *Crux Math.*, **10** (1984), 168.
- [14] J. STEINIG, Inequalities concerning the inradius and circumradius of a triangle, *Elem. Math.*, **18** (1963), 127–131.
- [15] Y.D. WU, The best constant for a geometric inequality, *J. Inequal. Pure Appl. Math.*, **6**(4) (2005), Art. 111. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=585>].

Fundamental Triangle Inequality

Shan-He Wu and Mihály Bencze

vol. 10, iss. 1, art. 16, 2009

Title Page

Contents



Page 15 of 15

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756