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FOUR INEQUALITIES SIMILAR TO HARDY-HILBERT'S INTEGRAL INEQUALITY

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ABSTRACT. Four new different types of inequalities similar to Hardy-Hilbert's inequality are given.

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1. INTRODUCTION

Suppose that f and g are real functions, such that $0 < \int_0^\infty f^2(t)dt < \infty$ and $0 < \int_0^\infty g^2(t)dt < \infty$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} < \pi \left(\int_0^\infty f^2(t)dt \int_0^\infty g^2(t)dt \right)^{\frac{1}{2}}, \tag{1.1}$$

where π is best possible. If (a_n) and (b_n) are sequences of real numbers such that $0 < \sum_{n=1}^\infty a_n^2 < \infty$ and $0 < \sum_{n=1}^\infty b_n^2 < \infty$, then

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{m+n} < \pi \left(\sum_{n=1}^\infty a_n^2 \sum_{n=1}^\infty b_n^2 \right)^{\frac{1}{2}}. \tag{1.2}$$

The inequalities (1.1) and (1.2) are called Hilbert's inequalities. These inequalities play an important role in analysis (cf. [1, Chap. 9]). In their recent papers Hu [5] and Gao [3] gave two distinct improvements of (1.1) and Gao [4] gave a strengthened version of (1.2).

The following definitions are given:

$$\varphi_\lambda(r) = \frac{r+\lambda-2}{r} \; (r = p, q), \quad k_\lambda(p) = B\left(\varphi_\lambda(p), \varphi_\lambda(q)\right),$$

and B is the beta function.

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Recently, by introducing some parameters, Yang and Debnath [2] gave the following extensions:

Theorem A. *If* $f, g \geq 0$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 2 - \min\{p, q\}$, *such that*

$$0 < \int_0^\infty t^{1-\lambda} f^p(t)dt < \infty \quad \textit{and} \quad 0 < \int_0^\infty t^{1-\lambda} g^q(t)dt < \infty,$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(Ax+By)^\lambda} dxdy < \frac{k_\lambda(p)}{A^{\varphi_\lambda(p)} B^{\varphi_\lambda(q)}} \left(\int_0^\infty x^{1-\lambda} f^p(x)dx \right)^{\frac{1}{p}} \left(\int_0^\infty y^{1-\lambda} g^q(y)dy \right)^{\frac{1}{q}}, \tag{1.3}$$

where the constant factor $[k_\lambda(p)/A^{\varphi_\lambda(p)} B^{\varphi_\lambda(q)}]$ *is the best possible.*

Theorem B. *If* $f \geq 0$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 2 - \min\{p, q\}$, $A, B > 0$ *such that* $0 < \int_0^\infty t^{1-\lambda} f^p(t)dt < \infty$, *then*

$$\int_0^\infty y^{(\lambda-1)(p-1)} \left(\int_0^\infty \frac{f(x)}{(Ax+By)^\lambda} dx \right)^p dy < \left(\frac{k_\lambda(p)}{A^{\varphi_\lambda(p)} B^{\varphi_\lambda(q)}} \right)^p \int_0^\infty x^{1-\lambda} f^p(x)dx, \tag{1.4}$$

where the constant factor $\left[k_\lambda(p)/A^{\varphi_\lambda(p)} B^{\varphi_\lambda(q)}\right]^p$ *is the best possible. The inequalities (1.3) and (1.4) are equivalent.*

Theorem C. *If* $a_n, b_n > 0$ $(n \in \mathbb{N})$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $2 - \min\{p, q\} < \lambda < 2$, $A, B > 0$ *such that*

$$0 < \sum_{n=1}^\infty n^{1-\lambda} a_n^p < \infty \quad \textit{and} \quad 0 < \sum_{n=1}^\infty n^{1-\lambda} b_n^q < \infty,$$

then

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{(Am+Bn)^\lambda} < \frac{k_\lambda(p)}{A^{\varphi_\lambda(p)} B^{\varphi_\lambda(q)}} \left(\sum_{n=1}^\infty n^{1-\lambda} a_n^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^\infty n^{1-\lambda} b_n^q \right)^{\frac{1}{q}}, \tag{1.5}$$

where the constant factor $\left[k_\lambda(p)/A^{\varphi_\lambda(p)} B^{\varphi_\lambda(q)}\right]$ *is the best possible.*

Theorem D. *If* $a_n \geq 0$ $(n \in \mathbb{N})$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $2 - \min\{p, q\} < \lambda \leq 2$, $A, B > 0$ *such that* $0 < \sum_{n=1}^\infty n^{1-\lambda} a_n^p < \infty$, *then*

$$\sum_{n=1}^\infty n^{(\lambda-1)(p-1)} \left(\sum_{m=1}^\infty \frac{a_m}{(Am+Bn)^\lambda} \right)^p < \left(\frac{k_\lambda(p)}{A^{\varphi_\lambda(p)} B^{\varphi_\lambda(q)}} \right)^p \sum_{n=1}^\infty n^{1-\lambda} a_n^p, \tag{1.6}$$

where the constant factor $\left[k_\lambda(p)/A^{\varphi_\lambda(p)} B^{\varphi_\lambda(q)}\right]^p$ *is the best possible. The inequalities (1.5) and (1.6) are equivalent.*

2. NEW INEQUALITIES

The aim of this paper is to give the following results:

Theorem 2.1. *Let* $\ln f$*,* $\ln g$ *be convex for nonnegative functions* f *and* g *such that* $f(0) = g(0) = 0$*,* $f(\infty) = g(\infty) = \infty$*,* $f'(s) \geq 0$*,* $g'(s) \geq 0$*,* $s \in \{x^p, y^q\}$*. Let* $\lambda > \max\{p,q\}$*,* $p > 1$*,* $\frac{1}{p} + \frac{1}{q} = 1$*. Let*

$$0 < \int_0^\infty \frac{t^{-p^2/q^2}\left[f(t^p)\right]^{2-\lambda+p/q}}{\left[f'(t)\right]^{\frac{p}{q}}}dt < \infty, \quad 0 < \int_0^\infty \frac{t^{-q^2/p^2}\left[g(t^q)\right]^{2-\lambda+q/p}}{\left[g'(t)\right]^{\frac{q}{p}}}dt < \infty,$$

then we have

$$\begin{aligned}(2.1)\quad \int_0^\infty\int_0^\infty \frac{f(xy)g(xy)}{\left(f(x^p)+g(y^q)\right)^\lambda}dxdy &\leq \frac{1}{\sqrt[p]{p}\sqrt[q]{q}}B^{\frac{1}{p}}(p,\lambda-p)\,B^{\frac{1}{q}}(q,\lambda-q)\\ &\quad\times\left(\int_0^\infty \frac{t^{-p^2/q^2}\left[f(t^p)\right]^{2-\lambda+p/q}}{\left[f'(t)\right]^{\frac{p}{q}}}dt\right)^{\frac{1}{p}}\left(\int_0^\infty \frac{t^{-q^2/p^2}\left[g(t^q)\right]^{2-\lambda+q/p}}{\left[g'(t)\right]^{\frac{q}{p}}}dt\right)^{\frac{1}{q}}.\end{aligned}$$

Proof. Since $\ln f$ is convex and $xy \leq \frac{x^p}{p} + \frac{y^q}{q}$, then

$$f(xy) = e^{\ln f(xy)} \leq e^{\ln f\left(\frac{x^p}{p}+\frac{y^q}{q}\right)} \leq e^{\frac{\ln f(x^p)}{p}+\frac{\ln f(y^q)}{q}} = f^{\frac{1}{p}}(x^p)f^{\frac{1}{q}}(y^q).$$

Therefore, we have

$$\begin{aligned}\int_0^\infty\int_0^\infty &\frac{f(xy)g(xy)}{\left(f(x^p)+g(y^q)\right)^\lambda}dxdy\\ &\leq \int_0^\infty\int_0^\infty \frac{f^{\frac{1}{p}}(x^p)g^{\frac{1}{q}}(y^q)\frac{[g'(y^q)]^{\frac{1}{p}}}{[f'(x^p)]^{\frac{1}{q}}}\frac{y^{\frac{q-1}{p}}}{x^{\frac{p-1}{q}}}}{\left(f(x^p)+g(y^q)\right)^{\frac{\lambda}{p}}}\frac{f^{\frac{1}{q}}(y^q)g^{\frac{1}{p}}(x^p)\frac{[f'(x^p)]^{\frac{1}{q}}}{[g'(y^q)]^{\frac{1}{p}}}\frac{x^{\frac{p-1}{q}}}{y^{\frac{q-1}{p}}}}{\left(f(x^p)+g(y^q)\right)^{\frac{\lambda}{q}}}dxdy\\ &\leq \left(\int_0^\infty\int_0^\infty \frac{f(x^p)g^{p/q}(y^q)g'(y^q)y^{q-1}}{x^{(p-1)p/q}\left[f'(x^p)\right]^{\frac{p}{q}}\left(f(x^p)+g(y^q)\right)^\lambda}dxdy\right)^{\frac{1}{p}}\\ &\qquad\times\left(\int_0^\infty\int_0^\infty \frac{f(y^q)g^{q/p}(x^p)f'(x^p)x^{p-1}}{y^{(q-1)q/p}\left[g'(y^q)\right]^{\frac{q}{p}}\left(f(x^p)+g(y^q)\right)^\lambda}dxdy\right)^{\frac{1}{q}}\\ &= M^{\frac{1}{p}}N^{\frac{1}{q}}, \quad \text{say.}\end{aligned}$$

Then

$$\begin{aligned}M &= \frac{1}{q}\int_0^\infty \frac{x^{(1-p)p/q}\left[f(x^p)\right]^{2-\lambda+p/q}}{\left[f'(x)\right]^{\frac{p}{q}}}dx\int_0^\infty \frac{\left(\frac{g(y^q)}{f(x^p)}\right)^{\frac{p}{q}}g'(y^q)\frac{qy^{q-1}}{f(x^p)}}{\left(1+\frac{g(y^q)}{f(x^p)}\right)^\lambda}dy\\ &= \frac{1}{q}\int_0^\infty \frac{x^{-p^2/q^2}\left[f(x^p)\right]^{2-\lambda+p/q}}{\left[f'(x)\right]^{\frac{p}{q}}}dx\int_0^\infty \frac{u^{p/q}}{(1+u)^\lambda}du\\ &= \frac{1}{q}B(p,\lambda-p)\int_0^\infty \frac{x^{-p^2/q^2}\left[f(x^p)\right]^{2-\lambda+p/q}}{\left[f'(x)\right]^{\frac{p}{q}}}dx.\end{aligned}$$

Similarly,

$$N = \frac{1}{p}B(q,\lambda-q)\int_0^\infty \frac{y^{-q^2/p^2}\left[g(y^q)\right]^{2-\lambda+q/p}}{\left[g'(y)\right]^{\frac{q}{p}}}dy.$$

Therefore

$$\int_0^\infty \int_0^\infty \frac{f(xy)g(xy)}{(f(x^p)+g(y^q))^\lambda}dxdy$$
$$\leq \frac{1}{\sqrt[q]{p}\sqrt[p]{q}}B^{\frac{1}{p}}(p,\lambda-p)\,B^{\frac{1}{q}}(q,\lambda-q)$$
$$\times\left(\int_0^\infty t\frac{t^{-p^2/q^2}\left[f(t^p)\right]^{2-\lambda+p/q}}{\left[f'(t)\right]^{\frac{p}{q}}}dt\right)^{\frac{1}{p}}\left(\int_0^\infty \frac{t^{-q^2/p^2}\left[g(t^q)\right]^{2-\lambda+q/p}}{\left[g'(t)\right]^{\frac{q}{p}}}dt\right)^{\frac{1}{q}}.$$

□

Theorem 2.2. *Let* f,g,h *be nonnegative functions,* $h(x,y)$ *is homogeneous of order* n *such that* $h(0,1)=h(1,0)=0$*,* $h(\infty,1)=h(1,\infty)=\infty$*. Let* $p>1$*,* $\frac{1}{p}+\frac{1}{q}=1$*,*

$$0<1+\mu-r<\lambda,\quad r\in\left\{\frac{p}{q},\frac{q}{p}\right\},\quad h_x(x,y)\geq 0,\quad h_y(x,y)\geq 0,$$

where $h_x=dh/dx$, $0<\int_0^\infty t^{p/q}f^p(t)dt<\infty$, $0<\int_0^\infty t^{q/p}g^q(t)dt<\infty$, *then*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)h^\mu(x,y)}{(1+h(x,y))^\lambda}dxdy \tag{2.2}$$
$$\leq \frac{1}{n}B^{\frac{1}{p}}\left(1+\mu-\frac{p}{q},\lambda-1-\mu+\frac{p}{q}\right)B^{\frac{1}{q}}\left(1+\mu-\frac{q}{p},\lambda-1-\mu-\frac{q}{p}\right)$$
$$\times\left(\int_0^\infty t^{\frac{p}{q-1}}f^p(t)dt\right)^{\frac{1}{p}}\left(\int_0^\infty t^{\frac{q}{p-1}}g^q(t)dt\right)^{\frac{1}{q}}.$$

Proof. Observe that

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)h^\mu(x,y)}{(1+h(x,y))^\lambda}dxdy$$
$$\leq\left(\int_0^\infty \int_0^\infty \frac{f^p(x)h^\mu(x,y)h_y(x,y)}{h_x^{p/q}(x,y)\,(1+h(x,y))^\lambda}dxdy\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^\infty \int_0^\infty \frac{g^q(y)h^\mu(x,y)h_x(x,y)}{h_y^{q/p}(x,y)\,(1+h(x,y))^\lambda}dxdy\right)^{\frac{1}{q}}$$
$$=M^{\frac{1}{p}}N^{\frac{1}{q}},\quad \text{say.}$$

Then

$$M=\int_0^\infty f^p(x)dx\int_0^\infty \frac{h^\mu(x,y)h_y(x,y)}{h_x^{p/q}(x,y)\,(1+h(x,y))^\lambda}dy.$$

Let $y=xv$, $dy=xdv$ and hence

$$h_y(x,y)=\frac{dh(x,xv)}{dy}=x^n\frac{dh(1,v)}{dy}=x^{n-1}\frac{dh(1,v)}{dv}=x^{n-1}h_v(1,v),$$
$$h_x(x,y)=\frac{dh(x,xv)}{dx}=\frac{d}{dx}x^nh(1,v)=nx^{n-1}h(1,v),$$

therefore

$$\begin{aligned} M &= \frac{1}{n^{p/q}} \int_0^\infty x^{p/q-1} f^p(x)dx \int_0^\infty \frac{[x^n h(1,v)]^{\mu-p/q}\, x^n h_v(1,v)}{(1+x^n h(1,v))^\lambda} dv \\ &= \frac{1}{n^{p/q}} B\left(1+\mu-\frac{p}{q}, \lambda-1-\mu+\frac{p}{q}\right) \int_0^\infty x^{p/q-1} f^p(x)dx. \end{aligned}$$

Similarly,

$$N = \frac{1}{n^{q/p}} B\left(1+\mu-\frac{q}{p}, \lambda-1-\mu+\frac{q}{p}\right) \int_0^\infty y^{q/p-1} g^q(y)dy.$$

This implies

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{f(x)g(y)h^\mu(x,y)}{(1+h(x,y))^\lambda} dxdy \\ &\qquad \le \frac{1}{n} B^{\frac{1}{p}}\left(1+\mu-\frac{p}{q}, \lambda-1-\mu+\frac{p}{q}\right) B^{\frac{1}{q}}\left(1+\mu-\frac{q}{p}, \lambda-1-\mu+\frac{q}{p}\right) \\ &\qquad\qquad \times \left(\int_0^\infty t^{p/q-1} f^p(t)dt\right)^{\frac{1}{p}} \left(\int_0^\infty t^{q/p-1} g^q(t)dt\right)^{\frac{1}{q}}. \end{aligned}$$

□

The following lemma is needed for the coming result.

Lemma 2.3. *Let* $s \ge 1$, $0 < 1+\mu \le \min\{\alpha, \lambda\}$ *and define*

$$f(s) = s^{-\alpha} \int_0^s \frac{t^\mu}{(1+t)^\lambda} dt,$$

then $f(s) \le f(1)$.

Proof. We have

$$\begin{aligned} f'(s) &= s^{-\alpha} \frac{s^\mu}{(1+s)^\lambda} + \int_0^s \frac{t^\mu}{(1+t)^\lambda} dt\,(-\alpha)\, s^{-\alpha-1} \\ &\le \frac{s^{\mu-\alpha}}{(1+s)^\lambda} - \frac{\alpha s^{-\alpha-1}}{(1+s)^\lambda} \int_0^s t^\mu dt \\ &= \frac{s^{\mu-\alpha}}{(1+s)^\lambda} \left(1 - \frac{\alpha}{1+\mu}\right) \le 0. \end{aligned}$$

This shows that f is nonincreasing and hence $f(s) \le f(1)$. □

Theorem 2.4. *Let* f, g, F, G *be nonnegative functions such that* $F(s) = \int_0^s f(t)dt$, $G(s) = \int_0^s g(t)dt$, *let,* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $(1-\lambda/2)r \le \lambda/2 + \alpha \le 2\alpha$, $r \in \left\{\frac{p}{q}, \frac{q}{p}\right\}$,

$$\begin{aligned} 0 &< \int_0^x (x-t) t^{(1-\lambda/2)p/q - \lambda/2 - \alpha} F^{p-1}(t) f(t) dt < \infty, \\ 0 &< \int_0^x (x-t) t^{(1-\lambda/2)q/p - \lambda/2 - \alpha} G^{q-1}(t) g(t) dt < \infty, \end{aligned}$$

then

$$\int_0^x\int_0^x \frac{F(s)G(t)}{(s+t)^\lambda}dsdt \leq \frac{\sqrt[p]{p}\sqrt[q]{q}}{2}x^\alpha B\left(\tfrac{\lambda}{2},\tfrac{\lambda}{2}\right)\left(\int_0^x (x-t)t^{(1-\lambda/2)p/q-\lambda/2-\alpha}F^{p-1}(t)f(t)dt\right)^{\frac{1}{p}} \times\left(\int_0^x (x-t)t^{(1-\lambda/2)q/p-\lambda/2-\alpha}G^{q-1}(t)g(t)dt\right)^{\frac{1}{q}}. \tag{2.3}$$

Proof. Observe that

$$\begin{aligned}
&\int_0^x\int_0^x \frac{F(s)G(t)}{(s+t)^\lambda}dsdt\\
&= \int_0^x\int_0^x \frac{F(s)\left(\frac{t^{1/p}}{s^{1/q}}\right)^{\lambda/2-1}}{(s+t)^{\lambda/p}}\cdot\frac{G(t)\left(\frac{s^{1/q}}{t^{1/p}}\right)^{\lambda/2-1}}{(s+t)^{\lambda/q}}dsdt\\
&\leq \left(\int_0^x\int_0^x \frac{F^p(s)t^{\lambda/2-1}}{(s+t)^\lambda s^{(\lambda/2-1)p/q}}dsdt\right)^{\frac{1}{p}}\left(\int_0^x\int_0^x \frac{G^q(t)s^{\lambda/2-1}}{(s+t)^\lambda t^{(\lambda/2-1)q/p}}dsdt\right)^{\frac{1}{q}}\\
&= M^{\frac{1}{p}}N^{\frac{1}{q}}, \qquad \text{say.}
\end{aligned}$$

Then

$$\begin{aligned}
M &= \int_0^x s^{(1-\lambda/2)p/q-\lambda/2}F^p(s)ds\int_0^x \frac{\left(\frac{t}{s}\right)^{\lambda/2-1}\frac{1}{s}}{\left(1+\frac{t}{s}\right)^\lambda}dt\\
&= \int_0^x s^{(1-\lambda/2)p/q-\lambda/2}F^p(s)ds\left(\frac{x}{s}\right)^\alpha\left(\frac{x}{s}\right)^{-\alpha}\int_0^{x/s}\frac{u^{\lambda/2-1}}{(1+u)^\lambda}du\\
&\leq \int_0^x s^{(1-\lambda/2)p/q-\lambda/2}F^p(s)ds\left(\frac{x}{s}\right)^\alpha\int_0^1\frac{u^{\lambda/2-1}}{(1+u)^\lambda}du\\
&= \frac{x^\alpha}{2}B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\int_0^x s^{(1-\lambda/2)p/q-\lambda/2-\alpha}F^p(s)ds,
\end{aligned}$$

by virtue of the lemma.

As

$$F^p(s) = p\int_0^s F^{p-1}(u)f(u)du,$$

then

$$\begin{aligned}
M &= \frac{p}{2}x^\alpha B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\int_0^x s^{(1-\lambda/2)p/q-\lambda/2-\alpha}ds\int_0^s F^{p-1}(u)f(u)du\\
&= \frac{p}{2}x^\alpha B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\int_0^x\int_0^s u^{(1-\lambda/2)p/q-\lambda/2-\alpha}F^{p-1}(u)f(u)dsdu\\
&= \frac{p}{2}x^\alpha B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\int_0^x (x-s)s^{(1-\lambda/2)p/q-\lambda/2-\alpha}F^{p-1}(s)f(s)ds.
\end{aligned}$$

Similarly,

$$N = \frac{q}{2}x^\alpha B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\int_0^x (x-t)t^{(1-\lambda/2)q/p-\lambda/2-\alpha}G^{q-1}(t)g(t)dt.$$

Therefore, we have

$$\int_0^x \int_0^x \frac{F(s)G(t)}{(s+t)^\lambda} dsdt \\ \leq \frac{\sqrt[p]{p}\sqrt[q]{q}}{2} x^\alpha B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left(\int_0^x (x-t)t^{(1-\lambda/2)p/q-\lambda/2-\alpha}F^{p-1}(t)f(t)dt\right)^{\frac{1}{p}} \\ \times\left(\int_0^x (x-t)t^{(1-\lambda/2)q/p-\lambda/2-\alpha}G^{q-1}(t)g(t)dt\right)^{\frac{1}{q}}.$$

□

Theorem 2.5. *Let* f, g *be nonnegative functions,* f *is submultiplicative and* g *is concave non-increasing,* $f'(x), g'(y) \geq 0$, $f(0) = g(0) = 0$, $f(\infty) = g(\infty) = \infty$, $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $0 < a+1 < \lambda, 0 < b+1 < \lambda$,

$$0 < \int_0^\infty \frac{[f(x)]^{\mu p - bp/q}[g(x)]^{1+a-\lambda}}{[f'(x)]^{\frac{p}{q}}}dx < \infty, \quad 0 < \int_0^\infty \frac{[g(y)]^{\mu q - aq/p}[f(y)]^{1+b-\lambda}}{[g'(y)]^{\frac{q}{p}}}dy < \infty,$$

then

$$\int_0^\infty \int_0^\infty \frac{f^\mu(xy)}{g^{\lambda/2}(xy)}dxdy \leq 2^{\lambda/2} B^{\frac{1}{p}}(a+1, \lambda-a-1)\, B^{\frac{1}{q}}(b+1,\lambda-b-1) \\ \times\left(\int_0^\infty \frac{[f(t)]^{\mu p}[g(t)]^{1+a-\lambda-bp/q}}{[g'(t)]^{\frac{p}{q}}}dt\right)^{\frac{1}{p}}\left(\int_0^\infty \frac{[f(t)]^{\mu q}[g(t)]^{1+b-\lambda-aq/p}}{[g'(t)]^{\frac{q}{p}}}dt\right)^{\frac{1}{q}}. \tag{2.4}$$

Proof. Since $\sqrt{xy} \leq \frac{x+y}{2}$, then

$$g(xy) = g\left((\sqrt{xy})^2\right) \geq (g(\sqrt{xy}))^2 \geq \left(g\left(\frac{x+y}{2}\right)\right)^2 \geq \left(\frac{g(x)+g(y)}{2}\right)^2,$$

and hence

$$\int_0^\infty\int_0^\infty \frac{f^\mu(xy)}{g^{\lambda/2}(xy)}dxdy \\ \leq 2^{\lambda/2}\int_0^\infty\int_0^\infty \frac{f^\mu(x)f^\mu(y)}{(g(x)+g(y))^\lambda}dxdy \\ = 2^{\lambda/2}\int_0^\infty\int_0^\infty \frac{f^\mu(x)\frac{[g(y)]^{a/p}}{[g(x)]^{b/q}}\frac{[g'(y)]^{1/p}}{[g'(x)]^{1/q}}}{(g(x)+g(y))^{\frac{\lambda}{p}}} \cdot \frac{f^\mu(y)\frac{[g(x)]^{b/q}}{[g(y)]^{a/p}}\cdot\frac{[g'(x)]^{1/q}}{[g'(y)]^{1/p}}}{(g(x)+g(y))^{\frac{\lambda}{q}}} \\ \leq 2^{\lambda/2}\left(\int_0^\infty\int_0^\infty \frac{f^{\mu p}(x)g^a(y)g'(y)}{[g(x)]^{bp/q}[g'(x)]^{\frac{p}{q}}(g(x)+g(y))^\lambda}dxdy\right)^{\frac{1}{p}} \\ \times\left(\int_0^\infty\int_0^\infty \frac{f^{\mu q}(y)g^b(x)g'(x)}{[g(y)]^{aq/p}[g'(y)]^{\frac{q}{p}}(g(x)+g(y))^\lambda}dxdy\right)^{\frac{1}{q}} \\ = 2^{\lambda/2}M^{\frac{1}{p}}N^{\frac{1}{q}}, \qquad \text{say.}$$

Then

$$M = \int_0^\infty \frac{[f(x)]^{\mu p} [g(x)]^{1+a-\lambda-bp/q}}{[g'(x)]^{\frac{p}{q}}} dx \int_0^\infty \frac{\left(\frac{g(y)}{g(x)}\right)^a \frac{g'(y)}{g(x)}}{\left(1+\frac{g(y)}{g(x)}\right)^\lambda} dy$$

$$= B(a+1, \lambda-a-1) \int_0^\infty \frac{[f(x)]^{\eta p} [g(x)]^{1+a-\lambda-bp/q}}{[g'(x)]^{\frac{p}{q}}} dx.$$

Similarly, we can show that

$$N = B(b+1, \lambda-b-1) \int_0^\infty \frac{[f(y)]^{\mu q} [g(y)]^{1+b-\lambda-aq/p}}{[g'(y)]^{\frac{q}{p}}} dy.$$

The result follows. □

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