Journal of Inequalities in Pure and Applied Mathematics

A NOTE ON ABSOLUTE NÖRLUND SUMMABILITY

HÜSEYİN BOR

Department of Mathematics
Erciyes University
38039 Kayseri, Turkey

EMail: bor@erciyes.edu.tr

Abstract

Contents

◀◀   ▶▶

◀   ▶

Home Page

Go Back

Close

Quit

Abstract

In this paper a main theorem on $|N, p_n|_k$ summability factors, which generalizes a result of Bor [2] on $|N, p_n|$ summability factors, has been proved.

Contents

Title Page

Contents

◀◀   ▶▶

◀   ▶

Go Back

Close

Quit

J. Ineq. Pure and Appl. Math. **6**(4) Art. 115, 2005

http://jipam.vu.edu.au

1. Introduction

A positive sequence (b_n) is said to be almost increasing if there exist a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). A positive sequence (γ_n) is said to be a quasi β-power increasing sequence if there exists a constant $K = K(\beta, \gamma) \geq 1$ such that

$$(1.1) \qquad Kn^\beta \gamma_n \geq m^\beta \gamma_m$$

holds for all $n \geq m \geq 1$. It should be noted that every almost increasing sequence is a quasi β-power increasing sequence for any nonnegative β, but the converse need not be true as can be seen by taking the example, say $\gamma_n = n^{-\beta}$ for $\beta > 0$. We denote by $\mathcal{BV}_\mathcal{O}$ the $\mathcal{BV} \cap \mathcal{C}_\mathcal{O}$, where $\mathcal{C}_\mathcal{O}$ and $\mathcal{BV}$ are the null sequences and sequences with bounded variation, respectively.

Let $\sum a_n$ be a given infinite series with the sequence of partial sums (s_n) and $w_n = na_n$. By u_n^α and t_n^α we denote the n-th Cesàro means of order α, with $\alpha > -1$, of the sequences (s_n) and (w_n), respectively.

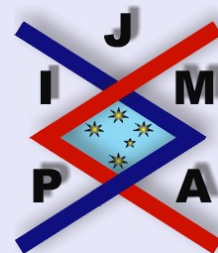The series $\sum a_n$ is said to be summable $|C, \alpha|_k$, $k \geq 1$, if (see [4])

$$(1.2) \qquad \sum_{n=1}^\infty n^{k-1} \left| u_n^\alpha - u_{n-1}^\alpha \right|^k = \sum_{n=1}^\infty \frac{1}{n} \left| t_n^\alpha \right|^k < \infty.$$

Let (p_n) be a sequence of constants, real or complex, and let us write

$$(1.3) \qquad P_n = p_0 + p_1 + p_2 + \cdots + p_n \neq 0, \ (n \geq 0).$$

The sequence-to-sequence transformation

$$(1.4) \qquad \sigma_n = \frac{1}{P_n} \sum_{v=0}^n p_{n-v} s_v$$

Title Page

Contents

◀◀   ▶▶

◀   ▶

Go Back

Close

Quit

defines the sequence (σ_n) of the Nörlund mean of the sequence (s_n), generated by the sequence of coefficients (p_n). The series $\sum a_n$ is said to be summable $|N, p_n|_k$, $k \geq 1$, if (see [3])

$$(1.5) \qquad \sum_{n=1}^{\infty} n^{k-1} |\sigma_n - \sigma_{n-1}|^k < \infty.$$

In the special case when

$$(1.6) \qquad p_n = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}, \ \alpha \geq 0$$

the Nörlund mean reduces to the (C, α) mean and $|N, p_n|_k$ summability becomes $|C, \alpha|_k$ summability. For $p_n = 1$ and $P_n = n$, we get the $(C, 1)$ mean and then $|N, p_n|_k$ summability becomes $|C, 1|_k$ summability. For any sequence (λ_n), we write $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$.

The known results. Concerning the $|C, 1|_k$ and $|N, p_n|_k$ summabilities Varma [6] has proved the following theorem.

Theorem A. *Let $p_0 > 0$, $p_n \geq 0$ and (p_n) be a non-increasing sequence. If $\sum a_n$ is summable $|C, 1|_k$, then the series $\sum a_n P_n (n+1)^{-1}$ is summable $|N, p_n|_k$, $k \geq 1$.*

Quite recently Bor [2] has proved the following theorem.

Theorem B. *Let (p_n) be as in Theorem A, and let (X_n) be a quasi β-power increasing sequence with some $0 < \beta < 1$. If*

$$(1.7) \qquad \sum_{v=1}^{n} \frac{1}{v} |t_v| = O(X_n) \quad as \ n \to \infty,$$
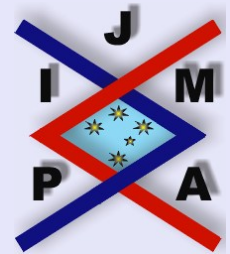
A Note on Absolute Nörlund Summability

Hüseyin Bor

Title Page

Contents

◀◀   ▶▶

◀   ▶

Go Back

Close

Quit

Page 4 of 10

and the sequences (λ_n) *and* (β_n) *satisfy the following conditions*

(1.8) $$X_n \lambda_n = O(1),$$

(1.9) $$|\Delta \lambda_n| \leq \beta_n,$$

(1.10) $$\beta_n \to 0,$$

(1.11) $$\sum n X_n |\Delta \beta_n| < \infty,$$

then the series $\sum a_n P_n \lambda_n (n+1)^{-1}$ *is summable* $|N, p_n|$.
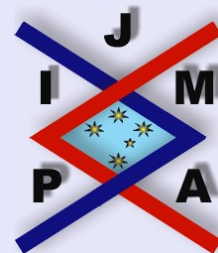
Title Page

Contents

◀◀   ▶▶

◀   ▶

Go Back

Close

Quit

2. Main Result

The aim of this paper is to generalize Theorem B for $|N, p_n|_k$ summability. Now we shall prove the following theorem.

Theorem 2.1. *Let (p_n) be as in Theorem A, and let (X_n) be a quasi β-power increasing sequence with some $0 < \beta < 1$. If*

$$(2.1) \qquad \sum_{v=1}^{n} \frac{1}{v} |t_v|^k = O(X_n) \quad as \ n \to \infty,$$

and the sequences (λ_n) and (β_n) satisfy the conditions from (1.8) to (1.11) of Theorem B; further suppose that

$$(2.2) \qquad (\lambda_n) \in \mathcal{BV_O},$$

*then the series $\sum a_n P_n \lambda_n (n+1)^{-1}$ is summable $|N, p_n|_k$, $k \geq 1$.*

Remark 1. *It should be noted that if we take $k = 1$, then we get Theorem B. In this case condition (2.2) is not needed.*

We need the following lemma for the proof of our theorem.

Lemma 2.2 ([5]). *Except for the condition (2.2), under the conditions on (X_n), (λ_n) and (β_n) as taken in the statement of the theorem, the following conditions hold when (1.11) is satisfied:*

$$(2.3) \qquad n\beta_n X_n = O(1) \ as \ n \to \infty,$$

$$(2.4) \qquad \sum_{n=1}^{\infty} \beta_n X_n < \infty.$$
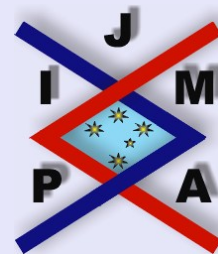
Title Page

Contents

◀◀　　▶▶

◀　　▶

Go Back

Close

Quit

3. Proof of Theorem 2.1

In order to prove the theorem, we need consider only the special case in which (N, p_n) is $(C, 1)$, that is, we shall prove that $\sum a_n \lambda_n$ is summable $|C, 1|_k$. Our theorem will then follow by means of Theorem A. Let T_n be the $n-$th $(C, 1)$ mean of the sequence $(n a_n \lambda_n)$, that is,

$$(3.1) \qquad T_n = \frac{1}{n+1} \sum_{v=1}^{n} v a_v \lambda_v.$$

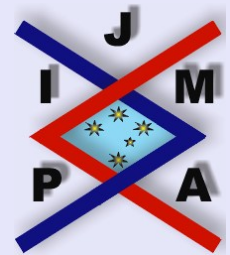Using Abel's transformation, we have

$$T_n = \frac{1}{n+1} \sum_{v=1}^{n} v a_v \lambda_v = \frac{1}{n+1} \sum_{v=1}^{n-1} \Delta \lambda_v (v+1) t_v + \lambda_n t_n$$
$$= T_{n,1} + T_{n,2}, \text{ say.}$$

To complete the proof of the theorem, it is sufficient to show that

$$(3.2) \qquad \sum_{n=1}^{\infty} \frac{1}{n} |T_{n,r}|^k < \infty \quad \text{for } r = 1, 2, \text{ by (1.2).}$$

Now, we have that

$$\sum_{n=2}^{m+1} \frac{1}{n} |T_{n,1}|^k \leq \sum_{n=2}^{m+1} \frac{1}{n(n+1)^k} \left\{ \sum_{v=1}^{n-1} \frac{v+1}{v} v |\Delta \lambda_v| |t_v| \right\}^k$$
$$= O(1) \sum_{n=2}^{m+1} \frac{1}{n^{k+1}} \left\{ \sum_{v=1}^{n-1} v |\Delta \lambda_v| |t_v| \right\}^k$$

Title Page

Contents

◀◀   ▶▶

◀   ▶

Go Back

Close

Quit

$$= O(1) \sum_{n=2}^{m+1} \frac{1}{n^2} \left\{ \sum_{v=1}^{n-1} v \left| \Delta \lambda_v \right| \left| t_v \right|^k \right\} \times \left\{ \frac{1}{n} \sum_{v=1}^{n-1} v \left| \Delta \lambda_v \right| \right\}^{k-1}$$

$$= O(1) \sum_{n=2}^{m+1} \frac{1}{n^2} \sum_{v=1}^{n-1} v \left| \Delta \lambda_v \right| \left| t_v \right|^k \quad \text{(by (2.2))}$$

$$= O(1) \sum_{n=2}^{m+1} \frac{1}{n^2} \left\{ \sum_{v=1}^{n-1} v \beta_v \left| t_v \right|^k \right\} \quad \text{(by (1.9))}$$

$$= O(1) \sum_{v=1}^{m} v \beta_v \left| t_v \right|^k \sum_{n=v+1}^{m+1} \frac{1}{n^2} = O(1) \sum_{v=1}^{m} v \beta_v \frac{\left| t_v \right|^k}{v}$$

$$= O(1) \sum_{v=1}^{m-1} \Delta(v \beta_v) \sum_{r=1}^{v} \frac{\left| t_r \right|^k}{r} + O(1) m \beta_m \sum_{v=1}^{m} \frac{\left| t_v \right|^k}{v}$$

$$= O(1) \sum_{v=1}^{m-1} \left| \Delta(v \beta_v) \right| X_v + O(1) m \beta_m X_m \quad \text{(by (2.1))}$$

$$= O(1) \sum_{v=1}^{m-1} \left| (v+1) \Delta \beta_v - \beta_v \right| X_v + O(1) m \beta_m X_m$$

$$= O(1) \sum_{v=1}^{m-1} v \left| \Delta \beta_v \right| X_v + O(1) \sum_{v=1}^{m-1} \left| \beta_v \right| X_v + O(1) m \beta_m X_m$$

$$= O(1) \quad \text{as } m \to \infty,$$

in view of (1.11), (2.3) and (2.4).
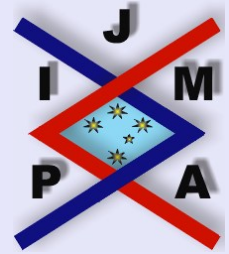
A Note on Absolute Nörlund Summability

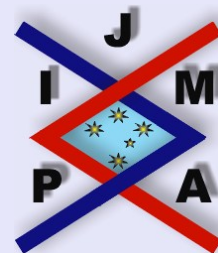Hüseyin Bor

Title Page

Contents

◀◀　　▶▶

◀　　▶

Go Back

Close

Quit

Page 8 of 10

J. Ineq. Pure and Appl. Math. **6**(4) Art. 115, 2005

http://jipam.vu.edu.au

Again

$$\sum_{n=1}^{m} \frac{1}{n} |T_{n,2}|^k = \sum_{n=1}^{m} |\lambda_n|^k \frac{|t_n|^k}{n}$$

$$= \sum_{n=1}^{m} |\lambda_n|^{k-1} |\lambda_n| \frac{|t_n|^k}{n} = O(1) \sum_{n=1}^{m} |\lambda_n| \frac{|t_n|^k}{n} \quad \text{(by (2.2))}$$

$$= O(1) \sum_{n=1}^{m-1} \Delta |\lambda_n| \sum_{v=1}^{n} \frac{|t_v|^k}{v} + O(1) |\lambda_m| \sum_{n=1}^{m} \frac{|t_n|^k}{n}$$

$$= O(1) \sum_{n=1}^{m-1} |\Delta \lambda_n| X_n + O(1) |\lambda_m| X_m \quad \text{(by (2.1))}$$

$$= O(1) \sum_{n=1}^{m-1} \beta_n X_n + O(1) |\lambda_m| X_m = O(1) \quad \text{as } m \to \infty,$$

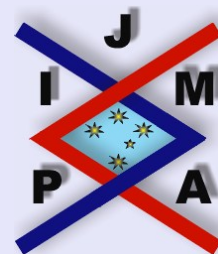by virtue of (1.8) and (2.4). This completes the proof of the theorem. $\qquad \square$

References

[1] S. ALJANČIĆ AND D. ARANDELOVIĆ, *O*-regularly varying functions, *Publ. Inst. Math.*, **22** (1977), 5–22.

[2] H. BOR, Absolute Nörlund summability factors, *J. Inequal. Pure Appl. Math.*, **6**(3) (2005), Art. 62. [ONLINE http://jipam.vu.edu.au/article.php?sid=535].

[3] D. BORWEIN AND F.P. CASS, Strong Nörlund summability, *Math. Zeith.*, **103** (1968), 94–111.

[4] T.M. FLETT, On an extension of absolute summability and some theorems of Littlewood and Paley, *Proc. London Math. Soc.*, **7** (1957), 113–141.

[5] L. LEINDLER, A new application of quasi power increasing sequences, *Publ. Math. Debrecen*, **58** (2001), 791–796.

[6] R.S. VARMA, On the absolute Nörlund summability factors, *Riv. Math. Univ. Parma* (4), **3** (1977), 27–33.

A Note on Absolute Nörlund Summability

Hüseyin Bor

Title Page

Contents

◀◀   ▶▶

◀   ▶

Go Back

Close

Quit