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A PROOF OF HÖLDER'S INEQUALITY USING THE CAUCHY-SCHWARZ INEQUALITY

YUAN-CHUAN LI AND SEN-YEN SHAW

Department of Applied Mathematics
National Chung-Hsing University
Taichung, 402 Taiwan
EMail: ycli@amath.nchu.edu.tw

Graduate School of Engineering
Lunghwa University of Science and Technology
Taoyuan, 333 Taiwan
EMail: shaw@math.ncu.edu.tw

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Abstract

In this note, Hölder's inequality is deduced directly from the Cauchy-Schwarz inequality.

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Let (Ω, μ) be a measure space and

$$L^p(\mu) \equiv L^p(\Omega, \mu) := \{f : \Omega \to \mathbb{C}; \|f\|^p < \infty\}$$

be a Lebesgue space with the L^p-norm $\|f\|_p := \left(\int_\Omega |f|^p d\mu\right)^{\frac{1}{p}}$ for $1 \le p < \infty$ and $\|f\|_\infty := \operatorname{ess\,sup}_{x\in\Omega} |f(x)|$. **Hölder's Inequality** states that:

If $p, q \ge 1$ *be such that* $\frac{1}{p} + \frac{1}{q} = 1$, *and if* $f \in L^p(\mu)$ *and* $g \in L^q(\mu)$, *then* $fg \in L^1(\mu)$ *and* $||fg||_1 \le ||f||_p||g||_q$.

The special case that $p = 1$ and $q = \infty$ is obvious, and the special case $p = q = 2$ is the **Cauchy-Schwarz inequality**: $||fg||_1 \le ||f||_2||g||_2$, which actually holds in all inner-product spaces.

Hölder's inequality can be easily proved (cf. [1, p. 457], [3, pp. 63-64]) by using the arithmetic-geometric mean inequality (or Young's inequality) $ab \le \frac{1}{p}a^p + \frac{1}{q}b^q$, $\frac{1}{p} + \frac{1}{q} = 1$ (which follows from Jensen's inequality, a consequence of the convexity of a function). It is also known that the Cauchy-Schwarz inequality implies Lyapunov's inequality (cf. [1, p. 462]), and from the latter follows the arithmetic-geometric mean inequality. Thus, in a sense, the arithmetic-geometric mean inequality, Hölder's inequality, the Cauchy-Schwarz inequality, and Lyapunov's inequality are all equivalent [1, p. 457]. In the following, we will see that by using the property of convexity one can also deduce Hölder's inequality directly from the Cauchy-Schwarz inequality.

It suffices to assume $f, g \geq 0$ and $1 < p, q < \infty$. If $fg = 0$ a.e. $[\mu]$, the inequality is obvious. Therefore we may assume $g > 0$ on Ω and $fg \neq 0$. Define the function

$$F(t) := \int_\Omega f^{pt} g^{q(1-t)} d\mu = \int_\Omega (g^q)(f^p g^{-q})^t d\mu, \quad t \in D_F,$$

with the domain D_F consisting of all those $t \in \mathbb{R}$ for which the integral exists. Then $0, 1 \in D_F$ and $F(1) = \|f\|_p^p$ and $F(0) = \|g\|_q^q$.

For every $\omega \in \Omega$, $(g^q)(\omega)[(f^p g^{-q})(\omega)]^t$ is convex on $\mathbb{R}$. Therefore for every $t_1, t_2 \in \mathbb{R}$, $0 < \lambda < 1$ and $\omega \in \Omega$,

$$\begin{aligned}(g^q)(\omega)&[(f^p g^{-q})(\omega)]^{\lambda t_1 + (1-\lambda) t_2} \\ &\leq \lambda (g^q)(\omega)[(f^p g^{-q})(\omega)]^{t_1} + (1-\lambda)(g^q)(\omega)[(f^p g^{-q})(\omega)]^{t_2}.\end{aligned}$$

By integration with respect to μ, we obtain that for $t_1, t_2 \in D_F$ and $0 < \lambda < 1$

$$F(\lambda t_1 + (1-\lambda) t_2) \leq \lambda F(t_1) + (1-\lambda) F(t_2),$$

i.e., F is convex on D_F. Hence D_F is an interval containing $[0, 1]$.

It is known (cf. [2, Ch. VII]) that *a function* $h : (a, b) \to \mathbb{R}$ *is convex if and only if* h *is continuous and midconvex on* (a, b). Hence F is continuous on

$(0,1)$. Since $fg \neq 0$, we must have that $F(t) \in (0,\infty)$ for all $t \in [0,1]$ and so $\ln F$ is well-defined on $[0,1]$ and is continuous on $(0,1)$. Let $t_1, t_2 \in (0,1)$ be arbitrary. The functions $u = [(g^q)(f^p g^{-q})^{t_1}]^{\frac{1}{2}}$ and $v = [(g^q)(f^p g^{-q})^{t_2}]^{\frac{1}{2}}$ belong to $L^2(\mu)$ because $\|u\|_2^2 = F(t_1) < \infty$ and $\|v\|_2^2 = F(t_2) < \infty$. Hence we can apply the Cauchy-Schwarz inequality to u and v and obtain

$$\begin{aligned} F\left(\frac{1}{2}t_1 + \frac{1}{2}t_2\right) &= \int_\Omega (g^q)(f^p g^{-q})^{\frac{1}{2}t_1 + \frac{1}{2}t_2} d\mu \\ &= \int_\Omega [(g^q)(f^p g^{-q})^{t_1}]^{\frac{1}{2}} [(g^q)(f^p g^{-q})^{t_2}]^{\frac{1}{2}} d\mu \\ &\leq \left(\int_\Omega (g^q)(f^p g^{-q})^{t_1} d\mu\right)^{\frac{1}{2}} \left(\int_\Omega (g^q)(f^p g^{-q})^{t_2} d\mu\right)^{\frac{1}{2}} \\ &= F(t_1)^{\frac{1}{2}} F(t_2)^{\frac{1}{2}}. \end{aligned}$$

Then we have

$$\ln F\left(\frac{1}{2}t_1 + \frac{1}{2}t_2\right) \leq \frac{1}{2}\ln F(t_1) + \frac{1}{2}\ln F(t_2),$$

i.e., $\ln F$ is midconvex on $(0,1)$. By the above remark we have that $\ln F$ is convex on $(0,1)$. Therefore

$$\begin{aligned} \ln F\left(\frac{1}{p}t + \frac{1}{q}(1-t)\right) &\leq \frac{1}{p}\ln F(t) + \frac{1}{q}\ln F(1-t) \\ &= \ln\left(F(t)^{1/p} F(1-t)^{1/q}\right), \end{aligned}$$

so that

$$F\left(\frac{1}{p}t+\frac{1}{q}(1-t)\right) \le F(t)^{1/p}F(1-t)^{1/q}$$

for all $t \in (0,1)$. Since F is continuous on $(0,1)$ and convex on $[0,1]$, we have

$$\begin{aligned} F\left(\frac{1}{p}\right) &= \lim_{t\uparrow 1} F\left(\frac{1}{p}t+\frac{1}{q}(1-t)\right) \\ &\le \limsup_{t\uparrow 1} F(t)^{1/p} \limsup_{t\uparrow 1} F(1-t)^{1/q} \\ &\le F(1)^{1/p}F(0)^{1/q}, \end{aligned}$$

and so $||fg||_1 \le ||f||_p ||g||_q$.

References

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