

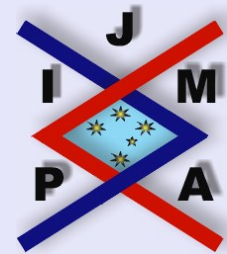
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## A PROOF OF HÖLDER'S INEQUALITY USING THE CAUCHY-SCHWARZ INEQUALITY

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Abstract

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## Abstract

In this note, Hölder's inequality is deduced directly from the Cauchy-Schwarz inequality.

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*Key words:* Hölder's inequality, Cauchy-Schwarz inequality.

Let  $(\Omega, \mu)$  be a measure space and

$$L^p(\mu) \equiv L^p(\Omega, \mu) := \{f : \Omega \rightarrow \mathbb{C}; \|f\|^p < \infty\}$$

be a Lebesgue space with the  $L^p$ -norm  $\|f\|_p := (\int_{\Omega} |f|^p d\mu)^{\frac{1}{p}}$  for  $1 \leq p < \infty$  and  $\|f\|_{\infty} := \text{ess sup}_{x \in \Omega} |f(x)|$ . **Hölder's Inequality** states that:

*If  $p, q \geq 1$  be such that  $\frac{1}{p} + \frac{1}{q} = 1$ , and if  $f \in L^p(\mu)$  and  $g \in L^q(\mu)$ , then  $fg \in L^1(\mu)$  and  $\|fg\|_1 \leq \|f\|_p \|g\|_q$ .*

The special case that  $p = 1$  and  $q = \infty$  is obvious, and the special case  $p = q = 2$  is the **Cauchy-Schwarz inequality**:  $\|fg\|_1 \leq \|f\|_2 \|g\|_2$ , which actually holds in all inner-product spaces.

Hölder's inequality can be easily proved (cf. [1, p. 457], [3, pp. 63-64]) by using the arithmetic-geometric mean inequality (or Young's inequality)  $ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$ ,  $\frac{1}{p} + \frac{1}{q} = 1$  (which follows from Jensen's inequality, a consequence of the convexity of a function). It is also known that the Cauchy-Schwarz inequality implies Lyapunov's inequality (cf. [1, p. 462]), and from



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the latter follows the arithmetic-geometric mean inequality. Thus, in a sense, the arithmetic-geometric mean inequality, Hölder's inequality, the Cauchy-Schwarz inequality, and Lyapunov's inequality are all equivalent [1, p. 457]. In the following, we will see that by using the property of convexity one can also deduce Hölder's inequality directly from the Cauchy-Schwarz inequality.

It suffices to assume  $f, g \geq 0$  and  $1 < p, q < \infty$ . If  $fg = 0$  a.e.  $[\mu]$ , the inequality is obvious. Therefore we may assume  $g > 0$  on  $\Omega$  and  $fg \neq 0$ . Define the function

$$F(t) := \int_{\Omega} f^{pt} g^{q(1-t)} d\mu = \int_{\Omega} (g^q)(f^p g^{-q})^t d\mu, \quad t \in D_F,$$

with the domain  $D_F$  consisting of all those  $t \in \mathbb{R}$  for which the integral exists. Then  $0, 1 \in D_F$  and  $F(1) = \|f\|_p^p$  and  $F(0) = \|g\|_q^q$ .

For every  $\omega \in \Omega$ ,  $(g^q)(\omega)[(f^p g^{-q})(\omega)]^t$  is convex on  $\mathbb{R}$ . Therefore for every  $t_1, t_2 \in \mathbb{R}$ ,  $0 < \lambda < 1$  and  $\omega \in \Omega$ ,

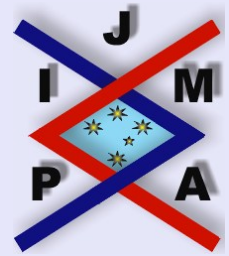
$$\begin{aligned} & (g^q)(\omega)[(f^p g^{-q})(\omega)]^{\lambda t_1 + (1-\lambda)t_2} \\ & \leq \lambda(g^q)(\omega)[(f^p g^{-q})(\omega)]^{t_1} + (1-\lambda)(g^q)(\omega)[(f^p g^{-q})(\omega)]^{t_2}. \end{aligned}$$

By integration with respect to  $\mu$ , we obtain that for  $t_1, t_2 \in D_F$  and  $0 < \lambda < 1$

$$F(\lambda t_1 + (1-\lambda)t_2) \leq \lambda F(t_1) + (1-\lambda)F(t_2),$$

i.e.,  $F$  is convex on  $D_F$ . Hence  $D_F$  is an interval containing  $[0, 1]$ .

It is known (cf. [2, Ch. VII]) that a function  $h : (a, b) \rightarrow \mathbb{R}$  is convex if and only if  $h$  is continuous and midconvex on  $(a, b)$ . Hence  $F$  is continuous on




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$(0, 1)$ . Since  $fg \neq 0$ , we must have that  $F(t) \in (0, \infty)$  for all  $t \in [0, 1]$  and so  $\ln F$  is well-defined on  $[0, 1]$  and is continuous on  $(0, 1)$ . Let  $t_1, t_2 \in (0, 1)$  be arbitrary. The functions  $u = [(g^q)(f^p g^{-q})^{t_1}]^{\frac{1}{2}}$  and  $v = [(g^q)(f^p g^{-q})^{t_2}]^{\frac{1}{2}}$  belong to  $L^2(\mu)$  because  $\|u\|_2^2 = F(t_1) < \infty$  and  $\|v\|_2^2 = F(t_2) < \infty$ . Hence we can apply the Cauchy-Schwarz inequality to  $u$  and  $v$  and obtain

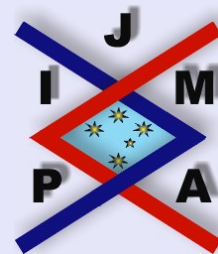
$$\begin{aligned} F\left(\frac{1}{2}t_1 + \frac{1}{2}t_2\right) &= \int_{\Omega} (g^q)(f^p g^{-q})^{\frac{1}{2}t_1 + \frac{1}{2}t_2} d\mu \\ &= \int_{\Omega} [(g^q)(f^p g^{-q})^{t_1}]^{\frac{1}{2}} [(g^q)(f^p g^{-q})^{t_2}]^{\frac{1}{2}} d\mu \\ &\leq \left(\int_{\Omega} (g^q)(f^p g^{-q})^{t_1} d\mu\right)^{\frac{1}{2}} \left(\int_{\Omega} (g^q)(f^p g^{-q})^{t_2} d\mu\right)^{\frac{1}{2}} \\ &= F(t_1)^{\frac{1}{2}} F(t_2)^{\frac{1}{2}}. \end{aligned}$$

Then we have

$$\ln F\left(\frac{1}{2}t_1 + \frac{1}{2}t_2\right) \leq \frac{1}{2} \ln F(t_1) + \frac{1}{2} \ln F(t_2),$$

i.e.,  $\ln F$  is midconvex on  $(0, 1)$ . By the above remark we have that  $\ln F$  is convex on  $(0, 1)$ . Therefore

$$\begin{aligned} \ln F\left(\frac{1}{p}t + \frac{1}{q}(1-t)\right) &\leq \frac{1}{p} \ln F(t) + \frac{1}{q} \ln F(1-t) \\ &= \ln (F(t)^{1/p} F(1-t)^{1/q}), \end{aligned}$$



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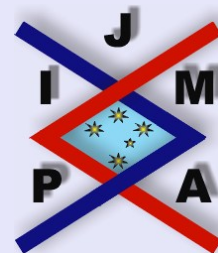
so that

$$F\left(\frac{1}{p}t + \frac{1}{q}(1-t)\right) \leq F(t)^{1/p}F(1-t)^{1/q}$$

for all  $t \in (0, 1)$ . Since  $F$  is continuous on  $(0, 1)$  and convex on  $[0, 1]$ , we have

$$\begin{aligned} F\left(\frac{1}{p}\right) &= \lim_{t \uparrow 1} F\left(\frac{1}{p}t + \frac{1}{q}(1-t)\right) \\ &\leq \limsup_{t \uparrow 1} F(t)^{1/p} \limsup_{t \uparrow 1} F(1-t)^{1/q} \\ &\leq F(1)^{1/p}F(0)^{1/q}, \end{aligned}$$

and so  $\|fg\|_1 \leq \|f\|_p \|g\|_q$ .



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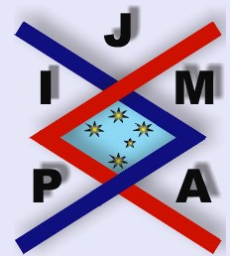
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