journal of inequalities in pure and applied mathematics

http://jipam.vu.edu.au

issn: 1443-5756

Volume 10 (2009), Issue 4, Article 94, 8 pp.

AN INEQUALITY FOR BI-ORTHOGONAL PAIRS

CHRISTOPHER MEANEY

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
MACQUARIE UNIVERSITY
NORTH RYDE NSW 2109, AUSTRALIA
chrism@maths.mq.edu.au

Received 26 November, 2009; accepted 14 December, 2009
Communicated by S.S. Dragomir

ABSTRACT. We use Salem's method [13, 14] to prove an inequality of Kwapień and Pełczyński concerning a lower bound for partial sums of series of bi-orthogonal vectors in a Hilbert space, or the dual vectors. This is applied to some lower bounds on L^1 norms for orthogonal expansions.

Key words and phrases: Bi-orthogonal pair, Bessel's inequality, Orthogonal expansion, Lebesgue constants.

2000 *Mathematics Subject Classification.* 42C15, 46C05.

1. INTRODUCTION

Suppose that H is a Hilbert space, $n \in \mathbb{N}$, and that $J = \{1, \dots, n\}$ or $J = \mathbb{N}$. A pair of sets $\{v_j : j \in J\}$ and $\{w_j : j \in J\}$ in H are said to be *a bi-orthogonal pair* when

$$\langle v_j, w_k \rangle_H = \delta_{jk}, \qquad \forall j, k \in J.$$

The inequality in Theorem 2.1 below comes from Section 6 of [6], where it was proved using Grothendieck's inequality, absolutely summing operators, and estimates on the Hilbert matrix. Here we present an alternate proof, based on earlier ideas from Salem [13, 14], where Bessel's inequality is combined with a result of Menshov [10]. Following the proof of Theorem 2.1, we will describe Salem's method of using L^2 inequalities to produce L^1 estimates on maximal functions. Such estimates are related to the stronger results of Olevskiĭ [11], Kashin and Szarek [4], and Bochkarev [1]. We conclude with an observation about the statement of Theorem 2.1 in a linear algebra setting. Some of these results were discussed in [9], where it was shown that Salem's methods emphasized the universality of the Rademacher-Menshov Theorem.

302-09

2. THE KWAPIEŃ-PEŁCZYŃSKI INEQUALITY

Theorem 2.1. *There is a positive constant* c *with the following property. For every* $n \geq 1$*, every Hilbert space* H*, and every bi-orthogonal pair* $\{v_1, \ldots, v_n\}$ *and* $\{w_1, \ldots, w_n\}$ *in* H*,*

$$\log n \leq c \max_{1 \leq m \leq n} \|w_m\|_H \max_{1 \leq k \leq n} \left\| \sum_{j=1}^{k} v_j \right\|_H . \tag{2.1}$$

Proof. Equip $[0,1]$ with a Lebesgue measure λ and let $V = L^2([0,1], H)$ be the space of H-valued square integrable functions on $[0,1]$, with inner product

$$\langle F, G \rangle_V = \int_0^1 \langle F(x), G(x) \rangle_H \, dx$$

and norm

$$\|F\|_V = \left(\int_0^1 \|F(x)\|_H^2 \, dx \right).$$

Suppose that $\{F_1, \ldots, F_n\}$ is an orthonormal set in $L^2([0,1])$ and define vectors $p_1, \ldots, p_n$ in V by

$$p_k(x) = F_k(x) w_k, \qquad 1 \leq k \leq n, x \in [0,1].$$

Then

$$\langle p_k(x), p_j(x) \rangle_H = F_k(x) \overline{F_j(x)} \, \langle w_k, w_j \rangle_H , \qquad 1 \leq j, k \leq n,$$

and so $\{p_1, \ldots, p_n\}$ is an orthogonal set in V. For every $P \in V$, Bessel's inequality states that

$$\sum_{k=1}^{n} \frac{|\langle P, p_k \rangle_V|^2}{\|w_k\|_H^2} \leq \|P\|_V^2 . \tag{2.2}$$

Note that here

$$\langle P, p_k \rangle_V = \int_0^1 \langle P(x), w_k \rangle_H \overline{F_k(x)} dx, \qquad 1 \leq k \leq n.$$

Now consider a decreasing sequence $f_1 \geq f_2 \geq \cdots \geq f_n \geq f_{n+1} = 0$ of characteristic functions of measurable subsets of $[0,1]$. For each scalar-valued $G \in L^2([0,1])$ define an element of V by setting

$$P_G(x) = G(x) \sum_{j=1}^{n} f_j(x) v_j.$$

The Abel transformation shows that

$$P_G(x) = G(x) \sum_{k=1}^{n} \Delta f_k(x) \sigma_k,$$

where $\Delta f_k = f_k - f_{k+1}$ and $\sigma_k = \sum_{j=1}^{k} v_j$, for $1 \leq k \leq n$. The functions $\Delta f_1, \ldots, \Delta f_n$ are characteristic functions of mutually disjoint subsets of $[0,1]$ and for each $0 \leq x \leq 1$ at most one of the values $\Delta f_k(x)$ is non-zero. Notice that

$$\|P_G(x)\|_H^2 = |G(x)|^2 \sum_{k=1}^{n} \Delta f_k(x) \, \|\sigma_k\|_H^2 .$$

Integrating over $[0,1]$ gives

$$\|P_G\|_V^2 \leq \|G\|_2^2 \max_{1 \leq k \leq n} \|\sigma_k\|_H^2 .$$

Note that

$$\langle P_G(x), p_k(x) \rangle_H = G(x) f_k(x) \overline{F_k(x)} \, \langle v_k, w_k \rangle_H , \quad 1 \leq k \leq n,$$

and

$$\langle P_G, p_k \rangle_V = \int_0^1 G(x) f_k(x) \overline{F_k(x)} dx \ \langle v_k, w_k \rangle_H \,, \quad 1 \le k \le n.$$

Combining this with Bessel's inequality (2.2), we arrive at the inequality

$$\sum_{k=1}^{n} \left| \int_{[0,1]} G f_k \overline{F_k} d\lambda \right|^2 \frac{1}{\|w_k\|_H^2} \le \|G\|_2^2 \max_{1\le k\le n} \|\sigma_k\|_H^2 \,. \tag{2.3}$$

This implies that

$$\left(\sum_{k=1}^{n} \left| \int_{[0,1]} G f_k \overline{F_k} d\lambda \right|^2 \right) \le \left(\max_{1\le j\le n} \|w_k\|_H^2 \right) \|G\|_2^2 \left(\max_{1\le k\le n} \|\sigma_k\|_H^2 \right) . \tag{2.4}$$

We now concentrate on the case where the functions $F_1, \dots, F_n$ are given by Menshov's result (Lemma 1 on page 255 of Kashin and Saakyan [3]). There is a constant $c_0 > 0$, independent of n, so that

$$\lambda \left(\left\{ x \in [0,1] \ : \ \max_{1\le j\le n} \left| \sum_{k=1}^{j} F_k(x) \right| > c_0 \log(n) \sqrt{n} \right\} \right) \ge \frac{1}{4}. \tag{2.5}$$

Let us use $\mathcal{M}(x)$ to denote the maximal function

$$\mathcal{M}(x) = \max_{1\le j\le n} \left| \sum_{k=1}^{j} F_k(x) \right| , \qquad 0 \le x \le 1.$$

Define an integer-valued function $m(x)$ on $[0,1]$ by

$$m(x) = \min \left\{ m \ : \ \left| \sum_{k=1}^{m} F_k(x) \right| = \mathcal{M}(x) \right\} .$$

Furthermore, let f_k be the characteristic function of the subset

$$\{ x \in [0,1] \ : \ m(x) \ge k \} \,.$$

Then

$$\sum_{k=1}^{n} f_k(x) F_k(x) = S_{m(x)}(x) = \sum_{k=1}^{m(x)} F_k(x), \qquad \forall 0 \le x \le 1.$$

For an arbitrary $G \in L^2([0,1])$ we have

$$\int_0^1 G(x) \overline{S_{m(x)}(x)} dx = \sum_{k=1}^{n} \int_0^1 G(x) f_k(x) \overline{F_k(x)} dx.$$

Using the Cauchy-Schwarz inequality on the right hand side, we have

$$\left| \int_0^1 G(x) \overline{S_{m(x)}(x)} dx \right| \le \sqrt{n} \left(\sum_{k=1}^{n} \left| \int_0^1 G f_k \overline{F_k} \, d\lambda \right|^2 \right)^{1/2} , \tag{2.6}$$

for all $G \in L^2([0,1])$. We will use the function G which has $|G(x)| = 1$ everywhere on $[0,1]$, with

$$G(x) \overline{S_{m(x)}(x)} = \mathcal{M}(x), \qquad \forall 0 \le x \le 1.$$

In this case, the left hand side of (2.6) is

$$\|\mathcal{M}\|_1 \ge \frac{c_0}{4} \log(n) \sqrt{n},$$

because of (2.5). Combining this with (2.6) we have

$$\frac{c_0}{4}\log(n)\sqrt{n} \le \sqrt{n}\left(\sum_{k=1}^{n}\left|\int_0^1 G f_k \overline{F_k}\, d\lambda\right|^2\right)^{1/2}.$$

This can be put back into (2.4) to obtain (2.1). Notice that $\|G\|_2 = 1$ on the right hand side of (2.3). ☐

3. APPLICATIONS

3.1. L^1 **estimates.** In this section we use $H = L^2(X,\mu)$, for a positive measure space (X,μ). Suppose we are given an orthonormal sequence of functions $(h_n)_{n=1}^{\infty}$ in $L^2(X,\mu)$, and suppose that each of the functions h_n is essentially bounded on X. Let $(a_n)_{n=1}^{\infty}$ be a sequence of non-zero complex numbers and set

$$M_n = \max_{1\le j\le n}\|h_j\|_\infty \text{ and } \mathcal{S}_n^*(x) = \max_{1\le k\le n}\left|\sum_{j=1}^{k} a_j h_j(x)\right|, \quad \text{for } x\in X,\ n\ge 1.$$

Lemma 3.1. *For a set of functions* $\{h_1,\dots,h_n\}\subset L^2(X,\mu)\cap L^\infty(X,\mu)$ *and maximal function*

$$\mathcal{S}_n^*(x) = \max_{1\le k\le n}\left|\sum_{j=1}^{k} a_j h_j(x)\right|,$$

we have

$$|a_j h_j(x)| \le 2\mathcal{S}_n^*(x), \qquad \forall x\in X, 1\le j\le n,$$

and

$$\frac{\left|\sum_{j=1}^{k} a_j h_j(x)\right|}{\mathcal{S}_n^*(x)} \le 1, \quad \forall 1\le k\le n \textit{ and } x \textit{ where } \mathcal{S}_n^*(x)\ne 0.$$

Proof. The first inequality follows from the triangle inequality and the fact that

$$a_j h_j(x) = \sum_{k=1}^{j} a_k h_k(x) - \sum_{k=1}^{j-1} a_k h_k(x)$$

for $2\le j\le n$. The second inequality is a consequence of the definition of $\mathcal{S}_n^*$.

Fix $n\ge 1$ and let

$$v_j(x) = a_j h_j(x)\left(\mathcal{S}_n^*(x)\right)^{-1/2} \text{ and } w_j(x) = a_j^{-1} h_j(x)\left(\mathcal{S}_n^*(x)\right)^{1/2}$$

for all $x\in X$ where $\mathcal{S}_n^*(x)\ne 0$ and $1\le j\le n$. From their definition,

$$\{v_1,\dots,v_n\} \text{ and } \{w_1,\dots,w_n\}$$

are a bi-orthogonal pair in $L^2(X,\mu)$. The conditions we have placed on the functions h_j give:

$$\|w_j\|_2^2 = |a_j|^{-2}\int_X |h_j|^2\left(\mathcal{S}_n^*\right)d\mu \le \frac{M_n^2}{\min_{1\le k\le n}|a_k|^2}\|\mathcal{S}_n^*\|_1$$

and

$$\left\|\sum_{j=1}^{k} v_j\right\|_2^2 = \int_X \frac{1}{(\mathcal{S}_n^*)}\left|\sum_{j=1}^{k} a_j h_j\right|^2 d\mu \le \left\|\sum_{j=1}^{k} a_j h_j\right\|_1.$$

We can put these estimates into (2.1) and find that

$$\log n \leq c \frac{M_n}{\min_{1\leq k\leq n}|a_k|} \|S_n^*\|_1^{1/2} \max_{1\leq k\leq n} \left\| \sum_{j=1}^{k} a_j h_j \right\|_1^{1/2} .$$

We could also say that

$$\max_{1\leq k\leq n} \left\| \sum_{j=1}^{k} a_j h_j \right\|_1 \leq \|\mathcal{S}_n^*\|_1$$

and so

$$\log(n) \leq c \frac{M_n}{\min_{1\leq k\leq n}|a_k|} \|\mathcal{S}_n^*\|_1 .$$

□

Corollary 3.2. *Suppose that $(h_n)_{n=1}^{\infty}$ is an orthonormal sequence in $L^2(X,\mu)$ consisting of essentially bounded functions. For each sequence $(a_n)_{n=1}^{\infty}$ of complex numbers and each $n \geq 1$,*

$$\left(\min_{1\leq k\leq n}|a_k|\ \log n\right)^2 \leq c\left(\max_{1\leq k\leq n}\|h_k\|_\infty\right)^2 \left\| \max_{1\leq k\leq n}\left|\sum_{j=1}^{k} a_j h_j\right| \right\|_1 \max_{1\leq k\leq n}\left\|\sum_{j=1}^{k} a_j h_j\right\|_1$$

and

$$\min_{1\leq k\leq n}|a_k|\ \log n \leq c\left(\max_{1\leq k\leq n}\|h_k\|_\infty\right) \left\| \max_{1\leq k\leq n}\left|\sum_{j=1}^{k} a_j h_j\right| \right\|_1 .$$

The constant c is independent of n, and the sequences involved here.

As observed in [4], this can also be obtained as a consequence of [11]. In addition, see [7].

The following is a paraphrase of the last page of [13]. For the special case of Fourier series on the unit circle, see Proposition 1.6.9 in [12].

Corollary 3.3. *Suppose that $(h_n)_{n=1}^{\infty}$ is an orthonormal sequence in $L^2(X,\mu)$ consisting of essentially bounded functions with $\|h_n\|_\infty \leq M$ for all $n \geq 1$. For each decreasing sequence $(a_n)_{n=1}^{\infty}$ of positive numbers and each $n \geq 1$,*

$$(a_n\ \log n)^2 \leq cM^2 \left\| \max_{1\leq k\leq n}\left|\sum_{j=1}^{k} a_j h_j\right| \right\|_1 \max_{1\leq k\leq n}\left\|\sum_{j=1}^{k} a_j h_j\right\|_1$$

and

$$a_n\ \log n \leq cM \left\| \max_{1\leq k\leq n}\left|\sum_{j=1}^{k} a_j h_j\right| \right\|_1 .$$

In particular, if $(a_n \log n)_{n=1}^{\infty}$ is unbounded then

$$\left(\left\| \max_{1\leq k\leq n}\left|\sum_{j=1}^{k} a_j h_j\right| \right\|_1 \right)_{n=1}^{\infty} \quad \textit{is unbounded.}$$

The constant c is independent of n, and the sequences involved here.

3.2. **Salem's Approach to the Littlewood Conjecture.** We concentrate on the case where $H = L^2(\mathbb{T})$ and the orthonormal sequence is a subset of $\{e^{inx} : n \in \mathbb{N}\}$. Let

$$m_1 < m_2 < m_3 < \cdots$$

be an increasing sequence of natural numbers and let

$$h_k(x) = e^{im_k x}$$

for all $k \geq 1$ and $x \in \mathbb{T}$. In addition, let

$$D_m(x) = \sum_{k=-m}^{m} e^{ikx}$$

be the m^{th} Dirichlet kernel. For all $N \geq m \geq 1$, there is the partial sum

$$\sum_{m_k \leq m} a_k h_k(x) = D_m * \left(\sum_{m_k \leq N} a_k h_k \right)(x).$$

It is a fact that D_m is an even function which satisfies the inequalities:

$$|D_m(x)| \leq \begin{cases} 2m+1 & \text{for all } x, \\ 1/|x| & \text{for } \frac{1}{2m+1} < x < 2\pi - \frac{1}{2m+1}. \end{cases} \tag{3.1}$$

Lemma 3.4. *If p is a trigonometric polynomial of degree N, then the maximal function of its Fourier partial sums*

$$S^* p(x) = \sup_{m \geq 1} |D_m * p(x)|$$

satisfies

$$\|S^* p\|_1 \leq c \log(2N+1) \|p\|_1 .$$

Proof. For such a trigonometric polynomial p, the partial sums are all partial sums of $p * D_N$, and all the Dirichlet kernels D_m for $1 \leq m \leq N$ are dominated by a function whose L^1 norm is of the order of $\log(2N+1)$. □

We can combine this with the inequalities in Corollary 3.2, since

$$\left\| \max_{1 \leq k \leq n} \left| \sum_{j=1}^{k} a_j h_j \right| \right\|_1 \leq c \log(2m_n + 1) \left\| \sum_{j=1}^{m} a_j h_j \right\|_1 .$$

We then arrive at the main result in [14].

Corollary 3.5. *For an increasing sequence $(m_n)_{n=1}^{\infty}$ of natural numbers and a sequence of non-zero complex numbers $(a_n)_{n=1}^{\infty}$ the partial sums of the trigonometric series*

$$\sum_{k=1}^{\infty} a_k e^{im_k x}$$

satisfy

$$\min_{1 \leq k \leq n} |a_k| \frac{\log n}{\sqrt{\log(2m_n + 1)}} \leq c \max_{1 \leq k \leq n} \left\| \sum_{j=1}^{k} a_j e^{im_j(\cdot)} \right\|_1 .$$

This was Salem's attempt at Littlewood's conjecture, which was subsequently settled in [5] and [8].

3.3. **Linearly Independent Sequences.** Notice that if $\{v_1, \dots, v_n\}$ is an arbitrary linearly independent subset of H then there is a unique subset

$$\left\{w_j^n \;:\; 1 \le j \le n\right\} \subseteq \text{span}\left(\{v_1, \dots, v_n\}\right)$$

so that $\{v_1, \dots, v_n\}$ and $\{w_1^n, \dots, w_n^n\}$ are a bi-orthogonal pair. See Theorem 15 in Chapter 3 of [2]. We can apply Theorem 2.1 to the pair in either order.

Corollary 3.6. *For each $n \ge 2$ and linearly independent subset $\{v_1, \dots, v_n\}$ in an inner-product space H, with dual basis $\{w_1^n, \dots, w_n^n\}$,*

$$\log n \le c \max_{1\le k\le n} \|w_k^n\|_H \max_{1\le k\le n} \left\| \sum_{j=1}^{k} v_j \right\|_H$$

and

$$\log n \le c \max_{1\le k\le n} \|v_k\|_H \max_{1\le k\le n} \left\| \sum_{j=1}^{k} w_j^n \right\|_H .$$

The constant $c > 0$ is independent of n, H, and the sets of vectors.

3.4. **Matrices.** Suppose that A is an invertible $n \times n$ matrix with complex entries and columns

$$a_1, \dots, a_n \in \mathbb{C}^n.$$

Let $b_1, \dots, b_n$ be the rows of A^{-1}. From their definition

$$\sum_{j=1}^{n} b_{ij} a_{jk} = \delta_{ik}$$

and so the two sets of vectors

$$\left\{\overline{b_1^T}, \dots, \overline{b_n^T}\right\} \quad \text{and} \quad \{a_1, \dots, a_n\}$$

are a bi-orthogonal pair in $\mathbb{C}^n$. Theorem 2.1 then says that

$$\log(n) \le c \max_{1\le k\le n} \|b_k\| \max_{1\le k\le n} \left\| \sum_{j=1}^{k} a_j \right\| .$$

The norm here is the finite dimensional ℓ^2 norm. This brings us back to the material in [6]. Note that [4] has logarithmic lower bounds for ℓ^1-norms of column vectors of orthogonal matrices.

REFERENCES

[1] S.V. BOCHKAREV, A generalization of Kolmogorov's theorem to biorthogonal systems, *Proceedings of the Steklov Institute of Mathematics*, **260** (2008), 37–49.

[2] K. HOFFMAN AND R.A. KUNZE, *Linear Algebra*, Second ed., Prentice Hall, 1971.

[3] B.S. KASHIN AND A.A. SAAKYAN, *Orthogonal Series*, Translations of Mathematical Monographs, vol. 75, American Mathematical Society, Providence, RI, 1989.

[4] B.S. KASHIN, A.A. SAAKYAN AND S.J. SZAREK, Logarithmic growth of the L^1-norm of the majorant of partial sums of an orthogonal series, *Math. Notes*, **58**(2) (1995), 824–832.

[5] S.V. KONYAGIN, On the Littlewood problem, *Izv. Akad. Nauk SSSR Ser. Mat.*, **45**(2) (1981), 243–265, 463.

[6] S. KWAPIEŃ AND A. PEŁCZYŃSKI, The main triangle projection in matrix spaces and its applications, *Studia Math.*, **34** (1970), 43–68. MR 0270118 (42 #5011)

[7] S. KWAPIEŃ, A. PEŁCZYŃSKI AND S.J. SZAREK, An estimation of the Lebesgue functions of biorthogonal systems with an application to the nonexistence of some bases in C and L^1, *Studia Math.*, **66**(2) (1979), 185–200.

[8] O.C. McGEHEE, L. PIGNO AND B. SMITH, Hardy's inequality and the L^1 norm of exponential sums, *Ann. of Math.* (2), **113**(3) (1981), 613–618.

[9] C. MEANEY, Remarks on the Rademacher-Menshov theorem, CMA/AMSI Research Symposium "Asymptotic Geometric Analysis, Harmonic Analysis, and Related Topics", *Proc. Centre Math. Appl. Austral. Nat. Univ.*, vol. 42, Austral. Nat. Univ., Canberra, 2007, pp. 100–110.

[10] D. MENCHOFF, Sur les séries de fonctions orthogonales, (Premiére Partie. La convergence.), *Fundamenta Math.*, **4** (1923), 82–105.

[11] A.M. OLEVSKIĬ, *Fourier series with respect to general orthogonal systems. Translated from the Russian by B. P. Marshall and H. J. Christoffers.*, Ergebnisse der Mathematik und ihrer Grenzgebiete. Band 86. Berlin-Heidelberg-New York: Springer-Verlag., 1975.

[12] M.A. PINSKY, *Introduction to Fourier Analysis and Wavelets*, Brooks/Cole, 2002.

[13] R. SALEM, A new proof of a theorem of Menchoff, *Duke Math. J.*, **8** (1941), 269–272.

[14] R. SALEM, On a problem of Littlewood, *Amer. J. Math.*, **77** (1955), 535–540.