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# A NOTE ON MULTIPLICATIVELY $e$-PERFECT NUMBERS

LE ANH VINH AND DANG PHUONG DUNG

School of Mathematics
University of New South Wales
Sydney 2052 NSW
vinh@maths.unsw.edu.au

School of Finance and Banking
University of New South Wales
Sydney 2052 NSW
phuogdug@yahoo.com

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Abstract. Let $T_e(n)$ denote the product of all exponential divisors of $n$. An integer $n$ is called multiplicatively $e$-perfect if $T_e(n) = n^2$ and multiplicatively $e$-superperfect if $T_e(T_e(n)) = n^2$. In this note, we give an alternative proof for characterization of multiplicatively $e$-perfect and multiplicatively $e$-superperfect numbers.

*Key words and phrases:* Perfect number, Exponential divisor, Multiplicatively perfect, Sum of divisors, Number of divisors.

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## 1. Introduction

Let $\sigma(n)$ be the sum of all divisors of $n$. An integer $n$ is called perfect if $\sigma(n) = 2n$ and superperfect if $\sigma(\sigma(n)) = 2n$. If $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ is the prime factorization of $n > 1$, a divisor $d \mid n$, called an exponential divisor ($e$-divisor) of $n$ is $d = p_1^{\beta_1} \cdots p_k^{\beta_k}$ with $\beta_i \mid \alpha_i$ for all $1 \leq i \leq k$. Let $T_e(n)$ denote the product of all exponential divisors of $n$. The concepts of multiplicatively $e$-perfect and multiplicatively $e$-superperfect numbers were first introduced by Sándor in [1].

**Definition 1.1.** An integer $n$ is called multiplicatively $e$-perfect if $T_e(n) = n^2$ and multiplicatively $e$-superperfect if $T_e(T_e(n)) = n^2$.

In [1], Sándor completely characterizes multiplicatively $e$-perfect and multiplicatively $e$-superperfect numbers.

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**Theorem 1.1** ([1]).

a) *An integer $n$ is multiplicatively $e$-perfect if and only if $n = p^\alpha$, where $p$ is prime and $\alpha$ is a perfect number.*

b) *An interger $n$ is multiplicatively $e$-superperfect if and only if $n = p^\alpha$, where $p$ is a prime, and $\alpha$ is a superperfect number.*

Sándor's proof is based on an explicit expression of $T_e(n)$. In this note, we offer an alternative proof of Theorem 1.1.

## 2. PROOF OF THEOREM 1.1

a) Suppose that $n$ is multiplicatively $e$-perfect; that is $T_e(n) = n^2$. If $n$ has more than one prime factor then $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ for some $k \geq 2$, $\alpha_i \geq 1$ and $p_1, \dots, p_k$ are $k$ distinct primes. We have three separate cases.

(1) Suppose that $\alpha_1 = \cdots = \alpha_k = 1$. Then $d$ is an exponential divisor of $n$ if and only if $d = p_1 \cdots p_k = n$. Hence $T_e(n) = n$, which is a contradiction.

(2) Suppose that two of $\alpha_1, \dots, \alpha_k$ are greater $1$. Without loss of generality, we may assume that $\alpha_1, \alpha_2 > 1$. Then $d_1 = p_1 p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, $d_2 = p_1^{\alpha_1} p_2 p_3^{\alpha_3} \cdots p_k^{\alpha_k}$, $d_3 = n$ are three distinct exponential divisors of $n$. Hence $d_1 d_2 d_3 \mid T_e(n)$. However, $p_1^{2\alpha_1+1} \mid d_1 d_2 d_3$ so $T_e(n) \neq n^2$, which is a contradiction.

(3) Suppose that there is exactly one of $\alpha_1, \dots, \alpha_k$ which is greater than $1$. Without loss of generality, we may assume that $\alpha_1 > 1$ and $\alpha_2 = \cdots = \alpha_k = 1$. We have that if $d$ is an exponential divisor of $n$ then $d = p_1^{\beta_1} p_2 \cdots p_k$ for some $\beta_1 \mid \alpha_1$. Hence if $n$ has more than two distinct exponential divisors then $p_2^3 \mid T_e(n) = p_1^{2\alpha_1} p_2^2 \cdots p_k^2$, which is a contradiction. However, $d_1 = p_1 p_2 \cdots p_k$, $d_2 = p_1^{\alpha_1} p_2 p_3 \cdots p_k$ are two distinct exponential divisors of $n$ so $d_1, d_2$ are all exponential divisors of $n$. Hence $T_e(n) = p_1^{\alpha_1+1} p_2^2 \cdots p_k^2 = p_1^{2\alpha_1} p_2^2 \cdots p_k^2$. This implies that $\alpha_1 = 1$, which is a contradiction.

Thus $n$ has only one prime factor; that is, $n = p^\alpha$ for some prime $p$. In this case then $T_e(n) = p^{\sigma(\alpha)}$. Hence $T_e(n) = n^2 = p^{2\alpha}$ if and only if $\sigma(\alpha) = 2\alpha$. This concludes the proof.

b) Suppose that $n$ is multiplicatively $e$-superperfect; that is $T_e(T_e(n)) = n^2$. If $n$ has more than one prime factor then $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ for some $k \geq 2$, $\alpha_i \geq 1$ and $p_1, \dots, p_k$ are $k$ distinct primes. We have two separate cases.

(1) Suppose that $\alpha_1 = \cdots = \alpha_k = 1$. Then $d$ is an exponential divisor of $n$ if and only if $d = p_1 \cdots p_k = n$. Hence $T_e(n) = n$ and $T_e(T_e(n)) = T_e(n) = n$ which is a contradiction.

(2) Suppose that there is at least one of $\alpha_1, \dots, \alpha_k$ which is greater $1$. Without loss of generality, we may assume that $\alpha_1 > 1$. Then $d_1 = p_1 p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, $d_2 = n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k}$, are two distinct exponential divisors of $n$. Hence $d_1 d_2 \mid T_e(n)$. However, $d_1 d_2 = p_1^{\alpha_1+1} p_2^{2\alpha_2} \cdots p_k^{2\alpha_k}$ so $T_e(n) = p_1^{\gamma_1} \cdots p_k^{\gamma_k}$ where $\gamma_1 \geq \alpha_1 + 1$, $\gamma_i \geq 2\alpha_i \geq 2$ for $i = 2, \dots, k$. Thus, $t_1 = p_1^{\gamma_1} p_2 p_3^{\gamma_3} \cdots p_k^{\gamma_k}$ and $t_2 = T_e(n) = p_1^{\gamma_1} p_2^{\gamma_2} p_3^{\gamma_3} \cdots p_k^{\gamma_k}$ are two distinct exponential divisors of $T_e(n)$. Hence $t_1 t_2 \mid T_e(T_e(n))$. However, $p_1^{2\gamma_1} \mid t_1 t_2$ and $\gamma_1 > \alpha_1$, which is a contradiction.

Thus $n$ has only one prime factor; that is $n = p^\alpha$ for some prime $p$. In this case then $T_e(n) = p^{\sigma(\alpha)}$ and $T_e(T_e(n)) = p^{\sigma(\sigma(n))}$. Hence $T_e(T_e(n)) = n^2 = p^{2\alpha}$ if and only if $\sigma(\sigma(\alpha)) = 2\alpha$. This concludes the proof.

**Remark 2.1.** In an e-mail message, Professor Sándor has provided the authors some more recent references related to the arithmetic function $T_e(n)$, as well as connected notions on $e$-perfect numbers and generalizations. These are [2], [3], and [4].

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