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ON JENSEN TYPE INEQUALITIES WITH ORDERED VARIABLES

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ABSTRACT. In this paper, we present some basic results concerning an extension of Jensen type inequalities with ordered variables to functions with inflection points, and then give several relevant applications of these results.

Key words and phrases: Convex function, k-arithmetic ordered variables, k-geometric ordered variables, Jensen's inequality, Karamata's inequality.

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1. BASIC RESULTS

An n-tuple of real numbers $X = (x_1, x_2, \dots, x_n)$ is said to be increasingly ordered if $x_1 \le x_2 \le \cdots \le x_n$. If $x_1 \ge x_2 \ge \cdots \ge x_n$, then X is decreasingly ordered.

In addition, a set $X = (x_1, x_2, \dots, x_n)$ with $\frac{x_1+x_2+\cdots+x_n}{n} = s$ is said to be k-arithmetic ordered if k of the numbers $x_1, x_2, \dots, x_n$ are smaller than or equal to s, and the other $n-k$ are greater than or equal to s. On the assumption that $x_1 \le x_2 \le \cdots \le x_n$, X is k-arithmetic ordered if

$$x_1 \le \cdots \le x_k \le s \le x_{k+1} \le \cdots \le x_n.$$

It is easily seen that

$$X_1 = (s - x_1 + x_{k+1}, s - x_2 + x_{k+2}, \dots, s - x_n + x_k)$$

is a k-arithmetic ordered set if X is increasingly ordered, and is an $(n-k)$-arithmetic ordered set if X is decreasingly ordered.

Similarly, an n-tuple of positive real numbers $A = (a_1, a_2, \dots, a_n)$ with $\sqrt[n]{a_1 a_2 \cdots a_n} = r$ is said to be k-geometric ordered if k of the numbers $a_1, a_2, \dots, a_n$ are smaller than or equal to r, and the other $n-k$ are greater than or equal to r. Notice that

$$A_1 = \left(\frac{a_{k+1}}{a_1}, \frac{a_{k+2}}{a_2}, \dots, \frac{a_k}{a_n}\right)$$

is a k-geometric ordered set if A is increasingly ordered, and is an $(n-k)$-geometric ordered set if A is decreasingly ordered.

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Theorem 1.1. *Let $n \geq 2$ and $1 \leq k \leq n-1$ be natural numbers, and let $f(u)$ be a function on a real interval I, which is convex for $u \geq s,\ s \in I$, and satisfies*

$$f(x) + kf(y) \geq (1+k)f(s)$$

for any $x, y \in I$ such that $x \leq y$ and $x + ky = (1+k)s$. If $x_1, x_2, \dots, x_n \in I$ such that

$$\frac{x_1 + x_2 + \cdots + x_n}{n} = S \geq s$$

and at least $n-k$ of $x_1, x_2, \dots, x_n$ are smaller than or equal to S, then

$$f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(S).$$

Proof. We will consider two cases: $S = s$ and $S > s$.

A. *Case* $S = s$. Without loss of generality, assume that $x_1 \leq x_2 \leq \cdots \leq x_n$. Since $x_1 + x_2 + \cdots + x_n = ns$, and at least $n-k$ of the numbers $x_1, x_2, \dots, x_n$ are smaller than or equal to s, there exists an integer $n-k \leq i \leq n-1$ such that $(x_1, x_2, \dots, x_n)$ is an i-arithmetic ordered set, i.e.

$$x_1 \leq \cdots \leq x_i \leq s \leq x_{i+1} \leq \cdots \leq x_n.$$

By Jensen's inequality for convex functions,

$$f(x_{i+1}) + f(x_{i+2}) + \cdots + f(x_n) \geq (n-i)f(z),$$

where

$$z = \frac{x_{i+1} + x_{i+2} + \cdots + x_n}{n-i}, \quad z \geq s,\ z \in I.$$

Thus, it suffices to prove that

$$f(x_1) + \cdots + f(x_i) + (n-i)f(z) \geq nf(s).$$

Let $y_1, y_2, \dots, y_i \in I$ be defined by

$$x_1 + ky_1 = (1+k)s,\ \ x_2 + ky_2 = (1+k)s,\ \dots,\ x_i + ky_i = (1+k)s.$$

We will show that $z \geq y_1 \geq y_2 \geq \cdots \geq y_i \geq s$. Indeed, we have

$$y_1 \geq y_2 \geq \cdots \geq y_i,$$
$$y_i - s = \frac{s - x_i}{k} \geq 0,$$

and

$$\begin{aligned} ky_1 &= (1+k)s - x_1 \\ &= (1+k-n)s + x_2 + \cdots + x_n \\ &\leq (k+i-n)s + x_{i+1} + \cdots + x_n \\ &= (k+i-n)s + (n-i)z \leq kz. \end{aligned}$$

Since $z \geq y_1 \geq y_2 \geq \cdots \geq y_i \geq s$ implies $y_1, y_2, \dots, y_i \in I$, by hypothesis we have

$$f(x_1) + kf(y_1) \geq (1+k)f(s),$$
$$f(x_2) + kf(y_2) \geq (1+k)f(s),$$
$$\cdots\cdots\cdots$$
$$f(x_i) + kf(y_i) \geq (1+k)f(s).$$

Adding all these inequalities, we get

$$f(x_1) + f(x_2) + \cdots + f(x_i) + k[f(y_1) + f(y_2) + \cdots + f(y_i)] \geq i(1+k)f(s).$$

Consequently, it suffices to show that

$$pf(z)+(i-p)f(s)\geq f(y_1)+f(y_2)+\cdots+f(y_i),$$

where $p=\frac{n-i}{k}\leq 1$. Let $t=pz+(1-p)s$, $s\leq t\leq z$. Since the decreasingly ordered vector $\vec{A_i}=(t,s,\dots,s)$ majorizes the decreasingly ordered vector $\vec{B_i}=(y_1,y_2,\dots,y_i)$, by Karamata's inequality for convex functions we have

$$f(t)+(i-1)f(s)\geq f(y_1)+f(y_2)+\cdots+f(y_i).$$

Adding this inequality to Jensen's inequality for the convex function

$$pf(z)+(1-p)f(s)\geq f(t),$$

the conclusion follows.

B. *Case* $S>s$. The function $f(u)$ is convex for $u\geq S$, $u\in I$. According to the result from Case A, it suffices to show that

$$f(x)+kf(y)\geq(1+k)f(S),$$

for any $x,y\in I$ such that $x<S<y$ and $x+ky=(1+k)S$.

For $x\geq s$, this inequality follows by Jensen's inequality for convex function.

For $x<s$, let z be defined by $x+kz=(1+k)s$. Since $k(z-s)=s-x>0$ and $k(y-z)=(1+k)(S-s)>0$, we have

$$x<s<z<y,\quad s<S<y.$$

Since $x+kz=(1+k)s$ and $x<z$, we have by hypothesis

$$f(x)+kf(z)\geq(1+k)f(s).$$

Therefore, it suffices to show that

$$k[f(y)-f(z)]\geq(1+k)[f(S)-f(s)],$$

which is equivalent to

$$\frac{f(y)-f(z)}{y-z}\geq\frac{f(S)-f(s)}{S-s}.$$

This inequality is true if

$$\frac{f(y)-f(z)}{y-z}\geq\frac{f(y)-f(s)}{y-s}\geq\frac{f(S)-f(s)}{S-s}.$$

The left inequality and the right inequality can be reduced to Jensen's inequalities for convex functions,

$$(y-z)f(s)+(z-s)f(y)\geq(y-s)f(z)$$

and

$$(S-s)f(y)+(y-S)f(s)\geq(y-s)f(S),$$

respectively. □

Remark 1.2. In the particular case $k=n-1$, if $f(x)+(n-1)f(y)\geq nf(s)$ for any $x,y\in I$ such that $x\leq y$ and $x+(n-1)y=ns$, then the inequality in Theorem 1.1,

$$f(x_1)+f(x_2)+\cdots+f(x_n)\geq nf(S),$$

holds for any $x_1,x_2,\dots,x_n\in I$ which satisfy $\frac{x_1+x_2+\cdots+x_n}{n}=S\geq s$. This result has been established in [1, p. 143] and [2].

Remark 1.3. In the particular case $k = 1$ (when $n-1$ of $x_1, x_2, \dots, x_n$ are smaller than or equal to S), the hypothesis $f(x) + kf(y) \geq (1+k)f(s)$ in Theorem 1.1 has a symmetric form:

$$f(x) + f(y) \geq 2f(s)$$

for any $x, y \in I$ such that $x + y = 2s$.

Remark 1.4. Let $g(u) = \frac{f(u)-f(s)}{u-s}$. In some applications it is useful to replace the hypothesis $f(x) + kf(y) \geq (1+k)f(s)$ in Theorem 1.1 by the equivalent condition:

$$g(x) \leq g(y) \quad \text{for any} \quad x, y \in I \quad \text{such that} \quad x < s < y \quad \text{and} \quad x + ky = (1+k)s.$$

Their equivalence follows from the following observation:

$$\begin{aligned} f(x) + kf(y) - (1+k)f(s) &= f(x) - f(s) + k(f(y) - f(s)) \\ &= (x-s)g(x) + k(y-s)g(y) \\ &= (x-s)(g(x) - g(y)). \end{aligned}$$

Remark 1.5. If f is differentiable on I, then Theorem 1.1 holds true by replacing the hypothesis $f(x) + kf(y) \geq (1+k)f(s)$ with the more restrictive condition:

$$f'(x) \leq f'(y) \quad \text{for any} \quad x, y \in I \quad \text{such that} \quad x \leq s \leq y \quad \text{and} \quad x + ky = (1+k)s.$$

To prove this assertion, we have to show that this condition implies $f(x)+kf(y) \geq (1+k)f(s)$ for any $x, y \in I$ such that $x \leq s \leq y$ and $x + ky = (1+k)s$. Let us denote

$$F(x) = f(x) + kf(y) - (1+k)f(s) = f(x) + kf\left(\frac{s + ks - x}{k}\right) - (1+k)f(s).$$

Since $F'(x) = f'(x) - f'(y) \leq 0$, $F(x)$ is decreasing for $x \in I$, $x \leq s$, and hence $F(x) \geq F(s) = 0$.

Remark 1.6. The inequality in Theorem 1.1 becomes equality for $x_1 = x_2 = \cdots = x_n = S$. In the particular case $S = s$, if there are $x, y \in I$ such that $x < s < y$, $x + ky = (k+1)s$ and $f(x) + kf(y) = (1+k)f(s)$, then equality holds again for $x_1 = x$, $x_2 = \cdots = x_{n-k} = s$ and $x_{n-k+1} = \cdots = x_n = y$.

Remark 1.7. Let i be an integer such that $n - k \leq i \leq n-1$. We may rewrite the inequality in Theorem 1.1 as either

$$f(S - a_1 + a_{n-i+1}) + f(S - a_2 + a_{n-i+2}) + \cdots + f(S - a_n + a_{n-i}) \geq nf(S)$$

with $a_1 \geq a_2 \geq \cdots \geq a_n$, or

$$f(S - a_1 + a_{i+1}) + f(S - a_2 + a_{i+2}) + \cdots + f(S - a_n + a_i) \geq nf(S)$$

with $a_1 \leq a_2 \leq \cdots \leq a_n$.

Corollary 1.8. *Let $n \geq 2$ and $1 \leq k \leq n-1$ be natural numbers, and let g be a function on $(0, \infty)$ such that $f(u) = g(e^u)$ is convex for $u \geq 0$, and*

$$g(x) + kg(y) \geq (1+k)g(1)$$

for any positive real numbers x and y with $x \leq y$ and $xy^k = 1$. If $a_1, a_2, \dots, a_n$ are positive real numbers such that $\sqrt[n]{a_1 a_2 \cdots a_n} = r \geq 1$ and at least $n-k$ of $a_1, a_2, \dots, a_n$ are smaller than or equal to r, then

$$g(a_1) + g(a_2) + \cdots + g(a_n) \geq ng(r).$$

Proof. We apply Theorem 1.1 to the function $f(u) = g(e^u)$. In addition, we set $s = 0$, $S = \ln r$, and replace x with $\ln x$, y with $\ln y$, and each x_i with $\ln a_i$. $\square$

Remark 1.9. If f is differentiable on $(0,\infty)$, then Corollary 1.8 holds true by replacing the hypothesis $g(x)+kg(y)\geq(1+k)g(1)$ with the more restrictive condition:

$$xg'(x)\leq yg'(y) \quad \text{for all} \quad x,y>0 \quad \text{such that} \quad x\leq 1\leq y \quad \text{and} \quad xy^k=1.$$

To prove this claim, it suffices to show that this condition implies $g(x)+kg(y)\geq(1+k)g(1)$ for all $x,y>0$ with $x\leq 1\leq y$ and $xy^k=1$. Let us define the function G by

$$G(x)=g(x)+kg(y)-(1+k)g(1)=g(x)+kg\left(\sqrt[k]{\frac{1}{x}}\right)-(1+k)g(1).$$

Since

$$G'(x)=g'(x)-\frac{1}{x\sqrt[k]{x}}g'(y)=\frac{xg'(x)-yg'(y)}{x}\leq 0,$$

$G(x)$ is decreasing for $x\leq 1$. Therefore, $G(x)\geq G(1)=0$ for $x\leq 1$, and hence $g(x)+kg(y)\geq(1+k)g(1)$.

Remark 1.10. Let i be an integer such that $n-k\leq i\leq n-1$. We may rewrite the inequality for $r=1$ in Corollary 1.8 as either

$$g\left(\frac{x_{n-i+1}}{x_1}\right)+g\left(\frac{x_{n-i+2}}{x_2}\right)+\cdots+g\left(\frac{x_{n-i}}{x_n}\right)\geq ng(1)$$

for $x_1\geq x_2\geq\cdots\geq x_n>0$, or

$$g\left(\frac{x_{i+1}}{x_1}\right)+g\left(\frac{x_{i+2}}{x_2}\right)+\cdots+g\left(\frac{x_i}{x_n}\right)\geq ng(1)$$

for $0<x_1\leq x_2\leq\cdots\leq x_n$.

Theorem 1.11. *Let $n\geq 2$ and $1\leq k\leq n-1$ be natural numbers, and let $f(u)$ be a function on a real interval I, which is concave for $u\leq s,\ s\in I$, and satisfies*

$$kf(x)+f(y)\leq(k+1)f(s)$$

for any $x,y\in I$ such that $x\leq y$ and $kx+y=(k+1)s$. If $x_1,x_2,\ldots,x_n\in I$ such that $\frac{x_1+x_2+\cdots+x_n}{n}=S\leq s$ and at least $n-k$ of $x_1,x_2,\ldots,x_n$ are greater than or equal to S, then

$$f(x_1)+f(x_2)+\cdots+f(x_n)\leq nf(S).$$

Proof. This theorem follows from Theorem 1.1 by replacing $f(u)$ by $-f(-u)$, s by $-s$, S by $-S$, x by $-y$, y by $-x$, and each x_i by $-x_{n-i+1}$ for all i. □

Remark 1.12. In the particular case $k=n-1$, if $(n-1)f(x)+f(y)\leq nf(s)$ for any $x,y\in I$ such that $x\leq y$ and $(n-1)x+y=ns$, then the inequality in Theorem 1.11,

$$f(x_1)+f(x_2)+\cdots+f(x_n)\leq nf(S),$$

holds for any $x_1,x_2,\ldots,x_n\in I$ which satisfy $\frac{x_1+x_2+\cdots+x_n}{n}=S\leq s$. This result has been established in [1, p. 147] and [2].

Remark 1.13. In the particular case $k=1$ (when $n-1$ of $x_1,x_2,\ldots,x_n$ are greater than or equal to S), the hypothesis $kf(x)+f(y)\leq(k+1)f(s)$ in Theorem 1.11 has a symmetric form: $f(x)+f(y)\leq 2f(s)$ for any $x,y\in I$ such that $x+y=2s$.

Remark 1.14. Let $g(u)=\frac{f(u)-f(s)}{u-s}$. The hypothesis $kf(x)+f(y)\leq(k+1)f(s)$ in Theorem 1.11 is equivalent to

$$g(x)\geq g(y) \quad \text{for any} \quad x,y\in I \quad \text{such that} \quad x<s<y \quad \text{and} \quad kx+y=(k+1)s.$$

Remark 1.15. If f is differentiable on I, then Theorem 1.11 holds true if we replace the hypothesis $kf(x)+f(y)\le(k+1)f(s)$ with the more restrictive condition

$$f'(x)\ge f'(y) \quad \text{for any} \quad x,y\in I \quad \text{such that} \quad x\le s\le y \quad \text{and} \quad kx+y=(k+1)s.$$

Remark 1.16. The inequality in Theorem 1.11 becomes equality for $x_1=x_2=\cdots=x_n=S$. In the particular case $S=s$, if there are $x,y\in I$ such that $x<s<y$, $kx+y=(k+1)s$ and $kf(x)+f(y)=(1+k)f(s)$, then equality holds again for $x_1=\cdots=x_k=x$, $x_{k+1}=\cdots=x_{n-1}=s$ and $x_n=y$.

Remark 1.17. Let i be an integer such that $1\le i\le k$. We may rewrite the inequality in Theorem 1.11 as either

$$f(S-a_1+a_{i+1})+f(S-a_2+a_{i+2})+\cdots+f(S-a_n+a_i)\le nf(S)$$

with $a_1\le a_2\le\cdots\le a_n$, or

$$f(S-a_1+a_{n-i+1})+f(S-a_2+a_{n-i+2})+\cdots+f(S-a_n+a_{n-i})\le nf(S)$$

with $a_1\ge a_2\ge\cdots\ge a_n$.

Corollary 1.18. *Let $n\ge 2$ and $1\le k\le n-1$ be natural numbers, and let g be a function on $(0,\infty)$ such that $f(u)=g(e^u)$ is concave for $u\le 0$, and*

$$kg(x)+g(y)\le(k+1)g(1)$$

for any positive real numbers x and y with $x\le y$ and $x^ky=1$. If $a_1,a_2,\ldots,a_n$ are positive real numbers such that $\sqrt[n]{a_1a_2\cdots a_n}=r\le 1$ and at least $n-k$ of $a_1,a_2,\ldots,a_n$ are greater than or equal to r, then

$$g(a_1)+g(a_2)+\cdots+g(a_n)\le ng(r).$$

Proof. We apply Theorem 1.11 to the function $f(u)=g(e^u)$. In addition, we set $s=0$, $S=\ln r$, and replace x with $\ln x$, y with $\ln y$, and each x_i with $\ln a_i$. □

Remark 1.19. If f is differentiable on $(0,\infty)$, then Corollary 1.18 holds true by replacing the hypothesis $kg(x)+g(y)\le(k+1)g(1)$ with the more restrictive condition:

$$xg'(x)\ge yg'(y) \quad \text{for all} \quad x,y>0 \quad \text{such that} \quad x\le 1\le y \quad \text{and} \quad x^ky=1.$$

Remark 1.20. Let i be an integer such that $1\le i\le k$. We may rewrite the inequality for $r=1$ in Corollary 1.18 as either

$$g\left(\frac{x_{i+1}}{x_1}\right)+g\left(\frac{x_{i+2}}{x_2}\right)+\cdots+g\left(\frac{x_i}{x_n}\right)\le ng(1)$$

for $0<x_1\le x_2\le\cdots\le x_n$, or

$$g\left(\frac{x_{n-i+1}}{x_1}\right)+g\left(\frac{x_{n-i+2}}{x_2}\right)+\cdots+g\left(\frac{x_{n-i}}{x_n}\right)\le ng(1)$$

for $x_1\ge x_2\ge\cdots\ge x_n>0$.

2. APPLICATIONS

Proposition 2.1. *Let $n\ge 2$ and $1\le k\le n-1$ be natural numbers, and let $x_1,x_2,\ldots,x_n$ be nonnegative real numbers such that $x_1+x_2+\cdots+x_n=n$.*

(a) *If at least $n-k$ of $x_1,x_2,\ldots,x_n$ are smaller than or equal to 1, then*

$$k(x_1^3+x_2^3+\cdots+x_n^3)+(1+k)n\ge(1+2k)(x_1^2+x_2^2+\cdots+x_n^2);$$

(b) *If at least* $n-k$ *of* $x_1, x_2, \dots, x_n$ *are greater than or equal to* 1*, then*

$$x_1^3 + x_2^3 + \cdots + x_n^3 + (k+1)n \le (k+2)(x_1^2 + x_2^2 + \cdots + x_n^2).$$

Proof. (a) The inequality is equivalent to $f(x_1) + f(x_2) + \cdots + f(x_n) \ge nf(S)$, where $S = \frac{x_1+x_2+\cdots+x_n}{n} = 1$ and $f(u) = ku^3 - (1+2k)u^2$. For $u \ge 1$,

$$f''(u) = 2(3ku - 1 - 2k) \ge 2(k-1) \ge 0.$$

Therefore, f is convex for $u \ge s = 1$. According to Theorem 1.1 and Remark 1.4, we have to show that $g(x) \le g(y)$ for any nonnegative real numbers $x < y$ such that $x + ky = 1 + k$, where

$$g(u) = \frac{f(u) - f(1)}{u-1} = ku^2 - (1+k)u - 1 - k.$$

Indeed,

$$g(y) - g(x) = (k-1)x(y-x) \ge 0.$$

Equality occurs for $x_1 = x_2 = \cdots = x_n = 1$. On the assumption that $x_1 \le x_2 \le \cdots \le x_n$, equality holds again for $x_1 = 0$, $x_2 = \cdots = x_{n-k} = 1$ and $x_{n-k+1} = \cdots = x_n = 1 + \frac{1}{k}$.

(b) Write the inequality as $f(x_1) + f(x_2) + \cdots + f(x_n) \le nf(S)$, where $S = \frac{x_1+x_2+\cdots+x_n}{n} = 1$ and $f(u) = u^3 - (k+2)u^2$. From the second derivative,

$$f''(u) = 2(3u - k - 2),$$

it follows that f is concave for $u \le s = 1$. According to Theorem 1.11 and Remark 1.14, we have to show that $g(x) \ge g(y)$ for any nonnegative real numbers $x < y$ such that $kx + y = k + 1$, where

$$g(u) = \frac{f(u) - f(1)}{u-1} = u^2 - (k+1)u - k - 1.$$

It is easy to see that

$$g(x) - g(y) = (k-1)x(y-x) \ge 0.$$

Equality occurs for $x_1 = x_2 = \cdots = x_n = 1$. On the assumption that $x_1 \le x_2 \le \cdots \le x_n$, equality holds again for $x_1 = \cdots = x_k = 0$, $x_{k+1} = \cdots = x_{n-1} = 1$ and $x_n = k + 1$. □

Remark 2.2. For $k = n - 1$, the inequalities above become as follows

$$(n-1)(x_1^3 + x_2^3 + \cdots + x_n^3) + n^2 \ge (2n-1)(x_1^2 + x_2^2 + \cdots + x_n^2)$$

and

$$x_1^3 + x_2^3 + \cdots + x_n^3 + n^2 \le (n+1)(x_1^2 + x_2^2 + \cdots + x_n^2),$$

respectively. By Remark 1.2 and Remark 1.12, these inequalities hold for any nonnegative real numbers $x_1, x_2, \dots, x_n$ which satisfy $x_1 + x_2 + \cdots + x_n = n$ (Problems 3.4.1 and 3.4.2 from [1, p. 154]).

Remark 2.3. For $k = 1$, we get the following statement:

Let $x_1, x_2, \dots, x_n$ be nonnegative real numbers such that $x_1 + x_2 + \cdots + x_n = n$.

(a) If $x_1 \le \cdots \le x_{n-1} \le 1 \le x_n$, then

$$x_1^3 + x_2^3 + \cdots + x_n^3 + 2n \ge 3(x_1^2 + x_2^2 + \cdots + x_n^2);$$

(b) If $x_1 \le 1 \le x_2 \le \cdots \le x_n$, then

$$x_1^3 + x_2^3 + \cdots + x_n^3 + 2n \le 3(x_1^2 + x_2^2 + \cdots + x_n^2).$$

Proposition 2.4. *Let $n \geq 2$ and $1 \leq k \leq n-1$ be natural numbers, and let $x_1, x_2, \dots, x_n$ be positive real numbers such that $x_1 + x_2 + \cdots + x_n = n$. If at least $n-k$ of $x_1, x_2, \dots, x_n$ are greater than or equal to 1, then*

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} - n \geq \frac{4k}{(k+1)^2}(x_1^2 + x_2^2 + \cdots + x_n^2 - n).$$

Proof. Rewrite the inequality as $f(x_1)+f(x_2)+\cdots+f(x_n) \leq nf(S)$, where $S = \frac{x_1+x_2+\cdots+x_n}{n} = 1$ and $f(u) = \frac{4ku^2}{(k+1)^2} - \frac{1}{u}$. For $0 < u \leq s = 1$, we have

$$f''(u) = \frac{8k}{(k+1)^2} - \frac{2}{u^3} \leq \frac{8k}{(k+1)^2} - 2 = \frac{-2(k-1)^2}{(k+1)^2} \leq 0;$$

therefore, f is concave on $(0,1]$. By Theorem 1.11 and Remark 1.14, we have to show that $g(x) \geq g(y)$ for any positive real numbers $x < y$ such that $kx + y = k+1$, where

$$g(u) = \frac{f(u) - f(1)}{u-1} = \frac{4k(u+1)}{(k+1)^2} + \frac{1}{u}.$$

Indeed,

$$g(x) - g(y) = (y-x)\left[\frac{1}{xy} - \frac{4k}{(k+1)^2}\right] = \frac{(y-x)(2kx-k-1)^2}{(k+1)^2xy} \geq 0.$$

Equality occurs for $x_1 = x_2 = \cdots = x_n = 1$. Under the assumption that $x_1 \leq x_2 \leq \cdots \leq x_n$, equality holds again for $x_1 = \cdots = x_k = \frac{k+1}{2k}$, $x_{k+1} = \cdots = x_{n-1} = 1$ and $x_n = \frac{k+1}{2}$. □

Remark 2.5. For $k = n-1$, the inequality in Proposition 2.4 becomes as follows:

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} - n \geq \frac{4(n-1)}{n^2}(x_1^2 + x_2^2 + \cdots + x_n^2 - n).$$

By Remark 1.12, this inequality holds for any positive real numbers $x_1, x_2, \dots, x_n$ which satisfy $x_1 + x_2 + \cdots + x_n = n$ (Problems 3.4.5 from [1, p. 158]).

Remark 2.6. For $k = 1$, the following nice statement follows:

If $x_1, x_2, \dots, x_n$ are positive real numbers such that $x_1 \leq 1 \leq x_2 \leq \cdots \leq x_n$ and $x_1 + x_2 + \cdots + x_n = n$, then

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} \geq x_1^2 + x_2^2 + \cdots + x_n^2.$$

Proposition 2.7. *Let $n \geq 2$ and $1 \leq k \leq n-1$ be natural numbers, and let $x_1, x_2, \dots, x_n$ be nonnegative real numbers such that $x_1 + x_2 + \cdots + x_n = n$.*

(a) *If at least $n-k$ of $x_1, x_2, \dots, x_n$ are smaller than or equal to 1, then*

$$\frac{1}{k+1+kx_1^2} + \frac{1}{k+1+kx_2^2} + \cdots + \frac{1}{k+1+kx_n^2} \geq \frac{n}{2k+1};$$

(b) *If at least $n-k$ of $x_1, x_2, \dots, x_n$ are greater than or equal to 1, then*

$$\frac{1}{k^2+k+1+kx_1^2} + \frac{1}{k^2+k+1+kx_2^2} + \cdots + \frac{1}{k^2+k+1+kx_n^2} \leq \frac{n}{(k+1)^2}.$$

Proof. (a) We may write the inequality as $f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(S)$, where $S = \frac{x_1+x_2+\cdots+x_n}{n} = 1$ and $f(u) = \frac{1}{k+1+ku^2}$. Since the second derivative,

$$f''(u) = \frac{2k(3ku^2-k-1)}{(k+1+ku^2)^3},$$

is positive for $u \geq 1$, f is convex for $u \geq s = 1$. According to Theorem 1.1 and Remark 1.4, we have to show that $g(x) \leq g(y)$ for any nonnegative real numbers $x < y$ such that $x + ky = 1 + k$, where

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{-k(u+1)}{(2k+1)(k+1+ku^2)}.$$

Indeed, we have

$$g(y) - g(x) = \frac{k^2(y-x)}{(2k+1)(k+1+kx^2)(k+1+ky^2)}\left(xy + x + y - 1 - \frac{1}{k}\right) \geq 0,$$

since

$$xy + x + y - 1 - \frac{1}{k} = \frac{x(2k-1+y)}{k} \geq 0.$$

Equality occurs for $x_1 = x_2 = \cdots = x_n = 1$. On the assumption that $x_1 \leq x_2 \leq \cdots \leq x_n$, equality holds again for $x_1 = 0$, $x_2 = \cdots = x_{n-k} = 1$ and $x_{n-k+1} = \cdots = x_n = 1 + \frac{1}{k}$.

(b) We will apply Theorem 1.11 to the function $f(u) = \frac{1}{k^2+k+1+ku^2}$, for $s = S = 1$. Since the second derivative,

$$f''(u) = \frac{2k(3ku^2 - k^2 - k - 1)}{(k^2+k+1+ku^2)^3},$$

is negative for $0 \leq u < 1$, f is concave for $0 \leq u \leq 1$. According to Remark 1.14, we have to show that $g(x) \geq g(y)$ for any nonnegative real numbers $x < y$ such that $kx + y = k + 1$, where

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{-k(u+1)}{(k+1)^2(k^2+k+1+ku^2)}.$$

We have

$$g(x) - g(y) = \frac{k^2(y-x)}{(k+1)^2(k^2+k+1+kx^2)(k^2+k+1+ky^2)} \times \left(k + \frac{1}{k} + 1 - xy - x - y\right) \geq 0,$$

since

$$k + \frac{1}{k} + 1 - xy - x - y = k\left(x - \frac{1}{k}\right)^2 \geq 0.$$

Equality occurs for $x_1 = x_2 = \cdots = x_n = 1$. On the assumption that $x_1 \leq x_2 \leq \cdots \leq x_n$, equality holds again for $x_1 = \cdots = x_k = \frac{1}{k}$, $x_{k+1} = \cdots = x_{n-1} = 1$ and $x_n = k$. □

Remark 2.8. For $k = n - 1$, the inequalities in Proposition 2.7 become as follows:

$$\frac{1}{n+(n-1)x_1^2} + \frac{1}{n+(n-1)x_2^2} + \cdots + \frac{1}{n+(n-1)x_n^2} \geq \frac{n}{2n-1}$$

and

$$\frac{1}{n^2-n+1+(n-1)x_1^2} + \frac{1}{n^2-n+1+(n-1)x_2^2} + \cdots + \frac{1}{n^2-n+1+(n-1)x_n^2} \leq \frac{1}{n},$$

respectively. By Remark 1.2 and Remark 1.12, these inequalities hold for any nonnegative numbers $x_1, x_2, \ldots, x_n$ which satisfy $x_1 + x_2 + \cdots + x_n = n$ (Problems 3.4.3 and 3.4.4 from [1, p. 156]).

Remark 2.9. For $k = 1$, we get the following statement:

Let $x_1, x_2, \ldots, x_n$ be nonnegative real numbers such that $x_1 + x_2 + \cdots + x_n = n$.

(a) If $x_1 \le \cdots \le x_{n-1} \le 1 \le x_n$, then

$$\frac{1}{2+x_1^2} + \frac{1}{2+x_2^2} + \cdots + \frac{1}{2+x_n^2} \ge \frac{n}{3};$$

(b) If $x_1 \le 1 \le x_2 \le \cdots \le x_n$, then

$$\frac{1}{3+x_1^2} + \frac{1}{3+x_2^2} + \cdots + \frac{1}{3+x_n^2} \le \frac{n}{4}.$$

Remark 2.10. By Theorem 1.1 and Theorem 1.11, the following more general statement holds:

Let $n \ge 2$ and $1 \le k \le n-1$ be natural numbers, and let $x_1, x_2, \ldots, x_n$ be nonnegative real numbers such that $x_1 + x_2 + \cdots + x_n = nS$.

(a) If $S \ge 1$ and at least $n-k$ of $x_1, x_2, \ldots, x_n$ are smaller than or equal to S, then

$$\frac{1}{k+1+kx_1^2} + \frac{1}{k+1+kx_2^2} + \cdots + \frac{1}{k+1+kx_n^2} \ge \frac{n}{k+1+kS^2};$$

(b) If $S \le 1$ and at least $n-k$ of $x_1, x_2, \ldots, x_n$ are greater than or equal to S, then

$$\frac{1}{k^2+k+1+kx_1^2} + \frac{1}{k^2+k+1+kx_2^2} + \cdots + \frac{1}{k^2+k+1+kx_n^2} \le \frac{n}{k^2+k+1+kS^2}.$$

Proposition 2.11. *Let $n \ge 2$ and $1 \le k \le n-1$ be natural numbers, and let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 a_2 \cdots a_n = 1$.*

(a) *If at least $n-k$ of $x_1, x_2, \ldots, x_n$ are smaller than or equal to 1, then*

$$\frac{1}{1+ka_1} + \frac{1}{1+ka_2} + \cdots + \frac{1}{1+ka_n} \ge \frac{n}{1+k};$$

(b) *If at least $n-k$ of $x_1, x_2, \ldots, x_n$ are greater than or equal to 1, then*

$$\frac{1}{a_1+k} + \frac{1}{a_2+k} + \cdots + \frac{1}{a_n+k} \le \frac{n}{1+k}.$$

Proof. (a) We will apply Corollary 1.8 to the function $g(x) = \frac{1}{1+kx}$, for $r = 1$. The function $f(u) = g(e^u) = \frac{1}{1+ke^u}$ has the second derivative

$$f''(u) = \frac{ke^u(ke^u-1)}{(1+ke^u)^3},$$

which is positive for $u > 0$. Therefore, f is convex for $u \ge 0$. Thus, it suffices to show that $g(x) + kg(y) \ge (1+k)g(1)$ for any $x, y > 0$ such that $xy^k = 1$. The inequality $g(x) + kg(y) \ge (1+k)g(1)$ is equivalent to

$$\frac{y^k}{y^k+k} + \frac{k}{1+ky} \ge 1,$$

or, equivalently,

$$y^k + k - 1 \ge ky.$$

The last inequality immediately follows from the AM-GM inequality applied to the positive numbers $y^k, 1, \ldots, 1$. Equality occurs for $a_1 = a_2 = \cdots = a_n = 1$.

(b) We can obtain the required inequality either by replacing each number a_i with its reverse $\frac{1}{a_i}$ in the inequality in part (a), or by means of Corollary 1.18. Equality occurs for $a_1 = a_2 = \cdots = a_n = 1$. □

Remark 2.12. For $k = n-1$, we get the known inequalities

$$\frac{1}{1+(n-1)a_1} + \frac{1}{1+(n-1)a_2} + \cdots + \frac{1}{1+(n-1)a_n} \geq 1$$

and

$$\frac{1}{a_1+n-1} + \frac{1}{a_2+n-1} + \cdots + \frac{1}{a_n+n-1} \leq 1,$$

which hold for any positive numbers $a_1, a_2, \ldots, a_n$ such that $a_1 a_2 \cdots a_n = 1$.

Remark 2.13. Using the substitution $a_1 = \frac{x_{k+1}}{x_1},\ a_2 = \frac{x_{k+2}}{x_2}, \ldots,\ a_n = \frac{x_k}{x_n}$, we get the following statement:

Let $n \geq 2$ and $1 \leq k \leq n-1$ be natural numbers, and let $x_1, x_2, \ldots, x_n$ be positive real numbers.

(a) If $x_1 \geq x_2 \geq \cdots \geq x_n$, then

$$\frac{x_1}{x_1+kx_{k+1}} + \frac{x_2}{x_2+kx_{k+2}} + \cdots + \frac{x_n}{x_n+kx_k} \geq \frac{n}{1+k};$$

(b) If $x_1 \leq x_2 \leq \cdots \leq x_n$, then

$$\frac{x_1}{kx_1+x_{k+1}} + \frac{x_2}{kx_2+x_{k+2}} + \cdots + \frac{x_n}{kx_n+x_k} \leq \frac{n}{k+1}.$$

In the particular case $k = 1$, we get

$$\frac{x_1}{x_1+x_2} + \frac{x_2}{x_2+x_3} + \cdots + \frac{x_n}{x_n+x_1} \geq \frac{n}{2}$$

for $x_1 \geq x_2 \geq \cdots \geq x_n > 0$, and

$$\frac{x_1}{x_1+x_2} + \frac{x_2}{x_2+x_3} + \cdots + \frac{x_n}{x_n+x_1} \leq \frac{n}{2}$$

for $0 < x_1 \leq x_2 \leq \cdots \leq x_n$.

Remark 2.14. By Corollary 1.8 and Corollary 1.18, we can see that the following more general statement holds:

Let $n \geq 2$ and $1 \leq k \leq n-1$ be natural numbers, and let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $\sqrt[n]{a_1 a_2 \cdots a_n} = r$.

(a) If $r \geq 1$, and at least $n-k$ of $a_1, a_2, \ldots, a_n$ are smaller than or equal to r, then

$$\frac{1}{1+ka_1} + \frac{1}{1+ka_2} + \cdots + \frac{1}{1+ka_n} \geq \frac{n}{1+kr};$$

(b) If $r \leq 1$, and at least $n-k$ of $a_1, a_2, \ldots, a_n$ are greater than or equal to r, then

$$\frac{1}{a_1+k} + \frac{1}{a_2+k} + \cdots + \frac{1}{a_n+k} \leq \frac{n}{r+k}.$$

Proposition 2.15. *Let $a_1, a_2, \ldots, a_n$ be positive numbers such that $a_1 a_2 \cdots a_n = 1$.*

(a) *If $a_1 \leq \cdots \leq a_{n-1} \leq 1 \leq a_n$, then*

$$\frac{1}{\sqrt{1+3a_1}} + \frac{1}{\sqrt{1+3a_2}} + \cdots + \frac{1}{\sqrt{1+3a_n}} \geq \frac{n}{2};$$

(b) *If $a_1 \leq 1 \leq a_2 \leq \cdots \leq a_n$, then*

$$\frac{1}{\sqrt{1+2a_1}} + \frac{1}{\sqrt{1+2a_2}} + \cdots + \frac{1}{\sqrt{1+2a_n}} \leq \frac{n}{\sqrt{3}}.$$

Proof. (a) We will apply Corollary 1.8 (case $k = 1$ and $r = 1$) to the function $g(x) = \frac{1}{\sqrt{1+3x}}$. The function $f(u) = g(e^u) = \frac{1}{\sqrt{1+3e^u}}$ has the second derivative

$$f''(u) = \frac{1}{2}e^u(3e^u - 2)(1+3e^u)^{-\frac{5}{2}}.$$

Since $f'' > 0$ for $u \geq 0$, f is convex for $u \geq 0$. Therefore, to finish the proof, we have to show that $g(x) + g(y) \geq 2g(1)$ for any $x, y > 0$ with $xy = 1$. This inequality is equivalent to

$$\frac{1}{\sqrt{1+3x}} + \sqrt{\frac{x}{x+3}} \geq 1.$$

Using the substitution $\frac{1}{\sqrt{1+3x}} = t,\ 0 < t < 1$, transforms the inequality into

$$\sqrt{\frac{1-t^2}{8t^2+1}} \geq 1 - t.$$

By squaring, we get $t(1-t)(2t-1)^2 \geq 0$, which is clearly true. Equality occurs for $a_1 = a_2 = \cdots = a_n = 1$.

(b) We will apply Corollary 1.18 (case $k = 1$ and $r = 1$) to the function $g(x) = \frac{1}{\sqrt{1+2x}}$. The function $f(u) = g(e^u) = \frac{1}{\sqrt{1+2e^u}}$ is concave for $u \leq 0$, since

$$f'' = e^u(e^u - 1)(1+2e^u)^{-\frac{5}{2}} \leq 0.$$

Thus, it suffices to show that $g(x)+g(y) \leq 2g(1)$ for any $x, y > 0$ with $xy = 1$. This inequality follows from the Cauchy-Schwarz inequality, as follows

$$\sqrt{\frac{3}{1+2x}} + \sqrt{\frac{3}{1+2y}} \leq \sqrt{\left(\frac{3}{1+2x}+1\right)\left(1+\frac{3}{1+2y}\right)} = 2.$$

Equality occurs for $a_1 = a_2 = \cdots = a_n = 1$. □

Remark 2.16. Using the substitution $a_1 = \frac{x_2}{x_1},\ a_2 = \frac{x_3}{x_2}, \ldots,\ a_n = \frac{x_1}{x_n}$, we get the following statement:

Let $x_1, x_2, \ldots, x_n$ be positive real numbers.

(a) If $x_1 \geq x_2 \geq \cdots \geq x_n$, then

$$\sqrt{\frac{x_1}{x_1+3x_2}} + \sqrt{\frac{x_2}{x_2+3x_3}} + \cdots + \sqrt{\frac{x_n}{x_n+3x_1}} \geq \frac{n}{2};$$

(b) If $x_1 \leq x_2 \leq \cdots \leq x_n$, then

$$\sqrt{\frac{3x_1}{x_1+2x_2}} + \sqrt{\frac{3x_2}{x_2+2x_3}} + \cdots + \sqrt{\frac{3x_n}{x_n+2x_1}} \leq n.$$

Remark 2.17. By Corollary 1.8 and Corollary 1.18, the following more general statement holds:

Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $\sqrt[n]{a_1a_2\cdots a_n} = r$.

(a) If $r \geq 1$ and $a_1 \leq \cdots \leq a_{n-1} \leq r \leq a_n$, then

$$\frac{1}{\sqrt{1+3a_1}} + \frac{1}{\sqrt{1+3a_2}} + \cdots + \frac{1}{\sqrt{1+3a_n}} \geq \frac{n}{\sqrt{1+3r}};$$

(b) If $r \leq 1$ and $a_1 \leq r \leq a_2 \leq \cdots \leq a_n$, then

$$\frac{1}{\sqrt{1+2a_1}} + \frac{1}{\sqrt{1+2a_2}} + \cdots + \frac{1}{\sqrt{1+2a_n}} \leq \frac{n}{\sqrt{1+2r}}.$$

Proposition 2.18. *Let $a_1, a_2, \ldots, a_n$ be positive numbers such that $a_1a_2\cdots a_n = 1$.*

(a) *If $a_1 \le \cdots \le a_{n-1} \le 1 \le a_n$, then the following inequality holds for $0 \le p \le p_0$, where $p_0 \cong 1.5214$ is the positive root of the equation $p^3 - p - 2 = 0$:*

$$\frac{1}{(p+a_1)^2} + \frac{1}{(p+a_2)^2} + \cdots + \frac{1}{(p+a_n)^2} \ge \frac{n}{(p+1)^2};$$

(b) *If $a_1 \le 1 \le a_2 \le \cdots \le a_n$, then the following inequality holds for $p \ge 1+\sqrt{2}$:*

$$\frac{1}{(p+a_1)^2} + \frac{1}{(p+a_2)^2} + \cdots + \frac{1}{(p+a_n)^2} \le \frac{n}{(p+1)^2}.$$

Proof. (a) We will apply Corollary 1.8 (case $k = 1$ and $r = 1$) to the function $g(x) = \frac{1}{(p+x)^2}$. Notice that the function $f(u) = g(e^u) = \frac{1}{(p+e^u)^2}$ is convex for $u \ge 0$, because

$$f''(u) = \frac{2e^u(2e^u - p)}{(p+e^u)^4} > 0.$$

Consequently, we have to show that $g(x) + g(y) \ge 2g(1)$ for any $x, y > 0$ with $xy = 1$; that is

$$\frac{1}{(p+x)^2} + \frac{1}{(p+y)^2} \ge \frac{2}{(p+1)^2}.$$

Using the substitution $x + y = 2t,\ t \ge 1$, the inequality transforms into

$$\frac{2t^2 + 2pt + p^2 - 1}{(2pt + p^2 + 1)^2} \ge \frac{1}{(p+1)^2},$$

or, equivalently,

$$(t-1)[(1+2p-p^2)t + (1-p)(p^2+1)] \ge 0.$$

It is true, because $1 + 2p - p^2 > p(2-p) > 0$ and

$$\begin{aligned}(1+2p-p^2)t + (1-p)(p^2+1) &\ge (1+2p-p^2) + (1-p)(p^2+1)\\ &= 2 + p - p^3 \ge 0\end{aligned}$$

for $0 \le p \le p_0$. Equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

(b) We will apply Corollary 1.18 (case $k = 1$ and $r = 1$) to the function $g(x) = \frac{1}{(p+x)^2}$. The function $f(u) = g(e^u) = \frac{1}{(p+e^u)^2}$ is concave for $u \le 0$, since

$$f''(u) = \frac{2e^u(2e^u - p)}{(p+e^u)^4} < 0.$$

By Corollary 1.18, it suffices to show that $g(x) + g(y) \le 2g(1)$ for any $x, y > 0$ with $xy = 1$; that is

$$\frac{1}{(p+x)^2} + \frac{1}{(p+y)^2} \le \frac{2}{(p+1)^2}.$$

Using the notation $x + y = 2t,\ t \ge 1$, the inequality becomes

$$(t-1)[(p^2 - 2p - 1)t + (p-1)(p^2+1)] \ge 0.$$

It is true, since $p^2 - 2p - 1 \ge 0$ for $p \ge 1+\sqrt{2}$. Equality holds for $a_1 = a_2 = \cdots = a_n = 1$. ☐

Remark 2.19. Using the substitution $a_1 = \frac{x_2}{x_1},\ a_2 = \frac{x_3}{x_2}, \ldots,\ a_n = \frac{x_1}{x_n}$, we get the following statement:

Let $x_1, x_2, \ldots, x_n$ be positive real numbers.

(a) If $0 \le p \le p_0 \cong 1.5214$ and $x_1 \ge x_2 \ge \cdots \ge x_n$, then
$$\left(\frac{x_1}{px_1+x_2}\right)^2 + \left(\frac{x_2}{px_2+x_3}\right)^2 + \cdots + \left(\frac{x_n}{px_n+x_1}\right)^2 \ge \frac{n}{(p+1)^2};$$
(b) If $p \ge 1+\sqrt{2}$ and $x_1 \le x_2 \le \cdots \le x_n$, then
$$\left(\frac{x_1}{px_1+x_2}\right)^2 + \left(\frac{x_2}{px_2+x_3}\right)^2 + \cdots + \left(\frac{x_n}{px_n+x_1}\right)^2 \le \frac{n}{(p+1)^2}.$$

Remark 2.20. By Corollary 1.8 and Corollary 1.18, the following more general statement holds:

Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $\sqrt[n]{a_1 a_2 \cdots a_n} = r$.

(a) If $r \ge 1$ and $a_1 \le \cdots \le a_{n-1} \le r \le a_n$, then the following inequality holds for $0 \le p \le p_0$, where $p_0 \cong 1,5214$ is the positive root of the equation $p^3 - p - 2 = 0$:
$$\frac{1}{(p+a_1)^2} + \frac{1}{(p+a_2)^2} + \cdots + \frac{1}{(p+a_n)^2} \ge \frac{n}{(p+r)^2};$$
(b) If $r \le 1$ and $a_1 \le r \le a_2 \le \cdots \le a_n$, then the following inequality holds for $p \ge 1+\sqrt{2}$:
$$\frac{1}{(p+a_1)^2} + \frac{1}{(p+a_2)^2} + \cdots + \frac{1}{(p+a_n)^2} \le \frac{n}{(p+r)^2}.$$

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