ON A GENERALIZATION OF THE HERMITE-HADAMARD INEQUALITY II

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Received: 01 November, 2007

Accepted: 12 November, 2007

Communicated by: W.S. Cheung

2000 AMS Sub. Class.: Primary 26A51; Secondary 26A46, 26A48.

Key words: Convex function, Hermite-Hadamard inequality, Taylor's Formula.

Abstract: Generalized form of Hermite-Hadamard inequality for $(2n)$-convex Lebesgue integrable functions are obtained through generalization of Taylor's Formula.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

The classical Hermite-Hadamard inequality gives us an estimate, from below and from above, of the mean value of a convex function $f:[a,b]\to\mathbb{R}$ (see [1, pp. 137]):

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)\,dx\le\frac{f(a)+f(b)}{2}. \tag{HH}$$

In [2] the first author with Sabir Hussain proved the following two theorems

Theorem 1. *Assume that f is Lebesgue integrable and convex on (a,b). Then*

$$\frac{1}{b-a}\int_a^b f(y)dy+f'_+(x)\left(x-\frac{a+b}{2}\right)-f(x)$$
$$\ge\left|\frac{1}{b-a}\int_a^b |f(y)-f(x)|\,dy-|f'_+(x)|\,\frac{(x-a)^2+(b-x)^2}{2(b-a)}\right|$$

for all $x\in(a,b)$.

Theorem 2. *Assume that $f:[a,b]\to\mathbb{R}$ is a convex function. Then*

$$\frac{1}{2}\left[f(x)+\frac{f(b)(b-x)+f(a)(x-a)}{b-a}\right]-\frac{1}{b-a}\int_a^b f(y)dy$$
$$\ge\frac{1}{2}\left|\frac{1}{b-a}\int_a^b |f(x)-f(y)|\,dy-\frac{1}{b-a}\int_a^b |x-y|\,|f'(y)|\,dy\right|$$

for all $x\in(a,b)$.

Remark 1. For $x=\frac{a+b}{2}$ in Theorem 1 and $x=a$ or $x=b$ in Theorem 2, we obtain improvements of inequality (HH).

In this paper we will prove further generalizations of these results.

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issn: 1443-5756

Theorem 3. *Assume that* $f : [a,b] \to \mathbb{R}$ *is a* $(2n-1)$*-times differentiable and* $(2n)-$*convex function. Then*

$$\frac{1}{(b-a)}\int_a^b f(y)dy - (b-a)f(x) - \sum_1^{2n-1}\frac{(b-x)^{k+1}-(a-x)^{k+1}}{(k+1)!(b-a)}f^{(k)}(x)$$
$$\geq \left|\frac{1}{(b-a)}\int_a^b \left|f(y)-f(x)-\sum_1^{2n-2}\frac{(y-x)^k}{k!}f^{(k)}(x)\right|dy \right.$$
$$\left. - \left|f^{(2n-1)}(x)\frac{(b-x)^{2n}-(a-x)^{2n}}{(2n)!(b-a)}\right|\right|$$

for all $x \in (a,b)$.

Proof. It is well known that a continuous $(2n)-$convex function can be uniformly approximated by a $(2n)-$convex polynomial. So we can suppose that we have $(2n)-$derivatives of f. By Taylor's formula,

$$f(y) = f(x) + (y-x)f'(x) + \frac{(y-x)^2}{2!}f''(x) + \cdots$$
$$+ \frac{(y-x)^{2n-1}}{2n-1!}f^{(2n-1)}(x) + \frac{(y-x)^{2n}}{2n!}f^{(2n)}(\xi),$$

for $x, y \in [a,b]$, $\xi \in (a,b)$. Since f is $(2n)-$convex, we have $f^{(2n)}(x) \geq 0$. So

$$f(y) \geq f(x) + (y-x)f'(x) + \frac{(y-x)^2}{2!}f''(x) + \cdots + \frac{(y-x)^{2n-1}}{(2n-1)!}f^{(2n-1)}(x)$$

and we can write

$$f(y) - f(x) - (y-x)f'(x) - \frac{(y-x)^2}{2!}f''(x) - \cdots - \frac{(y-x)^{2n-1}}{2n-1!}f^{(2n-1)}(x) \geq 0,$$

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i.e.,

$$f(y)-f(x)-(y-x)f'(x)-\frac{(y-x)^2}{2!}f''(x)-\cdots-\frac{(y-x)^{2n-1}}{(2n-1)!}f^{(2n-1)}(x)$$
$$=\left|f(y)-f(x)-(y-x)f'(x)-\frac{(y-x)^2}{2!}f''(x)-\cdots-\frac{(y-x)^{2n-1}}{(2n-1)!}f^{(2n-1)}(x)\right|.$$

Now by using the triangle inequality

$$(1)\quad f(y)-f(x)-(y-x)f'(x)-\frac{(y-x)^2}{2!}f''(x)-\cdots-\frac{(y-x)^{2n-1}}{2n-1!}f^{(2n-1)}(x)$$
$$\geq\left|\left|f(y)-f(x)-(y-x)f'(x)-\cdots-\frac{(y-x)^{2n-2}}{2n-2!}f^{(2n-2)}(x)\right|\right.$$
$$\left.-\left|\frac{(y-x)^{2n-1}}{2n-1!}f^{(2n-1)}(x)\right|\right|.$$

Now integrating the last inequality with respect to y and using the triangle inequality for integrals, we get

$$\int_a^b f(y)dy-(b-a)f(x)-\sum_{1}^{2n-1}\frac{(b-x)^{k+1}-(a-x)^{k+1}}{(k+1)!}f^{(k)}(x)$$
$$\geq\left|\int_a^b\left|f(y)-f(x)-\sum_{1}^{2n-2}\frac{(y-x)^k}{k!}f^{(k)}(x)\right|dy\right.$$
$$\left.-\left|f^{(2n-1)}(x)\frac{(b-x)^{2n}-(a-x)^{2n}}{(2n)!}\right|\right|.$$

□

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Theorem 4. *Assume that* $f : [a,b] \to \mathbb{R}$ *is a* $(2n-1)-$*times differentiable and* $(2n)-$*convex function. Then*

$$f(x) - \frac{2n}{(b-a)}\int_a^b f(y)dy - \sum_1^{2n-1} \frac{2n-k}{k!(b-a)}[(x-b)^k f^{(k-1)}(b) - (x-a)^k f^{(k-1)}(a)]$$
$$\geq \left| \frac{1}{b-a}\int_a^b \left| f(x) - f(y) - \sum_1^{2n-2} \frac{(x-y)^k}{k!} f^{(k)}(y) \right| dy \right.$$
$$\left. - \frac{1}{b-a}\int_a^b \left| \frac{(x-y)^{2n-1}}{(2n-1)!} f^{(2n-1)}(y) \right| dy \right| .$$

Proof. Integrating (1) with respect to x and by using the triangle inequality for integrals, we get

$$(2)\quad (b-a)f(y) - \int_a^b f(x)dx - \int_a^b \sum_1^{2n-1} \frac{(y-x)^k}{k!} f^{(k)}(x)dx$$
$$\geq \left| \int_a^b \left| f(y) - f(x) - \sum_1^{2n-2} \frac{(y-x)^k}{k!} f^{(k)}(x) \right| dx \right.$$
$$\left. - \int_a^b \left| \frac{(y-x)^{2n-1}}{(2n-1)!} f^{(2n-1)}(x) \right| dx \right| .$$

By replacing x and y we obtain the required result. □

Corollary 5. *Suppose that* $f : [a,b] \to \mathbb{R}$ *is a* $(2n-1)-$*times differentiable and* $(2n)-$*convex function. Then*

$$\frac{1}{(b-a)}\int_a^b f(y)dy - (b-a)f\left(\frac{a+b}{2}\right) - \sum_1^{2n-1} \frac{\left(\frac{b-a}{2}\right)^{k+1} - \left(\frac{a-b}{2}\right)^{k+1}}{(k+1)!(b-a)} f^{(k)}\left(\frac{a+b}{2}\right)$$

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$$\geq \left| \frac{1}{(b-a)} \int_a^b \left| f(y) - f\left(\frac{a+b}{2}\right) - \sum_1^{2n-2} \frac{\left(y - \frac{a+b}{2}\right)^k}{k!} f^{(k)}\left(\frac{a+b}{2}\right) \right| dy \right.$$
$$\left. - \left| f^{(2n-1)}\left(\frac{a+b}{2}\right) \frac{(b-a)^{2n} - (a-b)^{2n}}{(2n)!(b-a)2^{2n}} \right| \right|.$$

Proof. Set $x = \frac{a+b}{2}$ in Theorem 3. □

Corollary 6. *Suppose that* $f : [a,b] \to \mathbb{R}$ *is a* $(2n-1)-$*times differentiable and* $(2n)-$*convex function. Then*

$$f(a) - \frac{2n}{(b-a)} \int_a^b f(y)dy - \sum_1^{2n-1} \frac{2n-k}{k!(b-a)} [(a-b)^k f^{(k-1)}(b)]$$
$$\geq \left| \frac{1}{b-a} \int_a^b \left| f(a) - f(y) - \sum_1^{2n-2} \frac{(a-y)^k}{k!} f^{(k)}(y) \right| dy \right.$$
$$\left. - \frac{1}{b-a} \int_a^b \left| \frac{(a-y)^{2n-1}}{(2n-1)!} f^{(2n-1)}(y) \right| dy \right|.$$

Proof. Set $x = a$ in Theorem 4. □

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issn: 1443-5756