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SOME EXTENSIONS OF HILBERT'S TYPE INEQUALITY AND ITS APPLICATIONS

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ABSTRACT. By introducing parameters λ and μ, we give a generalization of the Hilbert's type integral inequality. As applications, we give its equivalent form.

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1. INTRODUCTION

If $f, g \geq 0$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \int_0^\infty f^p(x)dx < \infty$ and $0 < \int_0^\infty g^q(x)dx < \infty$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\pi/p)} \left\{ \int_0^\infty f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty g^q(x)dx \right\}^{\frac{1}{q}}, \tag{1.1}$$

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y} dx \right)^p dy < \left[\frac{\pi}{\sin(\pi/p)} \right]^p \int_0^\infty f^p(x)dx, \tag{1.2}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Inequality (1.1) is known as Hardy-Hilbert's integral inequality (see [1]); it is important in analysis and its applications (see [4]). Under the same condition of (1.1), we have the Hardy-Hilbert type inequality similar to (1.1):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}} dxdy < pq \left\{ \int_0^\infty f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty g^q(x)dx \right\}^{\frac{1}{q}}, \tag{1.3}$$

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342-05

$$(1.4)\qquad \int_0^\infty \left(\int_0^\infty \frac{f(x)}{\max\{x,y\}}dx\right)^p dy < (pq)^p \int_0^\infty f^p(x)dx,$$

where the constant factor pq is the best possible. The corresponding inequality for series is:

$$(1.5)\qquad \sum_{n=1}^\infty\sum_{m=1}^\infty \frac{a_m b_n}{\max\{m,n\}} < pq\left(\sum_{n=1}^\infty a_n^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty b_n^q\right)^{\frac{1}{q}},$$

where the constant factor pq is also the best possible. In particular, when $p = q = 2$, we have the well-known Hilbert type inequality:

$$(1.6)\qquad \int_0^\infty\int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}}dxdy < 4\left\{\int_0^\infty f^2(x)dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty g^2(x)dx\right\}^{\frac{1}{2}}.$$

In recent years, Kuang (see [3]) gave a strengthened form as:

$$(1.7)\qquad \sum_{n=1}^\infty\sum_{m=1}^\infty \frac{a_m b_n}{\max\{m,n\}} < \left\{\sum_{n=1}^\infty [pq-G(p,n)]a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^\infty [pq-G(q,n)]b_n^q\right\}^{\frac{1}{q}},$$

where $G(r,n) = \frac{r+1/3r-4/3}{(2n+1)^{1/r}} > 0$ $(r = p,q)$.

Yang (see [5, 8]) gave: for $\lambda > 2-\min\{p,q\}$

$$(1.8)\qquad \int_0^\infty\int_0^\infty \frac{f(x)g(y)}{\max\{x^\lambda,y^\lambda\}}dxdy < \frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)} \times\left\{\int_0^\infty x^{(p-1)(2-\lambda)-1}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty x^{(q-1)(2-\lambda)-1}g^q(x)dx\right\}^{\frac{1}{q}}$$

and

$$(1.9)\qquad \int_0^\infty\int_0^\infty \frac{f(x)g(y)}{\max\{x^\lambda,y^\lambda\}}dxdy < \frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)} \times\left\{\int_0^\infty x^{1-\lambda}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty x^{1-\lambda}g^q(x)dx\right\}^{\frac{1}{q}}.$$

At the same time, Yang (see [6, 7]) considered the refinement of other types of Hilbert's inequalities.

In this paper, we give a generalization of Hilbert's type inequality and an improvement as:

$$\int_0^\infty\int_0^\infty \frac{f(x^\lambda)g(y^\mu)}{\max\{x,y\}}dxdy < \frac{pq}{\lambda^{\frac{1}{p}}\mu^{\frac{1}{q}}}\left\{\int_0^\infty x^{\frac{1}{\lambda}-1}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty x^{\frac{1}{\mu}-1}g^q(x)dx\right\}^{\frac{1}{q}},$$

where $\lambda > 0$ and $\mu > 0$.

2. MAIN RESULTS

Lemma 2.1. *Suppose* $r > 1, \frac{1}{s}+\frac{1}{r} = 1, \lambda,\mu,\varepsilon > 0$. *Then*

$$(2.1)\qquad \int_1^\infty x^{-\varepsilon(\frac{1}{s}+\frac{\mu}{\lambda r})-1}\int_0^{x^{-\frac{1}{\lambda}}} \frac{1}{\max\{1,t\}}t^{\frac{-1-\mu\varepsilon}{r}}dtdx = O(1)\quad (\varepsilon\to o^+).$$

Proof. There exist $n \in \mathbb{N}$ which is large enough, such that $1 + \frac{-1-\mu\varepsilon}{r} > 0$ for $\varepsilon \in (0, \frac{1}{\mu n}]$ and $x \geq 1$, we have

$$\int_0^{x^{-\frac{1}{\lambda}}} \frac{1}{\max\{1,t\}} t^{\frac{-1-\mu\varepsilon}{r}} dt = \int_0^{x^{-\frac{1}{\lambda}}} t^{\frac{-1-\mu\varepsilon}{r}} dt = \frac{1}{1+\frac{-1-\mu\varepsilon}{r}} \left(x^{-\frac{1}{\lambda}}\right)^{1+\frac{-1-\mu\varepsilon}{r}}.$$

Since for $a \geq 1$ the function $g(y) = \frac{1}{ya^y}$ $(y \in (0,\infty))$ is decreasing, we find

$$\frac{1}{1+\frac{-1-\mu\varepsilon}{r}} \left(x^{-\frac{1}{\lambda}}\right)^{1+\frac{-1-\mu\varepsilon}{r}} \leq \frac{1}{1+\frac{-1-1/n}{r}} \left(x^{-\frac{1}{\lambda}}\right)^{1+\frac{-1-1/n}{r}},$$

so

$$\begin{aligned}
0 &< \int_1^\infty x^{-\varepsilon(\frac{1}{s}+\frac{\mu}{\lambda r})-1} \int_0^{x^{-\frac{1}{\lambda}}} \frac{1}{\max\{1,t\}} t^{\frac{-1-\mu\epsilon}{r}} dt dx \\
&< \int_1^\infty x^{-1} \frac{1}{1+\frac{-1-1/n}{r}} \left(x^{-\frac{1}{\lambda}}\right)^{1+\frac{-1-1/n}{r}} \\
&= \frac{1}{\lambda}\left(\frac{1}{1+\frac{-1-1/n}{r}}\right)^2.
\end{aligned}$$

Hence relation (2.1) is valid. The lemma is proved. □

Now we study the following inequality:

Theorem 2.2. *Suppose* $f(x),\ g(x) \geq 0,\ p > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \lambda > 0, \mu > 0$ *and*

$$0 < \int_0^\infty x^{\frac{1}{\lambda}-1} f^p(x) dx < \infty, \quad 0 < \int_0^\infty x^{\frac{1}{\mu}-1} g^q(x) dx < \infty.$$

Then

$$\int_0^\infty \int_0^\infty \frac{f(x^\lambda) g(y^\mu)}{\max\{x,y\}} dx dy < \frac{pq}{\lambda^{\frac{1}{p}} \mu^{\frac{1}{q}}} \left\{\int_0^\infty x^{\frac{1}{\lambda}-1} f^p(x) dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty x^{\frac{1}{\mu}-1} g^q(x) dx\right\}^{\frac{1}{q}}, \tag{2.2}$$

where the constant factor $\frac{pq}{\lambda^{\frac{1}{p}}\mu^{\frac{1}{q}}}$ *is the best possible for* $\lambda = \mu$.

Proof. By Hölder's inequality, we have

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \frac{f(x^\lambda) g(y^\mu)}{\max\{x,y\}} dx dy \\
&= \frac{1}{\lambda}\frac{1}{\mu} \int_0^\infty \int_0^\infty \frac{\left[x^{\frac{1}{\lambda}-1} f(x)\right]\left[y^{\frac{1}{\mu}-1} g(y)\right]}{\max\left\{x^{\frac{1}{\lambda}}, y^{\frac{1}{\mu}}\right\}} dx dy \\
&= \frac{1}{\lambda}\frac{1}{\mu} \int_0^\infty \int_0^\infty \left[\frac{x^{\frac{1}{\lambda}-1} f(x)}{\left(\max\left\{x^{\frac{1}{\lambda}}, y^{\frac{1}{\mu}}\right\}\right)^{\frac{1}{p}}} \left(\frac{x^{(1-\frac{1}{\lambda})p/q}}{y^{1-\frac{1}{\mu}}}\right)^{\frac{1}{p}} \left(\frac{x^{\frac{1}{\lambda}}}{y^{\frac{1}{\mu}}}\right)^{\frac{1}{pq}}\right] \\
&\qquad \times \left[\frac{y^{\frac{1}{\mu}-1} g(y)}{\left(\max\left\{x^{\frac{1}{\lambda}}, y^{\frac{1}{\mu}}\right\}\right)^{\frac{1}{q}}} \left(\frac{y^{(1-\frac{1}{\mu})}}{x^{(1-\frac{1}{\lambda})p/q}}\right)^{\frac{1}{p}} \left(\frac{y^{\frac{1}{\mu}}}{x^{\frac{1}{\lambda}}}\right)^{\frac{1}{pq}}\right] dx dy
\end{aligned}$$

$$\leq \frac{1}{\lambda}\frac{1}{\mu}\left[\int_0^\infty\int_0^\infty x^{p(\frac{1}{\lambda}-1)}\frac{f^p(x)}{\max\left\{x^{\frac{1}{\lambda}},y^{\frac{1}{\mu}}\right\}}\frac{x^{(1-\frac{1}{\lambda})p/q}}{y^{1-\frac{1}{\mu}}}\left(\frac{x^{\frac{1}{\lambda}}}{y^{\frac{1}{\mu}}}\right)^{\frac{1}{q}}dxdy\right]^{\frac{1}{p}}$$
$$\times\left[\int_0^\infty\int_0^\infty y^{q(\frac{1}{\mu}-1)}\frac{g^q(y)}{\max\left\{x^{\frac{1}{\lambda}},y^{\frac{1}{\mu}}\right\}}\frac{y^{(1-\frac{1}{\mu})q/p}}{x^{1-\frac{1}{\lambda}}}\left(\frac{y^{\frac{1}{\mu}}}{x^{\frac{1}{\lambda}}}\right)^{\frac{1}{p}}dxdy\right]^{\frac{1}{q}}.$$

Define the weight function $\varphi(x)$, $\psi(y)$ as

$$\varphi(x):=\int_0^\infty\frac{1}{\max\left\{x^{\frac{1}{\lambda}},y^{\frac{1}{\mu}}\right\}}\cdot\frac{x^{(1-\frac{1}{\lambda})p/q}}{y^{1-\frac{1}{\mu}}}\left(\frac{x^{\frac{1}{\lambda}}}{y^{\frac{1}{\mu}}}\right)^{\frac{1}{q}}dy,\qquad x\in(0,\infty),$$
$$\psi(y):=\int_0^\infty\frac{1}{\max\left\{x^{\frac{1}{\lambda}},y^{\frac{1}{\mu}}\right\}}\cdot\frac{y^{(1-\frac{1}{\mu})q/p}}{x^{1-\frac{1}{\lambda}}}\left(\frac{y^{\frac{1}{\mu}}}{x^{\frac{1}{\lambda}}}\right)^{\frac{1}{p}}dx,\qquad y\in(0,\infty),$$

then above inequality yields

$$\int_0^\infty\int_0^\infty\frac{f(x^\lambda)g(y^\mu)}{\max\{x,y\}}dxdy$$
$$\leq\frac{1}{\lambda}\frac{1}{\mu}\left[\int_0^\infty\varphi(x)x^{p(\frac{1}{\lambda}-1)}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty\psi(y)y^{q(\frac{1}{\mu}-1)}g^q(y)dy\right]^{\frac{1}{q}}.$$

For fixed x, let $y^{\frac{1}{\mu}}=x^{\frac{1}{\lambda}}t$, we have

$$\begin{aligned}\varphi(x)&:=\mu x^{(p-1)(1-\frac{1}{\lambda})}\int_0^\infty\frac{1}{\max\{1,t\}}t^{\frac{1}{p}-1}dt\\&=\mu pqx^{(p-1)(1-\frac{1}{\lambda})}\\&=\mu pqx^{(p-1)(1-\frac{1}{\lambda})}.\end{aligned}$$

By the same token, $\psi(y)=\lambda pqy^{(q-1)(1-\frac{1}{\mu})}$, thus

$$\int_0^\infty\int_0^\infty\frac{f(x^\lambda)g(y^\mu)}{\max\{x,y\}}dxdy\leq\frac{pq}{\lambda^{\frac{1}{p}}\mu^{\frac{1}{q}}}\left\{\int_0^\infty x^{\frac{1}{\lambda}-1}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty x^{\frac{1}{\mu}-1}g^q(x)dx\right\}^{\frac{1}{q}}.\tag{2.3}$$

If (2.3) takes the form of the equality, then there exist constants c and d, such that Kuang (see [2])

$$c\frac{x^{p(\frac{1}{\lambda}-1)}f^p(x)}{\max\{x^{\frac{1}{\lambda}},y^{\frac{1}{\mu}}\}}\cdot\frac{x^{(1-\frac{1}{\lambda})p/q}}{y^{1-\frac{1}{\mu}}}\left(\frac{x^{\frac{1}{\lambda}}}{y^{\frac{1}{\mu}}}\right)^{\frac{1}{q}}=d\frac{y^{q(\frac{1}{\mu}-1)}g^q(y)}{\max\left\{x^{\frac{1}{\lambda}},y^{\frac{1}{\mu}}\right\}}\cdot\frac{y^{(1-\frac{1}{\mu})q/p}}{x^{1-\frac{1}{\lambda}}}\left(\frac{y^{\frac{1}{\mu}}}{x^{\frac{1}{\lambda}}}\right)^{\frac{1}{p}}$$
$$\text{a.e. on }(0,\infty)\times(0,\infty).$$

Then we have

$$cx^{\frac{1}{\lambda}}f^p(x)=dy^{\frac{1}{\mu}}g^q(y)$$
$$\text{a.e. on }(0,\infty)\times(0,\infty).$$

Hence we have

$$cx^{\frac{1}{\lambda}}f^p(x) = dy^{\frac{1}{\mu}}g^q(y) = \text{constant}$$
$$\text{a.e. on } (0,\infty)\times(0,\infty),$$

which contradicts the facts that

$$0 < \int_0^\infty x^{\frac{1}{\lambda}} f^p(x)dx < \infty \quad \text{and} \quad 0 < \int_0^\infty y^{\frac{1}{\mu}} g^q(x)dx < \infty.$$

Hence (2.3) takes the form of strict inequality. So we have (2.2).

For $0 < \varepsilon < \frac{1}{2}$, setting $f_\varepsilon(x) = x^{\frac{-\varepsilon-1/\lambda}{p}}$, for $x \in [1,\infty)$; $f_\varepsilon(x) = 0$, for $x \in (0,1)$, and $g_\varepsilon(y) = y^{\frac{-\varepsilon-1/\mu}{q}}$, for $y \in [1,\infty)$; $g_\varepsilon(y) = 0$, for $y \in (0,1)$. Assume that the constant factor $\frac{pq}{\lambda^{\frac{1}{p}}\mu^{\frac{1}{q}}}$ in (2.2) is not the best possible, then there exists a positive number K with $K < \frac{pq}{\lambda^{\frac{1}{p}}\mu^{\frac{1}{q}}}$, such that (2.2) is valid by changing $\frac{pq}{\lambda^{\frac{1}{p}}\mu^{\frac{1}{q}}}$ to K. We have

$$\int_0^\infty\int_0^\infty \frac{f(x^\lambda)g(y^\mu)}{\max\{x,y\}}dxdy < K\left\{\int_0^\infty x^{\frac{1}{\lambda}-1}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty x^{\frac{1}{\mu}-1}g^q(x)dx\right\}^{\frac{1}{q}} = \frac{K}{\varepsilon}.$$

Since

$$\int_0^\infty \frac{1}{\max\{1,t\}}t^{\frac{-1-\varepsilon\mu}{q}}dt = pq + o(1) \qquad (\varepsilon\to 0^+),$$

setting $y^{\frac{1}{\mu}} = x^{\frac{1}{\lambda}}t$, by (2.1), we find

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \frac{f(x^\lambda)g(y^\mu)}{\max\{x,y\}}dxdy \\
&= \frac{1}{\lambda}\frac{1}{\mu}\int_0^\infty\int_0^\infty \frac{\left[x^{\frac{1}{\lambda}-1}f(x)\right]\left[y^{\frac{1}{\mu}-1}g(y)\right]}{\max\left\{x^{\frac{1}{\lambda}},y^{\frac{1}{\mu}}\right\}}dxdy \\
&= \frac{1}{\lambda}\frac{1}{\mu}\int_1^\infty\int_1^\infty \frac{x^{(\frac{1}{\lambda}-1)+\frac{-\varepsilon-\frac{1}{\lambda}}{p}}y^{(\frac{1}{\mu}-1)+\frac{-\varepsilon-\frac{1}{\mu}}{q}}}{\max\left\{x^{\frac{1}{\lambda}},y^{\frac{1}{\mu}}\right\}}dxdy \\
&= \frac{1}{\lambda}\frac{1}{\mu}\int_1^\infty\int_{x^{-\frac{1}{\lambda}}}^\infty \frac{x^{(\frac{1}{\lambda}-1)+\frac{-\varepsilon-\frac{1}{\lambda}}{p}}(t^\mu x^{\frac{\mu}{\lambda}})^{(\frac{1}{\mu}-1)+\frac{-\varepsilon-\frac{1}{\mu}}{q}}}{\max\left\{x^{\frac{1}{\lambda}},tx^{\frac{1}{\lambda}}\right\}}x^{\frac{\mu}{\lambda}}\mu t^{\mu-1}dxdt \\
&= \frac{1}{\lambda}\int_1^\infty x^{-\varepsilon(\frac{1}{p}+\frac{\mu}{\lambda q})-1}\int_{x^{-\frac{1}{\lambda}}}^\infty \frac{1}{\max\{1,t\}}t^{\frac{-1-\varepsilon\mu}{q}}dtdx \\
&= \frac{1}{\lambda}\int_1^\infty x^{-\varepsilon(\frac{1}{p}+\frac{\mu}{\lambda q})-1}\left(\int_0^\infty \frac{1}{\max\{1,t\}}t^{\frac{-1-\varepsilon\mu}{q}}dtdx - \int_0^{x^{-\frac{1}{\lambda}}} \frac{1}{\max\{1,t\}}t^{\frac{-1-\varepsilon\mu}{q}}dtdx\right) \\
&= \frac{1}{\lambda\varepsilon}\left[\frac{pq}{\frac{1}{p}+\frac{\mu}{\lambda q}} + o(1)\right].
\end{aligned}$$

Since for $\varepsilon > 0$ small enough, we have

$$\frac{1}{\lambda}\left[\frac{pq}{\frac{1}{p}+\frac{\mu}{\lambda q}} + o(1)\right] < K.$$

It is obvious that $\lambda^{\frac{1}{p}}\mu^{\frac{1}{q}} \le \frac{\lambda}{p} + \frac{\mu}{q}$ (i.e. $\frac{1}{\lambda}\frac{1}{\frac{1}{p}+\frac{\mu}{\lambda q}} \le \frac{1}{\lambda^{\frac{1}{p}}\mu^{\frac{1}{q}}}$) by Young's inequality. Consider the case of taking the form of the equality for Young's inequality, we get $\mu^{\frac{1}{q}} = \lambda^{\frac{p-1}{p}}$, i.e. $\lambda = \mu$, Then

$$\frac{1}{\lambda}\left[\frac{pq}{\frac{1}{p}+\frac{\mu}{\lambda q}} + o(1)\right] = \frac{pq}{\lambda^{\frac{1}{p}}\mu^{\frac{1}{q}}} + o(1) < K.$$

Thus we get $\frac{pq}{\lambda^{\frac{1}{p}}\mu^{\frac{1}{q}}} \le K$, which contradicts the hypothesis. Hence the constant factor $\frac{pq}{\lambda^{\frac{1}{p}}\mu^{\frac{1}{q}}}$ in (2.2) is best possible for $\lambda = \mu$. □

Remark 2.3. For $\lambda = \mu$, inequality (2.2) becomes

$$\int_0^\infty \int_0^\infty \frac{f(x^\lambda)g(y^\lambda)}{\max\{x,y\}}dxdy < \frac{pq}{\lambda}\left\{\int_0^\infty x^{\frac{1}{\lambda}-1}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty x^{\frac{1}{\lambda}-1}g^q(x)dx\right\}^{\frac{1}{q}}, \tag{2.4}$$

where the constant factor $\frac{pq}{\lambda}$ is the best possible.

Theorem 2.4. *Suppose $f \ge 0$, $p > 1$, $\lambda > 0$ and $0 < \int_0^\infty x^{\frac{1}{\lambda}-1}f^p(x)dx < \infty$. Then*

$$\int_0^\infty \left[\int_0^\infty \frac{f(x^\lambda)}{\max\{x,y\}}dx\right]^p dy < \frac{1}{\lambda}(pq)^p \int_0^\infty x^{\frac{1}{\lambda}-1}f(x)^p dx, \tag{2.5}$$

where the constant factor $\frac{1}{\lambda}(pq)^p$ is the best possible. Inequality (2.5) is equivalent to (2.4).

Proof. Setting $g(y)$ as

$$\left[\int_0^\infty \frac{x^{\frac{1}{\lambda}-1}f(x)}{\max\left\{x^{\frac{1}{\lambda}}, y^{\frac{1}{\lambda}}\right\}}dx\right]^{p-1} > 0, \qquad y \in (0,\infty).$$

then by (2.4), we find

$$\begin{aligned}\lambda^{-2}\int_0^\infty y^{\frac{1}{\lambda}-1}g^q(y)dy &= \lambda^{p-1}\int_0^\infty \left[\int_0^\infty \frac{f(x^\lambda)}{\max\{x,y\}}dx\right]^p dy \\ &= \int_0^\infty \int_0^\infty \frac{f(x^\lambda)g(y^\lambda)}{\max\{x,y\}}dxdy \\ &\le \frac{pq}{\lambda}\left\{\int_0^\infty x^{\frac{1}{\lambda}-1}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty y^{\frac{1}{\lambda}-1}g^q(y)dy\right\}^{\frac{1}{q}}. \end{aligned} \tag{2.6}$$

Hence we obtain

$$0 < \int_0^\infty y^{\frac{1}{\lambda}-1}g^q(y)dy \le \lambda^p(pq)^p \int_0^\infty x^{\frac{1}{\lambda}-1}f^p(x)dx < \infty. \tag{2.7}$$

By (2.4), both (2.6) and (2.7) take the form of strict inequality, so we have (2.5).

On the other hand, suppose that (2.5) is valid. By Hölder's inequality, we find

$$\begin{aligned}&\int_0^\infty \int_0^\infty \frac{f(x^\lambda)g(y^\lambda)}{\max\{x,y\}}dxdy \\ &\qquad= \int_0^\infty \left[\int_0^\infty \frac{f(x^\lambda)}{\max\{x,y\}}dx\right] g(y^\lambda)dy \\ &\qquad\le \left\{\int_0^\infty \left[\int_0^\infty \frac{f(x^\lambda)}{\max\{x,y\}}dx\right]^p dy\right\}^{\frac{1}{p}}\left\{\int_0^\infty g^q(y^\lambda)dy\right\}^{\frac{1}{q}}. \end{aligned} \tag{2.8}$$

Then by (2.5), we have (2.4). Thus (2.4) and (2.5) are equivalent.

If the constant $\frac{1}{\lambda}(pq)^p$ in (2.5) is not the best possible, by (2.8), we may get a contradiction that the constant factor in (2.4) is not the best possible. Thus we complete the proof of the theorem. □

Remark 2.5.

(i) For $\lambda = \mu = 1$, (2.2) and (2.5) reduce respectively to (1.3) and (1.4). It follows that (2.2) is a new extension of (1.6) and (1.3) with some parameters and the equivalent form (2.4) is a new extension of (1.4).

(ii) It is amazing that (2.4) and (1.9) are different, although both of them are the extensions of (1.6) with $(p, q)-$parameter and the best constant factor.

REFERENCES

[1] G.H. HARDY, J.E. LITTLEWOOD AND G. POLYA, *Inequalities*, Cambridge Univ Press Cambridge, 1952.

[2] J. KUANG, *Applied Inequalities* (Chinese), Hunan Jiaoyu Chubanshe, Changsha, 1993.

[3] J. KUANG AND L. DEBNATH, On new generalizations of Hilbert's inequality and their applications, *J. Math. Anal. Appl.*, **245**(1) (2000), 248–265.

[4] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Inequalities Involving Functions and their Integrals and Derivatives*, Mathematics and its Applications, 53. Kluwer Academic Publishers Group, Dordrecht, 1991.

[5] B. YANG, Generalization of a Hilbert type inequality and its applications (Chinese), *Gongcheng Shuxue Xuebao*, **21**(5) (2004), 821–824.

[6] B. YANG, On a Hilbert's type inequality and its applications, *Math. Appl.* (Wuhan), **15**(1) (2002), 52–56.

[7] B. YANG, A new Hardy-Hilbert type inequality and its applications (Chinese), *Acta Math. Sinica*, **46**(6) (2003), 1079–1086.

[8] B. YANG, Best generalization of a Hilbert-type inequality (Chinese), *J. Jilin Univ. Sci.*, **42**(1) (2004), 30–34.