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SOME EXTENSIONS OF HILBERT'S TYPE INEQUALITY AND ITS APPLICATIONS

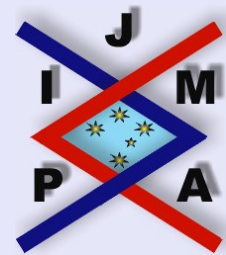
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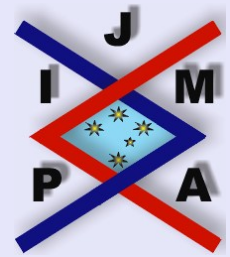


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Abstract

By introducing parameters λ and μ , we give a generalization of the Hilbert's type integral inequality. As applications, we give its equivalent form.

2000 Mathematics Subject Classification: 26D15.

Key words: Hilbert's integral inequality, Weight function.

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1. Introduction

If $f, g \geq 0$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \int_0^\infty f^p(x)dx < \infty$ and $0 < \int_0^\infty g^q(x)dx < \infty$, then

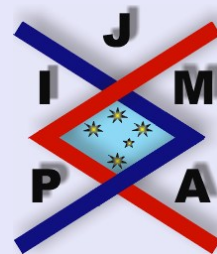
$$(1.1) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \frac{\pi}{\sin(\pi/p)} \left\{ \int_0^\infty f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty g^q(x)dx \right\}^{\frac{1}{q}},$$

$$(1.2) \quad \int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y} dx \right)^p dy < \left[\frac{\pi}{\sin(\pi/p)} \right]^p \int_0^\infty f^p(x)dx,$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Inequality (1.1) is known as Hardy-Hilbert's integral inequality (see [1]); it is important in analysis and its applications (see [4]). Under the same condition of (1.1), we have the Hardy-Hilbert type inequality similar to (1.1):

$$(1.3) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x, y\}} dx dy < pq \left\{ \int_0^\infty f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty g^q(x)dx \right\}^{\frac{1}{q}},$$

$$(1.4) \quad \int_0^\infty \left(\int_0^\infty \frac{f(x)}{\max\{x, y\}} dx \right)^p dy < (pq)^p \int_0^\infty f^p(x)dx,$$



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where the constant factor pq is the best possible. The corresponding inequality for series is:

$$(1.5) \quad \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m, n\}} < pq \left(\sum_{n=1}^{\infty} a_n^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q \right)^{\frac{1}{q}},$$

where the constant factor pq is also the best possible. In particular, when $p = q = 2$, we have the well-known Hilbert type inequality:

$$(1.6) \quad \int_0^{\infty} \int_0^{\infty} \frac{f(x)g(y)}{\max\{x, y\}} dx dy < 4 \left\{ \int_0^{\infty} f^2(x) dx \right\}^{\frac{1}{2}} \left\{ \int_0^{\infty} g^2(x) dx \right\}^{\frac{1}{2}}.$$

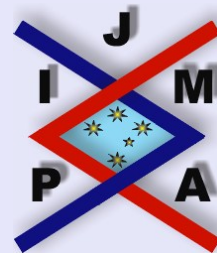
In recent years, Kuang (see [3]) gave a strengthened form as:

$$(1.7) \quad \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m, n\}} < \left\{ \sum_{n=1}^{\infty} [pq - G(p, n)] a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} [pq - G(q, n)] b_n^q \right\}^{\frac{1}{q}},$$

where $G(r, n) = \frac{r+1/3r-4/3}{(2n+1)^{1/r}} > 0$ ($r = p, q$).

Yang (see [5, 8]) gave: for $\lambda > 2 - \min\{p, q\}$

$$(1.8) \quad \int_0^{\infty} \int_0^{\infty} \frac{f(x)g(y)}{\max\{x^\lambda, y^\lambda\}} dx dy < \frac{pq\lambda}{(p + \lambda - 2)(q + \lambda - 2)} \times \left\{ \int_0^{\infty} x^{(p-1)(2-\lambda)-1} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^{\infty} x^{(q-1)(2-\lambda)-1} g^q(x) dx \right\}^{\frac{1}{q}}$$



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and

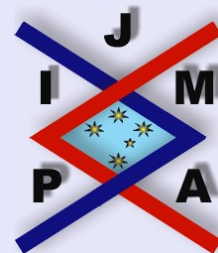
$$(1.9) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x^\lambda, y^\lambda\}} dx dy < \frac{pq\lambda}{(p + \lambda - 2)(q + \lambda - 2)} \\ \times \left\{ \int_0^\infty x^{1-\lambda} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty x^{1-\lambda} g^q(x) dx \right\}^{\frac{1}{q}}.$$

At the same time, Yang (see [6, 7]) considered the refinement of other types of Hilbert's inequalities.

In this paper, we give a generalization of Hilbert's type inequality and an improvement as:

$$\int_0^\infty \int_0^\infty \frac{f(x^\lambda)g(y^\mu)}{\max\{x, y\}} dx dy \\ < \frac{pq}{\lambda^{\frac{1}{p}} \mu^{\frac{1}{q}}} \left\{ \int_0^\infty x^{\frac{1}{\lambda}-1} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty x^{\frac{1}{\mu}-1} g^q(x) dx \right\}^{\frac{1}{q}},$$

where $\lambda > 0$ and $\mu > 0$.



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2. Main Results

Lemma 2.1. Suppose $r > 1, \frac{1}{s} + \frac{1}{r} = 1, \lambda, \mu, \varepsilon > 0$. Then

$$(2.1) \quad \int_1^\infty x^{-\varepsilon(\frac{1}{s} + \frac{\mu}{\lambda r}) - 1} \int_0^{x^{-\frac{1}{\lambda}}} \frac{1}{\max\{1, t\}} t^{\frac{-1-\mu\varepsilon}{r}} dt dx = O(1) \quad (\varepsilon \rightarrow o^+).$$

Proof. There exist $n \in \mathbb{N}$ which is large enough, such that $1 + \frac{-1-\mu\varepsilon}{r} > 0$ for $\varepsilon \in (0, \frac{1}{\mu n}]$ and $x \geq 1$, we have

$$\int_0^{x^{-\frac{1}{\lambda}}} \frac{1}{\max\{1, t\}} t^{\frac{-1-\mu\varepsilon}{r}} dt = \int_0^{x^{-\frac{1}{\lambda}}} t^{\frac{-1-\mu\varepsilon}{r}} dt = \frac{1}{1 + \frac{-1-\mu\varepsilon}{r}} \left(x^{-\frac{1}{\lambda}}\right)^{1 + \frac{-1-\mu\varepsilon}{r}}.$$

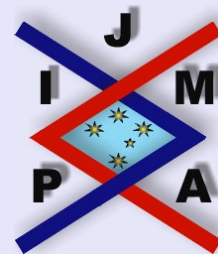
Since for $a \geq 1$ the function $g(y) = \frac{1}{ya^y}$ ($y \in (0, \infty)$) is decreasing, we find

$$\frac{1}{1 + \frac{-1-\mu\varepsilon}{r}} \left(x^{-\frac{1}{\lambda}}\right)^{1 + \frac{-1-\mu\varepsilon}{r}} \leq \frac{1}{1 + \frac{-1-1/n}{r}} \left(x^{-\frac{1}{\lambda}}\right)^{1 + \frac{-1-1/n}{r}},$$

so

$$\begin{aligned} 0 &< \int_1^\infty x^{-\varepsilon(\frac{1}{s} + \frac{\mu}{\lambda r}) - 1} \int_0^{x^{-\frac{1}{\lambda}}} \frac{1}{\max\{1, t\}} t^{\frac{-1-\mu\varepsilon}{r}} dt dx \\ &< \int_1^\infty x^{-1} \frac{1}{1 + \frac{-1-1/n}{r}} \left(x^{-\frac{1}{\lambda}}\right)^{1 + \frac{-1-1/n}{r}} \\ &= \frac{1}{\lambda} \left(\frac{1}{1 + \frac{-1-1/n}{r}} \right)^2. \end{aligned}$$

Hence relation (2.1) is valid. The lemma is proved. \square



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Now we study the following inequality:

Theorem 2.2. Suppose $f(x), g(x) \geq 0$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 0, \mu > 0$ and

$$0 < \int_0^\infty x^{\frac{1}{\lambda}-1} f^p(x) dx < \infty, \quad 0 < \int_0^\infty x^{\frac{1}{\mu}-1} g^q(x) dx < \infty.$$

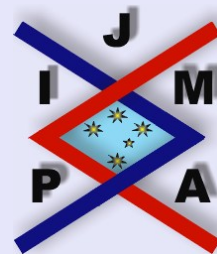
Then

$$(2.2) \quad \int_0^\infty \int_0^\infty \frac{f(x^\lambda)g(y^\mu)}{\max\{x, y\}} dx dy < \frac{pq}{\lambda^{\frac{1}{p}}\mu^{\frac{1}{q}}} \left\{ \int_0^\infty x^{\frac{1}{\lambda}-1} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty x^{\frac{1}{\mu}-1} g^q(x) dx \right\}^{\frac{1}{q}},$$

where the constant factor $\frac{pq}{\lambda^{\frac{1}{p}}\mu^{\frac{1}{q}}}$ is the best possible for $\lambda = \mu$.

Proof. By Hölder's inequality, we have

$$\begin{aligned} & \int_0^\infty \int_0^\infty \frac{f(x^\lambda)g(y^\mu)}{\max\{x, y\}} dx dy \\ &= \frac{1}{\lambda} \frac{1}{\mu} \int_0^\infty \int_0^\infty \frac{[x^{\frac{1}{\lambda}-1} f(x)] [y^{\frac{1}{\mu}-1} g(y)]}{\max\left\{x^{\frac{1}{\lambda}}, y^{\frac{1}{\mu}}\right\}} dx dy \\ &= \frac{1}{\lambda} \frac{1}{\mu} \int_0^\infty \int_0^\infty \left[\frac{x^{\frac{1}{\lambda}-1} f(x)}{\left(\max\left\{x^{\frac{1}{\lambda}}, y^{\frac{1}{\mu}}\right\}\right)^{\frac{1}{p}}} \left(\frac{x^{(1-\frac{1}{\lambda})p/q}}{y^{1-\frac{1}{\mu}}}\right)^{\frac{1}{p}} \left(\frac{x^{\frac{1}{\lambda}}}{y^{\frac{1}{\mu}}}\right)^{\frac{1}{pq}} \right] \end{aligned}$$



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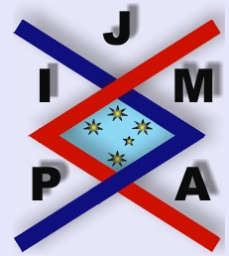


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$$\begin{aligned} & \times \left[\frac{y^{\frac{1}{\mu}-1}g(y)}{\left(\max\left\{x^{\frac{1}{\lambda}}, y^{\frac{1}{\mu}}\right\}\right)^{\frac{1}{q}}}\left(\frac{y^{(1-\frac{1}{\mu})}}{x^{(1-\frac{1}{\lambda})p/q}}\right)^{\frac{1}{p}}\left(\frac{y^{\frac{1}{\mu}}}{x^{\frac{1}{\lambda}}}\right)^{\frac{1}{pq}}\right] dx dy \\ \leq & \frac{1}{\lambda} \frac{1}{\mu} \left[\int_0^\infty \int_0^\infty x^{p(\frac{1}{\lambda}-1)} \frac{f^p(x)}{\max\left\{x^{\frac{1}{\lambda}}, y^{\frac{1}{\mu}}\right\}} \frac{x^{(1-\frac{1}{\lambda})p/q}}{y^{1-\frac{1}{\mu}}}\left(\frac{x^{\frac{1}{\lambda}}}{y^{\frac{1}{\mu}}}\right)^{\frac{1}{q}} dx dy \right]^{\frac{1}{p}} \\ & \times \left[\int_0^\infty \int_0^\infty y^{q(\frac{1}{\mu}-1)} \frac{g^q(y)}{\max\left\{x^{\frac{1}{\lambda}}, y^{\frac{1}{\mu}}\right\}} \frac{y^{(1-\frac{1}{\mu})q/p}}{x^{1-\frac{1}{\lambda}}}\left(\frac{y^{\frac{1}{\mu}}}{x^{\frac{1}{\lambda}}}\right)^{\frac{1}{p}} dx dy \right]^{\frac{1}{q}}. \end{aligned}$$

Define the weight function $\varphi(x), \psi(y)$ as

$$\varphi(x) := \int_0^\infty \frac{1}{\max\left\{x^{\frac{1}{\lambda}}, y^{\frac{1}{\mu}}\right\}} \cdot \frac{x^{(1-\frac{1}{\lambda})p/q}}{y^{1-\frac{1}{\mu}}}\left(\frac{x^{\frac{1}{\lambda}}}{y^{\frac{1}{\mu}}}\right)^{\frac{1}{q}} dy, \quad x \in (0, \infty),$$

$$\psi(y) := \int_0^\infty \frac{1}{\max\left\{x^{\frac{1}{\lambda}}, y^{\frac{1}{\mu}}\right\}} \cdot \frac{y^{(1-\frac{1}{\mu})q/p}}{x^{1-\frac{1}{\lambda}}}\left(\frac{y^{\frac{1}{\mu}}}{x^{\frac{1}{\lambda}}}\right)^{\frac{1}{p}} dx, \quad y \in (0, \infty),$$

then above inequality yields

$$\begin{aligned} & \int_0^\infty \int_0^\infty \frac{f(x^\lambda)g(y^\mu)}{\max\{x, y\}} dx dy \\ & \leq \frac{1}{\lambda} \frac{1}{\mu} \left[\int_0^\infty \varphi(x)x^{p(\frac{1}{\lambda}-1)} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty \psi(y)y^{q(\frac{1}{\mu}-1)} g^q(y) dy \right]^{\frac{1}{q}}. \end{aligned}$$

For fixed x , let $y^{\frac{1}{\mu}} = x^{\frac{1}{\lambda}}t$, we have

$$\begin{aligned}\varphi(x) &:= \mu x^{(p-1)(1-\frac{1}{\lambda})} \int_0^{\infty} \frac{1}{\max\{1, t\}} t^{\frac{1}{p}-1} dt \\ &= \mu pq x^{(p-1)(1-\frac{1}{\lambda})} \\ &= \mu pq x^{(p-1)(1-\frac{1}{\lambda})}.\end{aligned}$$

By the same token, $\psi(y) = \lambda pq y^{(q-1)(1-\frac{1}{\mu})}$, thus

$$(2.3) \quad \int_0^{\infty} \int_0^{\infty} \frac{f(x^{\lambda})g(y^{\mu})}{\max\{x, y\}} dx dy \leq \frac{pq}{\lambda^{\frac{1}{p}}\mu^{\frac{1}{q}}} \left\{ \int_0^{\infty} x^{\frac{1}{\lambda}-1} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^{\infty} x^{\frac{1}{\mu}-1} g^q(x) dx \right\}^{\frac{1}{q}}.$$

If (2.3) takes the form of the equality, then there exist constants c and d , such that Kuang (see [2])

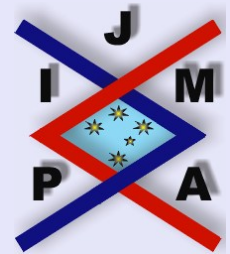
$$c \frac{x^{p(\frac{1}{\lambda}-1)} f^p(x)}{\max\{x^{\frac{1}{\lambda}}, y^{\frac{1}{\mu}}\}} \cdot \frac{x^{(1-\frac{1}{\lambda})p/q}}{y^{1-\frac{1}{\mu}}} \left(\frac{x^{\frac{1}{\lambda}}}{y^{\frac{1}{\mu}}} \right)^{\frac{1}{q}} = d \frac{y^{q(\frac{1}{\mu}-1)} g^q(y)}{\max\{x^{\frac{1}{\lambda}}, y^{\frac{1}{\mu}}\}} \cdot \frac{y^{(1-\frac{1}{\mu})q/p}}{x^{1-\frac{1}{\lambda}}} \left(\frac{y^{\frac{1}{\mu}}}{x^{\frac{1}{\lambda}}} \right)^{\frac{1}{p}}$$

a.e. on $(0, \infty) \times (0, \infty)$.

Then we have

$$cx^{\frac{1}{\lambda}} f^p(x) = dy^{\frac{1}{\mu}} g^q(y)$$

a.e. on $(0, \infty) \times (0, \infty)$.



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Hence we have

$$cx^{\frac{1}{\lambda}} f^p(x) = dy^{\frac{1}{\mu}} g^q(y) = \text{constant} \\ \text{a.e. on } (0, \infty) \times (0, \infty),$$

which contradicts the facts that

$$0 < \int_0^\infty x^{\frac{1}{\lambda}} f^p(x) dx < \infty \quad \text{and} \quad 0 < \int_0^\infty y^{\frac{1}{\mu}} g^q(y) dy < \infty.$$

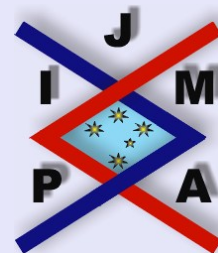
Hence (2.3) takes the form of strict inequality. So we have (2.2).

For $0 < \varepsilon < \frac{1}{2}$, setting $f_\varepsilon(x) = x^{\frac{-\varepsilon-1/\lambda}{p}}$, for $x \in [1, \infty)$; $f_\varepsilon(x) = 0$, for $x \in (0, 1)$, and $g_\varepsilon(y) = y^{\frac{-\varepsilon-1/\mu}{q}}$, for $y \in [1, \infty)$; $g_\varepsilon(y) = 0$, for $y \in (0, 1)$. Assume that the constant factor $\frac{pq}{\lambda^{\frac{1}{p}} \mu^{\frac{1}{q}}}$ in (2.2) is not the best possible, then there exists a positive number K with $K < \frac{pq}{\lambda^{\frac{1}{p}} \mu^{\frac{1}{q}}}$, such that (2.2) is valid by changing $\frac{pq}{\lambda^{\frac{1}{p}} \mu^{\frac{1}{q}}}$ to K . We have

$$\int_0^\infty \int_0^\infty \frac{f(x^\lambda)g(y^\mu)}{\max\{x, y\}} dx dy \\ < K \left\{ \int_0^\infty x^{\frac{1}{\lambda}-1} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty x^{\frac{1}{\mu}-1} g^q(x) dx \right\}^{\frac{1}{q}} = \frac{K}{\varepsilon}.$$

Since

$$\int_0^\infty \frac{1}{\max\{1, t\}} t^{\frac{-1-\varepsilon\mu}{q}} dt = pq + o(1) \quad (\varepsilon \rightarrow 0^+),$$



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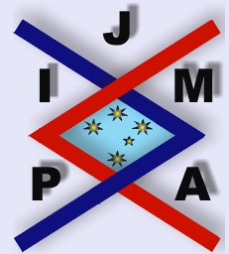
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setting $y^{\frac{1}{\mu}} = x^{\frac{1}{\lambda}}t$, by (2.1), we find

$$\begin{aligned}
 & \int_0^\infty \int_0^\infty \frac{f(x^\lambda)g(y^\mu)}{\max\{x, y\}} dx dy \\
 &= \frac{1}{\lambda} \frac{1}{\mu} \int_0^\infty \int_0^\infty \frac{\left[x^{\frac{1}{\lambda}-1}f(x)\right] \left[y^{\frac{1}{\mu}-1}g(y)\right]}{\max\left\{x^{\frac{1}{\lambda}}, y^{\frac{1}{\mu}}\right\}} dx dy \\
 &= \frac{1}{\lambda} \frac{1}{\mu} \int_1^\infty \int_1^\infty \frac{x^{(\frac{1}{\lambda}-1)+\frac{-\varepsilon-\frac{1}{\lambda}}{p}} y^{(\frac{1}{\mu}-1)+\frac{-\varepsilon-\frac{1}{\mu}}{q}}}{\max\left\{x^{\frac{1}{\lambda}}, y^{\frac{1}{\mu}}\right\}} dx dy \\
 &= \frac{1}{\lambda} \frac{1}{\mu} \int_1^\infty \int_{x^{-\frac{1}{\lambda}}}^\infty \frac{x^{(\frac{1}{\lambda}-1)+\frac{-\varepsilon-\frac{1}{\lambda}}{p}} (t^\mu x^{\frac{\mu}{\lambda}})^{(\frac{1}{\mu}-1)+\frac{-\varepsilon-\frac{1}{\mu}}{q}}}{\max\left\{x^{\frac{1}{\lambda}}, tx^{\frac{1}{\lambda}}\right\}} x^{\frac{\mu}{\lambda}} \mu t^{\mu-1} dx dt \\
 &= \frac{1}{\lambda} \int_1^\infty x^{-\varepsilon(\frac{1}{p}+\frac{\mu}{\lambda q})-1} \int_{x^{-\frac{1}{\lambda}}}^\infty \frac{1}{\max\{1, t\}} t^{\frac{-1-\varepsilon\mu}{q}} dt dx \\
 &= \frac{1}{\lambda} \int_1^\infty x^{-\varepsilon(\frac{1}{p}+\frac{\mu}{\lambda q})-1} \left(\int_0^\infty \frac{1}{\max\{1, t\}} t^{\frac{-1-\varepsilon\mu}{q}} dt dx \right. \\
 &\quad \left. - \int_0^{x^{-\frac{1}{\lambda}}} \frac{1}{\max\{1, t\}} t^{\frac{-1-\varepsilon\mu}{q}} dt dx \right) \\
 &= \frac{1}{\lambda\varepsilon} \left[\frac{pq}{\frac{1}{p} + \frac{\mu}{\lambda q}} + o(1) \right].
 \end{aligned}$$



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Since for $\varepsilon > 0$ small enough, we have

$$\frac{1}{\lambda} \left[\frac{pq}{\frac{1}{p} + \frac{\mu}{\lambda q}} + o(1) \right] < K.$$

It is obvious that $\lambda^{\frac{1}{p}} \mu^{\frac{1}{q}} \leq \frac{\lambda}{p} + \frac{\mu}{q}$ (i.e. $\frac{1}{\lambda^{\frac{1}{p}} + \frac{\mu}{\lambda q}} \leq \frac{1}{\lambda^{\frac{1}{p}} \mu^{\frac{1}{q}}}$) by Young's inequality.

Consider the case of taking the form of the equality for Young's inequality, we get $\mu^{\frac{1}{q}} = \lambda^{\frac{p-1}{p}}$, i.e. $\lambda = \mu$, Then

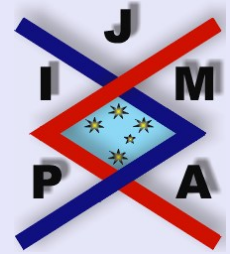
$$\frac{1}{\lambda} \left[\frac{pq}{\frac{1}{p} + \frac{\mu}{\lambda q}} + o(1) \right] = \frac{pq}{\lambda^{\frac{1}{p}} \mu^{\frac{1}{q}}} + o(1) < K.$$

Thus we get $\frac{pq}{\lambda^{\frac{1}{p}} \mu^{\frac{1}{q}}} \leq K$, which contradicts the hypothesis. Hence the constant factor $\frac{pq}{\lambda^{\frac{1}{p}} \mu^{\frac{1}{q}}}$ in (2.2) is best possible for $\lambda = \mu$. \square

Remark 1. For $\lambda = \mu$, inequality (2.2) becomes

$$(2.4) \quad \int_0^\infty \int_0^\infty \frac{f(x^\lambda)g(y^\lambda)}{\max\{x, y\}} dx dy < \frac{pq}{\lambda} \left\{ \int_0^\infty x^{\frac{1}{\lambda}-1} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty x^{\frac{1}{\lambda}-1} g^q(x) dx \right\}^{\frac{1}{q}},$$

where the constant factor $\frac{pq}{\lambda}$ is the best possible.



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Theorem 2.3. Suppose $f \geq 0$, $p > 1$, $\lambda > 0$ and $0 < \int_0^\infty x^{\frac{1}{\lambda}-1} f^p(x) dx < \infty$.
Then

$$(2.5) \quad \int_0^\infty \left[\int_0^\infty \frac{f(x^\lambda)}{\max\{x, y\}} dx \right]^p dy < \frac{1}{\lambda} (pq)^p \int_0^\infty x^{\frac{1}{\lambda}-1} f(x)^p dx,$$

where the constant factor $\frac{1}{\lambda} (pq)^p$ is the best possible. Inequality (2.5) is equivalent to (2.4).

Proof. Setting $g(y)$ as

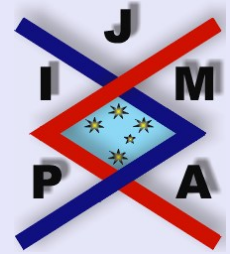
$$\left[\int_0^\infty \frac{x^{\frac{1}{\lambda}-1} f(x)}{\max\{x^{\frac{1}{\lambda}}, y^{\frac{1}{\lambda}}\}} dx \right]^{p-1} > 0, \quad y \in (0, \infty).$$

then by (2.4), we find

$$(2.6) \quad \begin{aligned} \lambda^{-2} \int_0^\infty y^{\frac{1}{\lambda}-1} g^q(y) dy &= \lambda^{p-1} \int_0^\infty \left[\int_0^\infty \frac{f(x^\lambda)}{\max\{x, y\}} dx \right]^p dy \\ &= \int_0^\infty \int_0^\infty \frac{f(x^\lambda) g(y^\lambda)}{\max\{x, y\}} dx dy \\ &\leq \frac{pq}{\lambda} \left\{ \int_0^\infty x^{\frac{1}{\lambda}-1} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty y^{\frac{1}{\lambda}-1} g^q(y) dy \right\}^{\frac{1}{q}}. \end{aligned}$$

Hence we obtain

$$(2.7) \quad 0 < \int_0^\infty y^{\frac{1}{\lambda}-1} g^q(y) dy \leq \lambda^p (pq)^p \int_0^\infty x^{\frac{1}{\lambda}-1} f^p(x) dx < \infty.$$



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By (2.4), both (2.6) and (2.7) take the form of strict inequality, so we have (2.5).

On the other hand, suppose that (2.5) is valid. By Hölder's inequality, we find

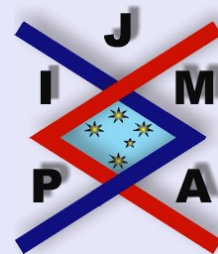
$$\begin{aligned}
 & \int_0^\infty \int_0^\infty \frac{f(x^\lambda)g(y^\lambda)}{\max\{x, y\}} dx dy \\
 &= \int_0^\infty \left[\int_0^\infty \frac{f(x^\lambda)}{\max\{x, y\}} dx \right] g(y^\lambda) dy \\
 (2.8) \quad &\leq \left\{ \int_0^\infty \left[\int_0^\infty \frac{f(x^\lambda)}{\max\{x, y\}} dx \right]^p dy \right\}^{\frac{1}{p}} \left\{ \int_0^\infty g^q(y^\lambda) dy \right\}^{\frac{1}{q}}.
 \end{aligned}$$

Then by (2.5), we have (2.4). Thus (2.4) and (2.5) are equivalent.

If the constant $\frac{1}{\lambda}(pq)^p$ in (2.5) is not the best possible, by (2.8), we may get a contradiction that the constant factor in (2.4) is not the best possible. Thus we complete the proof of the theorem. \square

Remark 2.

- (i) For $\lambda = \mu = 1$, (2.2) and (2.5) reduce respectively to (1.3) and (1.4). It follows that (2.2) is a new extension of (1.6) and (1.3) with some parameters and the equivalent form (2.4) is a new extension of (1.4).
- (ii) It is amazing that (2.4) and (1.9) are different, although both of them are the extensions of (1.6) with (p, q) -parameter and the best constant factor.



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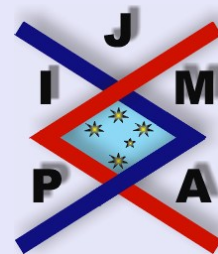
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