Journal of Inequalities in Pure and Applied Mathematics

SOME EXTENSIONS OF HILBERT'S TYPE INEQUALITY AND ITS APPLICATIONS

YONGJIN LI AND BING HE

Department of Mathematics
Sun Yat-sen University
Guangzhou, 510275 P. R. China.

EMail: stslyj@mail.sysu.edu.cn

EMail: hzs314@163.com

volume 7, issue 2, article 61,
2006.

Received 28 November, 2005;
accepted 19 January, 2006.

Communicated by: B. Yang

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ISSN (electronic): 1443-5756
342-05

Abstract

By introducing parameters λ and μ, we give a generalization of the Hilbert's type integral inequality. As applications, we give its equivalent form.

2000 Mathematics Subject Classification: 26D15.
Key words: Hilbert's integral inequality, Weight function.

The authors are grateful to the referee for the careful reading of the manuscript and for several useful remarks. The work was partially supported by the Foundation of Sun Yat-sen University Advanced Research Centre.

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1. Introduction

If $f, g \geq 0$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \int_0^\infty f^p(x)dx < \infty$ and $0 < \int_0^\infty g^q(x)dx < \infty$, then

$$(1.1)\qquad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\pi/p)} \left\{\int_0^\infty f^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty g^q(x)dx\right\}^{\frac{1}{q}},$$

$$(1.2)\qquad \int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y}dx\right)^p dy < \left[\frac{\pi}{\sin(\pi/p)}\right]^p \int_0^\infty f^p(x)dx,$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Inequality (1.1) is known as Hardy-Hilbert's integral inequality (see [1]); it is important in analysis and its applications (see [4]). Under the same condition of (1.1), we have the Hardy-Hilbert type inequality similar to (1.1):

$$(1.3)\qquad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}}dxdy < pq \left\{\int_0^\infty f^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty g^q(x)dx\right\}^{\frac{1}{q}},$$

$$(1.4)\qquad \int_0^\infty \left(\int_0^\infty \frac{f(x)}{\max\{x,y\}}dx\right)^p dy < (pq)^p \int_0^\infty f^p(x)dx,$$

where the constant factor pq is the best possible. The corresponding inequality for series is:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m,n\}} < pq\left(\sum_{n=1}^{\infty}a_n^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}, \tag{1.5}$$

where the constant factor pq is also the best possible. In particular, when $p = q = 2$, we have the well-known Hilbert type inequality:

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{\max\{x,y\}}dxdy < 4\left\{\int_0^{\infty}f^2(x)dx\right\}^{\frac{1}{2}}\left\{\int_0^{\infty}g^2(x)dx\right\}^{\frac{1}{2}}. \tag{1.6}$$

In recent years, Kuang (see [3]) gave a strengthened form as:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m,n\}} < \left\{\sum_{n=1}^{\infty}[pq-G(p,n)]a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}[pq-G(q,n)]b_n^q\right\}^{\frac{1}{q}}, \tag{1.7}$$

where $G(r,n) = \frac{r+1/3r-4/3}{(2n+1)^{1/r}} > 0$ $(r = p,q)$.

Yang (see [5, 8]) gave: for $\lambda > 2 - \min\{p,q\}$

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{\max\{x^{\lambda},y^{\lambda}\}}dxdy < \frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)} \times\left\{\int_0^{\infty}x^{(p-1)(2-\lambda)-1}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^{\infty}x^{(q-1)(2-\lambda)-1}g^q(x)dx\right\}^{\frac{1}{q}} \tag{1.8}$$

and

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x^\lambda, y^\lambda\}} dxdy < \frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)} \times \left\{\int_0^\infty x^{1-\lambda} f^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty x^{1-\lambda} g^q(x)dx\right\}^{\frac{1}{q}}. \tag{1.9}$$

At the same time, Yang (see [6, 7]) considered the refinement of other types of Hilbert's inequalities.

In this paper, we give a generalization of Hilbert's type inequality and an improvement as:

$$\int_0^\infty \int_0^\infty \frac{f(x^\lambda)g(y^\mu)}{\max\{x, y\}} dxdy < \frac{pq}{\lambda^{\frac{1}{p}}\mu^{\frac{1}{q}}} \left\{\int_0^\infty x^{\frac{1}{\lambda}-1} f^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty x^{\frac{1}{\mu}-1} g^q(x)dx\right\}^{\frac{1}{q}},$$

where $\lambda > 0$ and $\mu > 0$.

2. Main Results

Lemma 2.1. *Suppose* $r>1$, $\frac{1}{s}+\frac{1}{r}=1$, $\lambda,\mu,\varepsilon>0$. *Then*

$$\int_1^\infty x^{-\varepsilon(\frac{1}{s}+\frac{\mu}{\lambda r})-1}\int_0^{x^{-\frac{1}{\lambda}}}\frac{1}{\max\{1,t\}}t^{\frac{-1-\mu\varepsilon}{r}}dtdx=O(1)\quad(\varepsilon\to o^+). \tag{2.1}$$

Proof. There exist $n\in\mathbb{N}$ which is large enough, such that $1+\frac{-1-\mu\varepsilon}{r}>0$ for $\varepsilon\in(0,\frac{1}{\mu n}]$ and $x\geq 1$, we have

$$\int_0^{x^{-\frac{1}{\lambda}}}\frac{1}{\max\{1,t\}}t^{\frac{-1-\mu\varepsilon}{r}}dt=\int_0^{x^{-\frac{1}{\lambda}}}t^{\frac{-1-\mu\varepsilon}{r}}dt=\frac{1}{1+\frac{-1-\mu\varepsilon}{r}}\left(x^{-\frac{1}{\lambda}}\right)^{1+\frac{-1-\mu\varepsilon}{r}}.$$

Since for $a\geq 1$ the function $g(y)=\frac{1}{ya^y}$ $(y\in(0,\infty))$ is decreasing, we find

$$\frac{1}{1+\frac{-1-\mu\varepsilon}{r}}\left(x^{-\frac{1}{\lambda}}\right)^{1+\frac{-1-\mu\varepsilon}{r}}\leq\frac{1}{1+\frac{-1-1/n}{r}}\left(x^{-\frac{1}{\lambda}}\right)^{1+\frac{-1-1/n}{r}},$$

so

$$\begin{aligned}0&<\int_1^\infty x^{-\varepsilon(\frac{1}{s}+\frac{\mu}{\lambda r})-1}\int_0^{x^{-\frac{1}{\lambda}}}\frac{1}{\max\{1,t\}}t^{\frac{-1-\mu\epsilon}{r}}dtdx\\&<\int_1^\infty x^{-1}\frac{1}{1+\frac{-1-1/n}{r}}\left(x^{-\frac{1}{\lambda}}\right)^{1+\frac{-1-1/n}{r}}\\&=\frac{1}{\lambda}\left(\frac{1}{1+\frac{-1-1/n}{r}}\right)^2.\end{aligned}$$

Hence relation (2.1) is valid. The lemma is proved. □

Now we study the following inequality:

Theorem 2.2. *Suppose* $f(x),\ g(x) \geq 0,\ p > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \lambda > 0, \mu > 0$ *and*

$$0 < \int_0^\infty x^{\frac{1}{\lambda}-1} f^p(x)dx < \infty, \quad 0 < \int_0^\infty x^{\frac{1}{\mu}-1} g^q(x)dx < \infty.$$

Then

$$\int_0^\infty \int_0^\infty \frac{f(x^\lambda)g(y^\mu)}{\max\{x,y\}}dxdy < \frac{pq}{\lambda^{\frac{1}{p}}\mu^{\frac{1}{q}}} \left\{\int_0^\infty x^{\frac{1}{\lambda}-1} f^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty x^{\frac{1}{\mu}-1} g^q(x)dx\right\}^{\frac{1}{q}}, \tag{2.2}$$

where the constant factor $\frac{pq}{\lambda^{\frac{1}{p}}\mu^{\frac{1}{q}}}$ *is the best possible for* $\lambda = \mu$.

Proof. By Hölder's inequality, we have

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \frac{f(x^\lambda)g(y^\mu)}{\max\{x,y\}}dxdy \\
&= \frac{1}{\lambda}\frac{1}{\mu} \int_0^\infty \int_0^\infty \frac{\left[x^{\frac{1}{\lambda}-1} f(x)\right]\left[y^{\frac{1}{\mu}-1} g(y)\right]}{\max\left\{x^{\frac{1}{\lambda}}, y^{\frac{1}{\mu}}\right\}}dxdy \\
&= \frac{1}{\lambda}\frac{1}{\mu} \int_0^\infty \int_0^\infty \left[\frac{x^{\frac{1}{\lambda}-1} f(x)}{\left(\max\left\{x^{\frac{1}{\lambda}}, y^{\frac{1}{\mu}}\right\}\right)^{\frac{1}{p}}} \left(\frac{x^{(1-\frac{1}{\lambda})p/q}}{y^{1-\frac{1}{\mu}}}\right)^{\frac{1}{p}} \left(\frac{x^{\frac{1}{\lambda}}}{y^{\frac{1}{\mu}}}\right)^{\frac{1}{pq}}\right]
\end{aligned}$$

$$\times \left[\frac{y^{\frac{1}{\mu}-1} g(y)}{\left(\max\left\{x^{\frac{1}{\lambda}}, y^{\frac{1}{\mu}}\right\}\right)^{\frac{1}{q}}} \left(\frac{y^{(1-\frac{1}{\mu})}}{x^{(1-\frac{1}{\lambda})p/q}} \right)^{\frac{1}{p}} \left(\frac{y^{\frac{1}{\mu}}}{x^{\frac{1}{\lambda}}} \right)^{\frac{1}{pq}} \right] dxdy$$

$$\leq \frac{1}{\lambda}\frac{1}{\mu} \left[\int_0^\infty \int_0^\infty x^{p(\frac{1}{\lambda}-1)} \frac{f^p(x)}{\max\left\{x^{\frac{1}{\lambda}}, y^{\frac{1}{\mu}}\right\}} \frac{x^{(1-\frac{1}{\lambda})p/q}}{y^{1-\frac{1}{\mu}}} \left(\frac{x^{\frac{1}{\lambda}}}{y^{\frac{1}{\mu}}} \right)^{\frac{1}{q}} dxdy \right]^{\frac{1}{p}}$$

$$\times \left[\int_0^\infty \int_0^\infty y^{q(\frac{1}{\mu}-1)} \frac{g^q(y)}{\max\left\{x^{\frac{1}{\lambda}}, y^{\frac{1}{\mu}}\right\}} \frac{y^{(1-\frac{1}{\mu})q/p}}{x^{1-\frac{1}{\lambda}}} \left(\frac{y^{\frac{1}{\mu}}}{x^{\frac{1}{\lambda}}} \right)^{\frac{1}{p}} dxdy \right]^{\frac{1}{q}}.$$

Define the weight function $\varphi(x)$, $\psi(y)$ as

$$\varphi(x) := \int_0^\infty \frac{1}{\max\left\{x^{\frac{1}{\lambda}}, y^{\frac{1}{\mu}}\right\}} \cdot \frac{x^{(1-\frac{1}{\lambda})p/q}}{y^{1-\frac{1}{\mu}}} \left(\frac{x^{\frac{1}{\lambda}}}{y^{\frac{1}{\mu}}} \right)^{\frac{1}{q}} dy, \qquad x \in (0,\infty),$$

$$\psi(y) := \int_0^\infty \frac{1}{\max\left\{x^{\frac{1}{\lambda}}, y^{\frac{1}{\mu}}\right\}} \cdot \frac{y^{(1-\frac{1}{\mu})q/p}}{x^{1-\frac{1}{\lambda}}} \left(\frac{y^{\frac{1}{\mu}}}{x^{\frac{1}{\lambda}}} \right)^{\frac{1}{p}} dx, \qquad y \in (0,\infty),$$

then above inequality yields

$$\int_0^\infty \int_0^\infty \frac{f(x^\lambda) g(y^\mu)}{\max\{x, y\}} dxdy$$
$$\leq \frac{1}{\lambda}\frac{1}{\mu} \left[\int_0^\infty \varphi(x) x^{p(\frac{1}{\lambda}-1)} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty \psi(y) y^{q(\frac{1}{\mu}-1)} g^q(y) dy \right]^{\frac{1}{q}}.$$

For fixed x, let $y^{\frac{1}{\mu}} = x^{\frac{1}{\lambda}}t$, we have

$$\begin{aligned}\varphi(x) &:= \mu x^{(p-1)(1-\frac{1}{\lambda})}\int_0^\infty \frac{1}{\max\{1,t\}}t^{\frac{1}{p}-1}dt\\ &= \mu pq x^{(p-1)(1-\frac{1}{\lambda})}\\ &= \mu pq x^{(p-1)(1-\frac{1}{\lambda})}.\end{aligned}$$

By the same token, $\psi(y) = \lambda pq y^{(q-1)(1-\frac{1}{\mu})}$, thus

$$(2.3)\quad \int_0^\infty\int_0^\infty \frac{f(x^\lambda)g(y^\mu)}{\max\{x,y\}}dxdy \leq \frac{pq}{\lambda^{\frac{1}{p}}\mu^{\frac{1}{q}}}\left\{\int_0^\infty x^{\frac{1}{\lambda}-1}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty x^{\frac{1}{\mu}-1}g^q(x)dx\right\}^{\frac{1}{q}}.$$

If (2.3) takes the form of the equality, then there exist constants c and d, such that Kuang (see [2])

$$c\frac{x^{p(\frac{1}{\lambda}-1)}f^p(x)}{\max\{x^{\frac{1}{\lambda}},y^{\frac{1}{\mu}}\}}\cdot\frac{x^{(1-\frac{1}{\lambda})p/q}}{y^{1-\frac{1}{\mu}}}\left(\frac{x^{\frac{1}{\lambda}}}{y^{\frac{1}{\mu}}}\right)^{\frac{1}{q}} = d\frac{y^{q(\frac{1}{\mu}-1)}g^q(y)}{\max\left\{x^{\frac{1}{\lambda}},y^{\frac{1}{\mu}}\right\}}\cdot\frac{y^{(1-\frac{1}{\mu})q/p}}{x^{1-\frac{1}{\lambda}}}\left(\frac{y^{\frac{1}{\mu}}}{x^{\frac{1}{\lambda}}}\right)^{\frac{1}{p}}$$

a.e. on $(0,\infty)\times(0,\infty)$.

Then we have

$$cx^{\frac{1}{\lambda}}f^p(x) = dy^{\frac{1}{\mu}}g^q(y)$$

a.e. on $(0,\infty)\times(0,\infty)$.

Hence we have

$$cx^{\frac{1}{\lambda}}f^p(x) = dy^{\frac{1}{\mu}}g^q(y) = \text{constant}$$
$$\text{a.e. on } (0,\infty)\times(0,\infty),$$

which contradicts the facts that

$$0 < \int_0^\infty x^{\frac{1}{\lambda}}f^p(x)dx < \infty \quad \text{and} \quad 0 < \int_0^\infty y^{\frac{1}{\mu}}g^q(x)dx < \infty.$$

Hence (2.3) takes the form of strict inequality. So we have (2.2).

For $0 < \varepsilon < \frac{1}{2}$, setting $f_\varepsilon(x) = x^{\frac{-\varepsilon-1/\lambda}{p}}$, for $x \in [1,\infty)$; $f_\varepsilon(x) = 0$, for $x \in (0,1)$, and $g_\varepsilon(y) = y^{\frac{-\varepsilon-1/\mu}{q}}$, for $y \in [1,\infty)$; $g_\varepsilon(y) = 0$, for $y \in (0,1)$. Assume that the constant factor $\frac{pq}{\lambda^{\frac{1}{p}}\mu^{\frac{1}{q}}}$ in (2.2) is not the best possible, then there exists a positive number K with $K < \frac{pq}{\lambda^{\frac{1}{p}}\mu^{\frac{1}{q}}}$, such that (2.2) is valid by changing $\frac{pq}{\lambda^{\frac{1}{p}}\mu^{\frac{1}{q}}}$ to K. We have

$$\int_0^\infty\int_0^\infty \frac{f(x^\lambda)g(y^\mu)}{\max\{x,y\}}dxdy$$
$$< K\left\{\int_0^\infty x^{\frac{1}{\lambda}-1}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty x^{\frac{1}{\mu}-1}g^q(x)dx\right\}^{\frac{1}{q}} = \frac{K}{\varepsilon}.$$

Since

$$\int_0^\infty \frac{1}{\max\{1,t\}}t^{\frac{-1-\varepsilon\mu}{q}}dt = pq + o(1) \qquad (\varepsilon \to 0^+),$$

setting $y^{\frac{1}{\mu}} = x^{\frac{1}{\lambda}}t$, by (2.1), we find

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \frac{f(x^\lambda)g(y^\mu)}{\max\{x,y\}}dxdy \\
&= \frac{1}{\lambda}\frac{1}{\mu}\int_0^\infty \int_0^\infty \frac{\left[x^{\frac{1}{\lambda}-1}f(x)\right]\left[y^{\frac{1}{\mu}-1}g(y)\right]}{\max\left\{x^{\frac{1}{\lambda}}, y^{\frac{1}{\mu}}\right\}}dxdy \\
&= \frac{1}{\lambda}\frac{1}{\mu}\int_1^\infty \int_1^\infty \frac{x^{(\frac{1}{\lambda}-1)+\frac{-\varepsilon-\frac{1}{\lambda}}{p}}y^{(\frac{1}{\mu}-1)+\frac{-\varepsilon-\frac{1}{\mu}}{q}}}{\max\left\{x^{\frac{1}{\lambda}}, y^{\frac{1}{\mu}}\right\}}dxdy \\
&= \frac{1}{\lambda}\frac{1}{\mu}\int_1^\infty \int_{x^{-\frac{1}{\lambda}}}^\infty \frac{x^{(\frac{1}{\lambda}-1)+\frac{-\varepsilon-\frac{1}{\lambda}}{p}}(t^\mu x^{\frac{\mu}{\lambda}})^{(\frac{1}{\mu}-1)+\frac{-\varepsilon-\frac{1}{\mu}}{q}}}{\max\left\{x^{\frac{1}{\lambda}}, tx^{\frac{1}{\lambda}}\right\}}x^{\frac{\mu}{\lambda}}\mu t^{\mu-1}dxdt \\
&= \frac{1}{\lambda}\int_1^\infty x^{-\varepsilon(\frac{1}{p}+\frac{\mu}{\lambda q})-1}\int_{x^{-\frac{1}{\lambda}}}^\infty \frac{1}{\max\{1,t\}}t^{\frac{-1-\varepsilon\mu}{q}}dtdx \\
&= \frac{1}{\lambda}\int_1^\infty x^{-\varepsilon(\frac{1}{p}+\frac{\mu}{\lambda q})-1}\left(\int_0^\infty \frac{1}{\max\{1,t\}}t^{\frac{-1-\varepsilon\mu}{q}}dtdx\right. \\
&\qquad \left.-\int_0^{x^{-\frac{1}{\lambda}}} \frac{1}{\max\{1,t\}}t^{\frac{-1-\varepsilon\mu}{q}}dtdx\right) \\
&= \frac{1}{\lambda\varepsilon}\left[\frac{pq}{\frac{1}{p}+\frac{\mu}{\lambda q}}+o(1)\right].
\end{aligned}$$

Since for $\varepsilon > 0$ small enough, we have

$$\frac{1}{\lambda}\left[\frac{pq}{\frac{1}{p}+\frac{\mu}{\lambda q}}+o(1)\right] < K.$$

It is obvious that $\lambda^{\frac{1}{p}}\mu^{\frac{1}{q}} \le \frac{\lambda}{p}+\frac{\mu}{q}$ (i.e. $\frac{1}{\lambda}\frac{1}{\frac{1}{p}+\frac{\mu}{\lambda q}} \le \frac{1}{\lambda^{\frac{1}{p}}\mu^{\frac{1}{q}}}$) by Young's inequality. Consider the case of taking the form of the equality for Young's inequality, we get $\mu^{\frac{1}{q}} = \lambda^{\frac{p-1}{p}}$, i.e. $\lambda = \mu$, Then

$$\frac{1}{\lambda}\left[\frac{pq}{\frac{1}{p}+\frac{\mu}{\lambda q}}+o(1)\right] = \frac{pq}{\lambda^{\frac{1}{p}}\mu^{\frac{1}{q}}}+o(1) < K.$$

Thus we get $\frac{pq}{\lambda^{\frac{1}{p}}\mu^{\frac{1}{q}}} \le K$, which contradicts the hypothesis. Hence the constant factor $\frac{pq}{\lambda^{\frac{1}{p}}\mu^{\frac{1}{q}}}$ in (2.2) is best possible for $\lambda = \mu$. □

Remark 1. *For $\lambda = \mu$, inequality (2.2) becomes*

$$(2.4)\qquad \int_0^\infty\int_0^\infty \frac{f(x^\lambda)g(y^\lambda)}{\max\{x,y\}}dxdy < \frac{pq}{\lambda}\left\{\int_0^\infty x^{\frac{1}{\lambda}-1}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty x^{\frac{1}{\lambda}-1}g^q(x)dx\right\}^{\frac{1}{q}},$$

where the constant factor $\frac{pq}{\lambda}$ is the best possible.

Theorem 2.3. *Suppose* $f \geq 0$, $p > 1$, $\lambda > 0$ *and* $0 < \int_0^\infty x^{\frac{1}{\lambda}-1} f^p(x)dx < \infty$. *Then*

$$\int_0^\infty \left[\int_0^\infty \frac{f(x^\lambda)}{\max\{x,y\}}dx\right]^p dy < \frac{1}{\lambda}(pq)^p \int_0^\infty x^{\frac{1}{\lambda}-1} f(x)^p dx, \tag{2.5}$$

where the constant factor $\frac{1}{\lambda}(pq)^p$ *is the best possible. Inequality (2.5) is equivalent to (2.4).*

Proof. Setting $g(y)$ as

$$\left[\int_0^\infty \frac{x^{\frac{1}{\lambda}-1} f(x)}{\max\left\{x^{\frac{1}{\lambda}}, y^{\frac{1}{\lambda}}\right\}}dx\right]^{p-1} > 0, \qquad y \in (0,\infty).$$

then by (2.4), we find

$$\begin{aligned} \lambda^{-2}\int_0^\infty y^{\frac{1}{\lambda}-1} g^q(y)dy &= \lambda^{p-1}\int_0^\infty \left[\int_0^\infty \frac{f(x^\lambda)}{\max\{x,y\}}dx\right]^p dy \\ &= \int_0^\infty \int_0^\infty \frac{f(x^\lambda)g(y^\lambda)}{\max\{x,y\}}dxdy \\ &\leq \frac{pq}{\lambda}\left\{\int_0^\infty x^{\frac{1}{\lambda}-1} f^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty y^{\frac{1}{\lambda}-1} g^q(y)dy\right\}^{\frac{1}{q}}. \end{aligned} \tag{2.6}$$

Hence we obtain

$$0 < \int_0^\infty y^{\frac{1}{\lambda}-1} g^q(y)dy \leq \lambda^p (pq)^p \int_0^\infty x^{\frac{1}{\lambda}-1} f^p(x)dx < \infty. \tag{2.7}$$

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By (2.4), both (2.6) and (2.7) take the form of strict inequality, so we have (2.5).

On the other hand, suppose that (2.5) is valid. By Hölder's inequality, we find

$$\int_0^\infty \int_0^\infty \frac{f(x^\lambda)g(y^\lambda)}{\max\{x,y\}}dxdy$$
$$= \int_0^\infty \left[\int_0^\infty \frac{f(x^\lambda)}{\max\{x,y\}}dx\right] g(y^\lambda)dy$$
$$\leq \left\{\int_0^\infty \left[\int_0^\infty \frac{f(x^\lambda)}{\max\{x,y\}}dx\right]^p dy\right\}^{\frac{1}{p}} \left\{\int_0^\infty g^q(y^\lambda)dy\right\}^{\frac{1}{q}}. \tag{2.8}$$

Then by (2.5), we have (2.4). Thus (2.4) and (2.5) are equivalent.

If the constant $\frac{1}{\lambda}(pq)^p$ in (2.5) is not the best possible, by (2.8), we may get a contradiction that the constant factor in (2.4) is not the best possible. Thus we complete the proof of the theorem. □

Remark 2.

(i) *For $\lambda = \mu = 1$, (2.2) and (2.5) reduce respectively to (1.3) and (1.4). It follows that (2.2) is a new extension of (1.6) and (1.3) with some parameters and the equivalent form (2.4) is a new extension of (1.4).*

(ii) *It is amazing that (2.4) and (1.9) are different, although both of them are the extensions of (1.6) with $(p,q)-$parameter and the best constant factor.*

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