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ON ANALYTIC FUNCTIONS RELATED TO CERTAIN FAMILY OF INTEGRAL OPERATORS

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Abstract

Let $\mathcal{A}$ be the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \dots$, analytic in the open unit disc E. A certain integral operator is used to define some subclasses of $\mathcal{A}$ and their inclusion properties are studied.

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Contents

1. Introduction

Let $\mathcal{A}$ denote the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open disk $E = \{z : |z| < 1\}$. Let the functions f_i be defined for $i = 1, 2$, by

$$f_i(z) = z + \sum_{n=2}^{\infty} a_{n,i} z^n. \tag{1.2}$$

The modified Hadamard product (convolution) of f_1 and f_2 is defined here by

$$(f_1 \star f_2)(z) = z + \sum_{n=2}^{\infty} a_{n,1} a_{n,2} z^n.$$

Let $P_k(\beta)$ be the class of functions $h(z)$ analytic in the unit disc E satisfying the properties $h(0) = 1$ and

$$\int_0^{2\pi} \left| \operatorname{Re} \frac{h(z) - \beta}{1 - \beta} \right| d\theta \leq k\pi, \tag{1.3}$$

where $z = re^{i\theta}$, $k \geq 2$ and $0 \leq \beta < 1$, see [4]. For $\beta = 0$, we obtain the class P_k defined by Pinchuk [5]. The case $k = 2, \beta = 0$ gives us the class P of functions with positive real part, and $k = 2$, $P_2(\beta) = P(\beta)$ is the class of functions with positive real part greater than β.

Also we can write for $h \in P_k(\beta)$

(1.4) $$h(z) = \frac{1}{2}\int_0^{2\pi} \frac{1+(1-2\beta)ze^{-it}}{1-ze^{-it}} d\mu(t),$$

where $\mu(t)$ is a function with bounded variation on $[0, 2\pi]$ such that

(1.5) $$\int_0^{2\pi} d\mu(t) = 2 \quad \text{and} \quad \int_0^{2\pi} |d\mu(t)| \le k.$$

From (1.4) and (1.5), we can write, for $h \in P_k(\beta)$,

(1.6) $$h(z) = \left(\frac{k}{4}+\frac{1}{2}\right) h_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right) h_2(z), \quad h_1, h_2 \in P(\beta).$$

We have the following classes:

$$R_k(\alpha) = \left\{ f : f \in \mathcal{A} \quad \text{and} \quad \frac{zf'(z)}{f(z)} \in P_k(\alpha), \quad z \in E, \quad 0 \le \alpha < 1 \right\}.$$

We note that $R_2(\alpha) = S^\star(\alpha)$ is the class of starlike functions of order α.

$$V_k(\alpha) = \left\{ f : f \in \mathcal{A} \quad \text{and} \quad \frac{(zf'(z))'}{f'(z)} \in P_k(\alpha), \quad z \in E, \quad 0 \le \alpha < 1 \right\}.$$

Note that $V_2(\alpha) = C(\alpha)$ is the class of convex functions of order α.

$$T_k(\beta, \alpha) = \left\{ f : f \in \mathcal{A}, g \in R_2(\alpha) \right.$$
$$\left. \text{and} \quad \frac{zf'(z)}{g(z)} \in P_k(\beta), \quad z \in E, \quad 0 \le \alpha, \beta < 1 \right\}.$$

We note that $T_2(0,0)$ is the class K of close-to-convex univalent functions.

$$T_k^\star(\beta,\alpha) = \Big\{ f : f \in \mathcal{A}, g \in V_2(\alpha) \quad \text{and} \quad \frac{(zf'(z))'}{g'(z)} \in P_k(\beta), \quad z \in E, \quad 0 \le \alpha, \beta < 1 \Big\}.$$

In particular, the class $T_2^\star(\beta,\alpha) = C^\star(\beta,\alpha)$ was considered by Noor [3] and for $T_2^\star(0,0) = C^\star$ is the class of quasi-convex univalent functions which was first introduced and studied in [2].

It can be easily seen from the above definitions that

$$f(z) \in V_k(\alpha) \iff zf'(z) \in R_k(\alpha) \tag{1.7}$$

and

$$f(z) \in T_k^\star(\beta,\alpha) \iff zf'(z) \in T_k(\beta,\alpha). \tag{1.8}$$

We consider the following integral operator $L_\lambda^\mu : \mathcal{A} \longrightarrow \mathcal{A}$, for $\lambda > -1$; $\mu > 0$; $f \in \mathcal{A}$,

$$\begin{aligned} L_\lambda^\mu f(z) &= C_\lambda^{\lambda+\mu} \frac{\mu}{z^\lambda} \int_0^z t^{\lambda-1} \left(1 - \frac{t}{z}\right)^{\mu-1} f(t)dt \\ &= z + \frac{\Gamma(\lambda+\mu+1)}{\Gamma(\lambda+1)} \sum_{n=2}^{\infty} \frac{\Gamma(\lambda+n)}{\Gamma(\lambda+\mu+n)} a_n z^n, \end{aligned} \tag{1.9}$$

where Γ denotes the Gamma function. From (1.9), we can obtain the well-known generalized Bernadi operator as follows:

$$I_\mu f(z) = \frac{\mu+1}{z^\mu}\int_0^z t^{\mu-1} f(t)dt$$
$$= z + \sum_{n=2}^{\infty} \frac{\mu+1}{\mu+n} a_n z^n, \quad \mu > -1;\ f \in \mathcal{A}.$$

We now define the following subclasses of $\mathcal{A}$ by using the integral operator L_λ^μ.

Definition 1.1. *Let* $f \in \mathcal{A}$. *Then* $f \in R_k(\lambda,\mu,\alpha)$ *if and only if* $L_\lambda^\mu f \in R_k(\alpha)$, *for* $z \in E$.

Definition 1.2. *Let* $f \in \mathcal{A}$. *Then* $f \in V_k(\lambda,\mu,\alpha)$ *if and only if* $L_\lambda^\mu f \in V_k(\alpha)$, *for* $z \in E$.

Definition 1.3. *Let* $f \in \mathcal{A}$. *Then* $f \in T_k(\lambda,\mu,\beta,\alpha)$ *if and only if* $L_\lambda^\mu f \in T_k(\beta,\alpha)$, *for* $z \in E$.

Definition 1.4. *Let* $f \in \mathcal{A}$. *Then* $f \in T_k^\star(\lambda,\mu,\beta,\alpha)$ *if and only if* $L_\lambda^\mu f \in T_k^\star(\beta,\alpha)$, *for* $z \in E$.

We shall need the following result.

Lemma 1.1 ([1]). *Let* $u = u_1 + iu_2$ *and* $v = v_1 + iv_2$ *and let* Φ *be a complex-valued function satisfying the conditions:*

(i) $\Phi(u,v)$ *is continuous in a domain* $D \subset \mathbf{C}^2$,

(ii) $(1,0) \in D$ *and* $\Phi(1,0) > 0$.

(iii) $\operatorname{Re}\Phi(iu_2, v_1) \leq 0$, *whenever* $(iu_2, v_1) \in D$ *and* $v_1 \leq -\frac{1}{2}(1+u_2^2)$.

If $h(z) = 1 + \sum_{m=2}^{\infty} c_m z^m$ *is a function analytic in* E *such that* $(h(z), zh'(z)) \in D$ *and* $\operatorname{Re}\Phi(h(z), zh'(z)) > 0$ *for* $z \in E$, *then* $\operatorname{Re} h(z) > 0$ *in* E.

2. Main Results

Theorem 2.1. *Let* $f \in \mathcal{A}, \lambda > -1, \mu > 0$ *and* $\lambda + \mu > 0$. *Then* $R_k(\lambda, \mu, 0) \subset R_k(\lambda, \mu + 1, \alpha)$, *where*

$$\alpha = \frac{2}{(\beta + 1) + \sqrt{\beta^2 + 2\beta + 9}}, \quad \text{with} \quad \beta = 2(\lambda + \mu). \tag{2.1}$$

Proof. Let $f \in R_k(\lambda, \mu, 0)$ and let

$$\frac{\left(zL_\lambda^{\mu+1} f(z)\right)'}{L_\lambda^{\mu+1} f(z)} = p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z),$$

where $p(0) = 1$ and $p(z)$ is analytic in E. From (1.9), it can easily be seen that

$$z\left(L_\lambda^{\mu+1} f(z)\right)' = (\lambda + \mu + 1) L_\lambda^{\mu} f(z) - (\lambda + \mu) L_\lambda^{\mu+1} f(z). \tag{2.2}$$

Some computation and use of (2.2) yields

$$\frac{z\left(L_\lambda^{\mu} f(z)\right)'}{L_\lambda^{\mu} f(z)} = \left\{p(z) + \frac{zp'(z)}{p(z) + \lambda + \mu}\right\} \in P_k, \quad z \in E.$$

Let

$$\Phi_{\lambda,\mu}(z) = \sum_{j=1}^{\infty} \frac{(\lambda + \mu) + j}{\lambda + \mu + 1} z^j$$
$$= \left(\frac{\lambda + \mu}{\lambda + \mu + 1}\right) \frac{z}{1 - z} + \left(\frac{1}{\lambda + \mu + 1}\right) \frac{z}{(1 - z)^2}.$$

Then

$$p(z) \star \Phi_{\lambda,\mu}(z)$$
$$= p(z) + \frac{zp'(z)}{p(z)+\lambda+\mu}$$
$$= \left(\frac{k}{4}+\frac{1}{2}\right)[p_1(z) \star \Phi_{\lambda,\mu}(z)] - \left(\frac{k}{4}-\frac{1}{2}\right)[p_2(z) \star \Phi_{\lambda,\mu}(z)]$$
$$= \left(\frac{k}{4}+\frac{1}{2}\right)\left[p_1(z) + \frac{zp_1'(z)}{p_1(z)+\lambda+\mu}\right] - \left(\frac{k}{4}-\frac{1}{2}\right)\left[p_2(z) + \frac{zp_2'(z)}{p_2(z)+\lambda+\mu}\right],$$

and this implies that

$$\left(p_i(z) + \frac{zp_i'(z)}{p_i(z)+\lambda+\mu}\right) \in P, \quad z \in E.$$

We want to show that $p_i(z) \in P(\alpha)$, where α is given by (2.1) and this will show that $p \in P_k(\alpha)$ for $z \in E$. Let

$$p_i(z) = (1-\alpha)h_i(z) + \alpha, \quad i = 1, 2.$$

Then

$$\left\{(1-\alpha)h_i(z) + \alpha + \frac{(1-\alpha)zh_i'(z)}{(1-\alpha)h_i(z)+\alpha+\lambda+\mu}\right\} \in P.$$

We form the functional $\Psi(u,v)$ by choosing $u = h_i(z), \quad v = zh_i'$. Thus

$$\Psi(u,v) = (1-\alpha)u + \alpha + \frac{(1-\alpha)v}{(1-\alpha)u+(\alpha+\lambda+\mu)}.$$

The first two conditions of Lemma 1.1 are clearly satisfied. We verify the condition (iii) as follows.

$$\operatorname{Re}\Psi(iu_2, v_1) = \alpha + \frac{(1-\alpha)(\alpha+\lambda+\mu)v_1}{(\alpha+\lambda+\mu)^2 + (1-\alpha)^2 u_2^2}.$$

By putting $v_1 \le -\frac{(1+u_2^2)}{2}$, we obtain

$$\begin{aligned}
&\operatorname{Re}\Psi(iu_2, v_1) \\
&\le \alpha - \frac{1}{2}\frac{(1-\alpha)(\alpha+\lambda+\mu)(1+u_2^2)}{(\alpha+\lambda+\mu)^2 + (1-\alpha)^2 u_2^2} \\
&= \frac{2\alpha(\alpha+\lambda+\mu)^2 + 2\alpha(1-\alpha)^2 u_2^2 - (1-\alpha)(\alpha+\lambda+\mu) - (1-\alpha)(\alpha+\lambda+\mu)u_2^2}{2[(\alpha+\lambda+\mu)^2 + (1-\alpha)^2 u_2^2]} \\
&= \frac{A + Bu_2^2}{2C},
\end{aligned}$$

where

$$\begin{aligned}
A &= 2\alpha(\alpha+\lambda+\mu)^2 - (1-\alpha)(\alpha+\lambda+\mu), \\
B &= 2\alpha(1-\alpha)^2 - (1-\alpha)(\alpha+\lambda+\mu), \\
C &= (\alpha+\lambda+\mu)^2 + (1-\alpha)^2 u_2^2 > 0.
\end{aligned}$$

We note that $\operatorname{Re}\Psi(iu_2, v_1) \le 0$ if and only if, $A \le 0$ and $B \le 0$. From $A \le 0$, we obtain α as given by (2.1) and $B \le 0$ gives us $0 \le \alpha < 1$, and this completes the proof. □

Theorem 2.2. *For $\lambda > -1, \mu > 0$ and $(\lambda + \mu) > 0$, $\quad V_k(\lambda, \mu, 0) \subset V_k(\lambda, \mu + 1, \alpha)$, where α is given by (2.1).*

Proof. Let $f \in V_k(\lambda, \mu, 0)$. Then $L_\lambda^\mu f \in V_k(0) = V_k$ and, by (1.7) $z(L_\lambda^\mu)' \in R_k(0) = R_k$. This implies

$$L_\lambda^\mu(zf') \in R_k \quad \Longrightarrow \quad zf' \in R_k(\lambda, \mu, 0) \subset R_k(\lambda, \mu + 1, \alpha).$$

Consequently $f \in V_k(\lambda, \mu + 1, \alpha)$, where α is given by (2.1). □

Theorem 2.3. *Let $\lambda > -1,\ \mu > 0$ and $(\lambda + \mu) > 0$. Then*

$$T_k(\lambda, \mu, \beta, 0) \subset T_k(\lambda, \mu + 1, \gamma, \alpha),$$

where α is given by (2.1) and $\gamma \leq \beta$ is defined in the proof.

Proof. Let $f \in T_k(\lambda, \mu, 0)$. Then there exists $g \in R_2(\lambda, \mu, 0)$ such that $\left\{ \frac{z(L_\lambda^\mu f)'}{L_\lambda^\mu g} \right\} \in P_k(\beta)$, for $z \in E,\ 0 \leq \beta < 1$. Let

$$\begin{aligned} \frac{z(L_\lambda^{\mu+1} f(z))'}{L_\lambda^{\mu+1} g(z)} &= (1 - \gamma)p(z) + \gamma \\ &= \left(\frac{k}{4} + \frac{1}{2} \right) \{(1 - \gamma)p_1(z) + \gamma\} - \left(\frac{k}{4} - \frac{1}{2} \right) \{(1 - \gamma)p_2(z) + \gamma\}, \end{aligned}$$

where $p(0) = 1$, and $p(z)$ is analytic in E.

Making use of (2.2) and Theorem 2.1 with $k = 2$, we have

$$\left(\frac{z(L_\lambda^\mu f(z))'}{L_\lambda^\mu g(z)} - \beta\right) = \left\{(1-\gamma)p(z) + (\gamma - \beta) + \frac{(1-\gamma)zp'(z)}{(1-\alpha)q(z) + \alpha + \lambda + \mu}\right\} \in P_k, \tag{2.3}$$

and $q \in P$, where

$$(1-\alpha)q(z) + \alpha = \frac{z\left(L_\lambda^{\mu+1} g(z)\right)'}{L_\lambda^{\mu+1} g(z)}, \quad z \in E.$$

Using (1.6), we form the functional $\Phi(u,v)$ by taking $u = u_1 + iu_2 = p_i(z),\ v = v_1 + iv_2 = zp_i'$ in (2.3) as

$$\Phi(u,v) = (1-\gamma)u + (\gamma - \beta) + \frac{(1-\gamma)v}{(1-\alpha)q(z) + \alpha + \lambda + \mu}. \tag{2.4}$$

It can be easily seen that the function $\Phi(u,v)$ defined by (2.4) satisfies the conditions (i) and (ii) of Lemma 1.1. To verify the condition (iii), we proceed, with $q(z) = q_1 + iq_2$, as follows:

$$\begin{aligned} &\operatorname{Re}\left[\Phi(iu_2, v_1)\right] \\ &= (\gamma - \beta) + \operatorname{Re}\left\{\frac{(1-\gamma)v_1}{(1-\alpha)(q_1 + iq_2) + \alpha + \lambda + \mu}\right\} \\ &= (\gamma - \beta) + \frac{(1-\gamma)(1-\alpha)v_1 q_1 + (1-\gamma)(\alpha + \lambda + \mu)v_1}{\left[(1-\alpha)q_1 + \alpha + \lambda + \mu\right]^2 + (1-\alpha)^2 q_2^2} \end{aligned}$$

$$\leq (\gamma - \beta) - \frac{1}{2}\frac{(1-\gamma)(1-\alpha)(1+u_2^2)q_1 + (1-\gamma)(\alpha+\lambda+\mu)(1+u_2^2)}{[(1-\alpha)q_1+\alpha+\lambda+\mu]^2 + (1-\alpha)^2 q_2^2}$$
$$\leq 0, \quad \text{for} \quad \gamma \leq \beta < 1.$$

Therefore, applying Lemma 1.1, $p_i \in P$, $i = 1, 2$ and consequently $p \in P_k$ and thus $f \in T_k(\lambda, \mu+1, \gamma, \alpha)$. □

Using the same technique and relation (1.8) with Theorem 2.3, we have the following.

Theorem 2.4. *For $\lambda > -1$, $\mu > 0$, $\lambda + \mu > 0$, $T_k^\star(\lambda, \mu, \beta, 0) \subset T_k^\star(\lambda, \mu + 1, \gamma, \alpha)$, where γ and α are as given in Theorem 2.3.*

Remark 1. *For different choices of k, λ and μ, we obtain several interesting special cases of the results proved in this paper.*

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