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## ON ANALYTIC FUNCTIONS RELATED TO CERTAIN FAMILY OF INTEGRAL OPERATORS

KHALIDA INAYAT NOOR

Mathematics Department  
COMSATS Institute of Information Technology  
Islamabad, Pakistan

*EMail:* [khalidanoor@hotmail.com](mailto:khalidanoor@hotmail.com)

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Abstract

Contents



Home Page

Go Back

Close

Quit

## Abstract

Let  $\mathcal{A}$  be the class of functions  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \dots$ , analytic in the open unit disc  $E$ . A certain integral operator is used to define some subclasses of  $\mathcal{A}$  and their inclusion properties are studied.

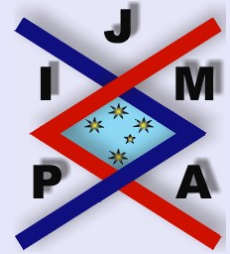
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## Contents

1	Introduction .....	3
2	Main Results .....	8
	References	



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**On Analytic Functions Related to Certain Family of Integral Operators**

Khalida Inayat Noor

---

Title Page

Contents



Go Back

Close

Quit

Page 2 of 14

# 1. Introduction

Let  $\mathcal{A}$  denote the class of functions

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open disk  $E = \{z : |z| < 1\}$ . Let the functions  $f_i$  be defined for  $i = 1, 2$ , by

$$(1.2) \quad f_i(z) = z + \sum_{n=2}^{\infty} a_{n,i} z^n.$$

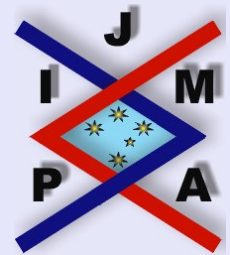
The modified Hadamard product (convolution) of  $f_1$  and  $f_2$  is defined here by

$$(f_1 \star f_2)(z) = z + \sum_{n=2}^{\infty} a_{n,1} a_{n,2} z^n.$$

Let  $P_k(\beta)$  be the class of functions  $h(z)$  analytic in the unit disc  $E$  satisfying the properties  $h(0) = 1$  and

$$(1.3) \quad \int_0^{2\pi} \left| \operatorname{Re} \frac{h(z) - \beta}{1 - \beta} \right| d\theta \leq k\pi,$$

where  $z = re^{i\theta}$ ,  $k \geq 2$  and  $0 \leq \beta < 1$ , see [4]. For  $\beta = 0$ , we obtain the class  $P_k$  defined by Pinchuk [5]. The case  $k = 2, \beta = 0$  gives us the class  $P$  of functions with positive real part, and  $k = 2, P_2(\beta) = P(\beta)$  is the class of functions with positive real part greater than  $\beta$ .



On Analytic Functions Related to Certain Family of Integral Operators

Khalida Inayat Noor

Title Page

Contents



Go Back

Close

Quit

Page 3 of 14

Also we can write for  $h \in P_k(\beta)$

$$(1.4) \quad h(z) = \frac{1}{2} \int_0^{2\pi} \frac{1 + (1 - 2\beta)ze^{-it}}{1 - ze^{-it}} d\mu(t),$$

where  $\mu(t)$  is a function with bounded variation on  $[0, 2\pi]$  such that

$$(1.5) \quad \int_0^{2\pi} d\mu(t) = 2 \quad \text{and} \quad \int_0^{2\pi} |d\mu(t)| \leq k.$$

From (1.4) and (1.5), we can write, for  $h \in P_k(\beta)$ ,

$$(1.6) \quad h(z) = \left(\frac{k}{4} + \frac{1}{2}\right) h_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) h_2(z), \quad h_1, h_2 \in P(\beta).$$

We have the following classes:

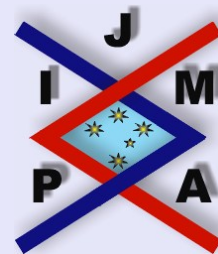
$$R_k(\alpha) = \left\{ f : f \in \mathcal{A} \quad \text{and} \quad \frac{zf'(z)}{f(z)} \in P_k(\alpha), \quad z \in E, \quad 0 \leq \alpha < 1 \right\}.$$

We note that  $R_2(\alpha) = S^*(\alpha)$  is the class of starlike functions of order  $\alpha$ .

$$V_k(\alpha) = \left\{ f : f \in \mathcal{A} \quad \text{and} \quad \frac{(zf'(z))'}{f'(z)} \in P_k(\alpha), \quad z \in E, \quad 0 \leq \alpha < 1 \right\}.$$

Note that  $V_2(\alpha) = C(\alpha)$  is the class of convex functions of order  $\alpha$ .

$$T_k(\beta, \alpha) = \left\{ f : f \in \mathcal{A}, g \in R_2(\alpha) \right. \\ \left. \text{and} \quad \frac{zf'(z)}{g(z)} \in P_k(\beta), \quad z \in E, \quad 0 \leq \alpha, \beta < 1 \right\}.$$



On Analytic Functions Related to Certain Family of Integral Operators

Khalida Inayat Noor

Title Page

Contents



Go Back

Close

Quit

Page 4 of 14

We note that  $T_2(0, 0)$  is the class  $K$  of close-to-convex univalent functions.

$$T_k^*(\beta, \alpha) = \left\{ f : f \in \mathcal{A}, g \in V_2(\alpha) \quad \text{and} \right. \\ \left. \frac{(zf'(z))'}{g'(z)} \in P_k(\beta), \quad z \in E, \quad 0 \leq \alpha, \beta < 1 \right\}.$$

In particular, the class  $T_2^*(\beta, \alpha) = C^*(\beta, \alpha)$  was considered by Noor [3] and for  $T_2^*(0, 0) = C^*$  is the class of quasi-convex univalent functions which was first introduced and studied in [2].

It can be easily seen from the above definitions that

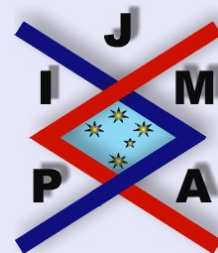
$$(1.7) \quad f(z) \in V_k(\alpha) \iff zf'(z) \in R_k(\alpha)$$

and

$$(1.8) \quad f(z) \in T_k^*(\beta, \alpha) \iff zf'(z) \in T_k(\beta, \alpha).$$

We consider the following integral operator  $L_\lambda^\mu : \mathcal{A} \longrightarrow \mathcal{A}$ , for  $\lambda > -1; \mu > 0$ ;  $f \in \mathcal{A}$ ,

$$(1.9) \quad L_\lambda^\mu f(z) = C_\lambda^{\lambda+\mu} \frac{\mu}{z^\lambda} \int_0^z t^{\lambda-1} \left(1 - \frac{t}{z}\right)^{\mu-1} f(t) dt \\ = z + \frac{\Gamma(\lambda + \mu + 1)}{\Gamma(\lambda + 1)} \sum_{n=2}^{\infty} \frac{\Gamma(\lambda + n)}{\Gamma(\lambda + \mu + n)} a_n z^n,$$



On Analytic Functions Related to Certain Family of Integral Operators

Khalida Inayat Noor

Title Page

Contents



Go Back

Close

Quit

Page 5 of 14

where  $\Gamma$  denotes the Gamma function. From (1.9), we can obtain the well-known generalized Bernadi operator as follows:

$$I_\mu f(z) = \frac{\mu + 1}{z^\mu} \int_0^z t^{\mu-1} f(t) dt$$

$$= z + \sum_{n=2}^{\infty} \frac{\mu + 1}{\mu + n} a_n z^n, \quad \mu > -1; f \in \mathcal{A}.$$

We now define the following subclasses of  $\mathcal{A}$  by using the integral operator  $L_\lambda^\mu$ .

**Definition 1.1.** Let  $f \in \mathcal{A}$ . Then  $f \in R_k(\lambda, \mu, \alpha)$  if and only if  $L_\lambda^\mu f \in R_k(\alpha)$ , for  $z \in E$ .

**Definition 1.2.** Let  $f \in \mathcal{A}$ . Then  $f \in V_k(\lambda, \mu, \alpha)$  if and only if  $L_\lambda^\mu f \in V_k(\alpha)$ , for  $z \in E$ .

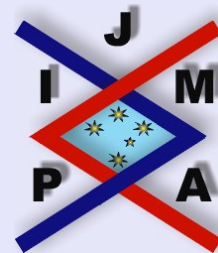
**Definition 1.3.** Let  $f \in \mathcal{A}$ . Then  $f \in T_k(\lambda, \mu, \beta, \alpha)$  if and only if  $L_\lambda^\mu f \in T_k(\beta, \alpha)$ , for  $z \in E$ .

**Definition 1.4.** Let  $f \in \mathcal{A}$ . Then  $f \in T_k^*(\lambda, \mu, \beta, \alpha)$  if and only if  $L_\lambda^\mu f \in T_k^*(\beta, \alpha)$ , for  $z \in E$ .

We shall need the following result.

**Lemma 1.1 ([1]).** Let  $u = u_1 + iu_2$  and  $v = v_1 + iv_2$  and let  $\Phi$  be a complex-valued function satisfying the conditions:

(i)  $\Phi(u, v)$  is continuous in a domain  $D \subset \mathbf{C}^2$ ,



## On Analytic Functions Related to Certain Family of Integral Operators

Khalida Inayat Noor

Title Page

Contents



Go Back

Close

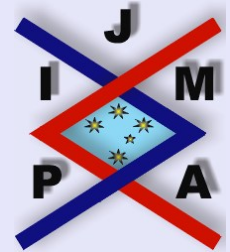
Quit

Page 6 of 14

(ii)  $(1, 0) \in D$  and  $\Phi(1, 0) > 0$ .

(iii)  $\operatorname{Re} \Phi(iu_2, v_1) \leq 0$ , whenever  $(iu_2, v_1) \in D$  and  $v_1 \leq -\frac{1}{2}(1 + u_2^2)$ .

If  $h(z) = 1 + \sum_{m=2}^{\infty} c_m z^m$  is a function analytic in  $E$  such that  $(h(z), zh'(z)) \in D$  and  $\operatorname{Re} \Phi(h(z), zh'(z)) > 0$  for  $z \in E$ , then  $\operatorname{Re} h(z) > 0$  in  $E$ .



---

**On Analytic Functions Related  
to Certain Family of Integral  
Operators**

Khalida Inayat Noor

---

Title Page

Contents



Go Back

Close

Quit

Page 7 of 14

## 2. Main Results

**Theorem 2.1.** Let  $f \in \mathcal{A}$ ,  $\lambda > -1$ ,  $\mu > 0$  and  $\lambda + \mu > 0$ . Then  $R_k(\lambda, \mu, 0) \subset R_k(\lambda, \mu + 1, \alpha)$ , where

$$(2.1) \quad \alpha = \frac{2}{(\beta + 1) + \sqrt{\beta^2 + 2\beta + 9}}, \quad \text{with } \beta = 2(\lambda + \mu).$$

*Proof.* Let  $f \in R_k(\lambda, \mu, 0)$  and let

$$\frac{(zL_\lambda^{\mu+1}f(z))'}{L_\lambda^{\mu+1}f(z)} = p(z) = \left(\frac{k}{4} + \frac{1}{2}\right)p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right)p_2(z),$$

where  $p(0) = 1$  and  $p(z)$  is analytic in  $E$ . From (1.9), it can easily be seen that

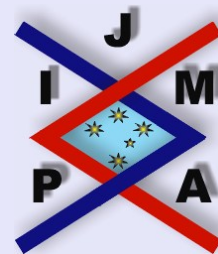
$$(2.2) \quad z(L_\lambda^{\mu+1}f(z))' = (\lambda + \mu + 1)L_\lambda^\mu f(z) - (\lambda + \mu)L_\lambda^{\mu+1}f(z).$$

Some computation and use of (2.2) yields

$$\frac{z(L_\lambda^\mu f(z))'}{L_\lambda^\mu f(z)} = \left\{ p(z) + \frac{zp'(z)}{p(z) + \lambda + \mu} \right\} \in P_k, \quad z \in E.$$

Let

$$\begin{aligned} \Phi_{\lambda, \mu}(z) &= \sum_{j=1}^{\infty} \frac{(\lambda + \mu) + j}{\lambda + \mu + 1} z^j \\ &= \left(\frac{\lambda + \mu}{\lambda + \mu + 1}\right) \frac{z}{1 - z} + \left(\frac{1}{\lambda + \mu + 1}\right) \frac{z}{(1 - z)^2}. \end{aligned}$$



On Analytic Functions Related  
to Certain Family of Integral  
Operators

Khalida Inayat Noor

Title Page

Contents



Go Back

Close

Quit

Page 8 of 14



Then

$$\begin{aligned}
 p(z) \star \Phi_{\lambda, \mu}(z) &= p(z) + \frac{zp'(z)}{p(z) + \lambda + \mu} \\
 &= \left(\frac{k}{4} + \frac{1}{2}\right) [p_1(z) \star \Phi_{\lambda, \mu}(z)] - \left(\frac{k}{4} - \frac{1}{2}\right) [p_2(z) \star \Phi_{\lambda, \mu}(z)] \\
 &= \left(\frac{k}{4} + \frac{1}{2}\right) \left[ p_1(z) + \frac{zp'_1(z)}{p_1(z) + \lambda + \mu} \right] - \left(\frac{k}{4} - \frac{1}{2}\right) \left[ p_2(z) + \frac{zp'_2(z)}{p_2(z) + \lambda + \mu} \right],
 \end{aligned}$$

and this implies that

$$\left( p_i(z) + \frac{zp'_i(z)}{p_i(z) + \lambda + \mu} \right) \in P, \quad z \in E.$$

We want to show that  $p_i(z) \in P(\alpha)$ , where  $\alpha$  is given by (2.1) and this will show that  $p \in P_k(\alpha)$  for  $z \in E$ . Let

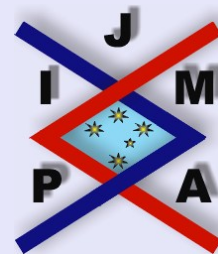
$$p_i(z) = (1 - \alpha)h_i(z) + \alpha, \quad i = 1, 2.$$

Then

$$\left\{ (1 - \alpha)h_i(z) + \alpha + \frac{(1 - \alpha)zh'_i(z)}{(1 - \alpha)h_i(z) + \alpha + \lambda + \mu} \right\} \in P.$$

We form the functional  $\Psi(u, v)$  by choosing  $u = h_i(z)$ ,  $v = zh'_i$ . Thus

$$\Psi(u, v) = (1 - \alpha)u + \alpha + \frac{(1 - \alpha)v}{(1 - \alpha)u + (\alpha + \lambda + \mu)}.$$



On Analytic Functions Related to Certain Family of Integral Operators

Khalida Inayat Noor

Title Page

Contents



Go Back

Close

Quit

Page 9 of 14

The first two conditions of Lemma 1.1 are clearly satisfied. We verify the condition (iii) as follows.

$$\operatorname{Re} \Psi(iu_2, v_1) = \alpha + \frac{(1 - \alpha)(\alpha + \lambda + \mu)v_1}{(\alpha + \lambda + \mu)^2 + (1 - \alpha)^2u_2^2}.$$

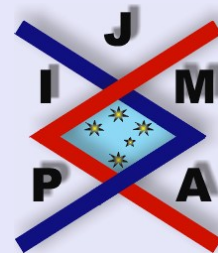
By putting  $v_1 \leq -\frac{(1+u_2^2)}{2}$ , we obtain

$$\begin{aligned} & \operatorname{Re} \Psi(iu_2, v_1) \\ & \leq \alpha - \frac{1}{2} \frac{(1 - \alpha)(\alpha + \lambda + \mu)(1 + u_2^2)}{(\alpha + \lambda + \mu)^2 + (1 - \alpha)^2u_2^2} \\ & = \frac{2\alpha(\alpha + \lambda + \mu)^2 + 2\alpha(1 - \alpha)^2u_2^2 - (1 - \alpha)(\alpha + \lambda + \mu) - (1 - \alpha)(\alpha + \lambda + \mu)u_2^2}{2[(\alpha + \lambda + \mu)^2 + (1 - \alpha)^2u_2^2]} \\ & = \frac{A + Bu_2^2}{2C}, \end{aligned}$$

where

$$\begin{aligned} A &= 2\alpha(\alpha + \lambda + \mu)^2 - (1 - \alpha)(\alpha + \lambda + \mu), \\ B &= 2\alpha(1 - \alpha)^2 - (1 - \alpha)(\alpha + \lambda + \mu), \\ C &= (\alpha + \lambda + \mu)^2 + (1 - \alpha)^2u_2^2 > 0. \end{aligned}$$

We note that  $\operatorname{Re} \Psi(iu_2, v_1) \leq 0$  if and only if,  $A \leq 0$  and  $B \leq 0$ . From  $A \leq 0$ , we obtain  $\alpha$  as given by (2.1) and  $B \leq 0$  gives us  $0 \leq \alpha < 1$ , and this completes the proof.  $\square$



On Analytic Functions Related to Certain Family of Integral Operators

Khalida Inayat Noor

Title Page

Contents

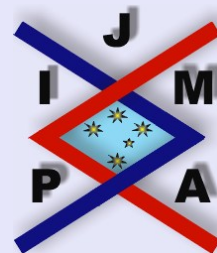


Go Back

Close

Quit

Page 10 of 14



Title Page

Contents



Go Back

Close

Quit

Page 11 of 14

**Theorem 2.2.** For  $\lambda > -1, \mu > 0$  and  $(\lambda + \mu) > 0$ ,  $V_k(\lambda, \mu, 0) \subset V_k(\lambda, \mu + 1, \alpha)$ , where  $\alpha$  is given by (2.1).

*Proof.* Let  $f \in V_k(\lambda, \mu, 0)$ . Then  $L_\lambda^\mu f \in V_k(0) = V_k$  and, by (1.7)  $z(L_\lambda^\mu)' \in R_k(0) = R_k$ . This implies

$$L_\lambda^\mu(zf') \in R_k \implies zf' \in R_k(\lambda, \mu, 0) \subset R_k(\lambda, \mu + 1, \alpha).$$

Consequently  $f \in V_k(\lambda, \mu + 1, \alpha)$ , where  $\alpha$  is given by (2.1). □

**Theorem 2.3.** Let  $\lambda > -1, \mu > 0$  and  $(\lambda + \mu) > 0$ . Then

$$T_k(\lambda, \mu, \beta, 0) \subset T_k(\lambda, \mu + 1, \gamma, \alpha),$$

where  $\alpha$  is given by (2.1) and  $\gamma \leq \beta$  is defined in the proof.

*Proof.* Let  $f \in T_k(\lambda, \mu, 0)$ . Then there exists  $g \in R_2(\lambda, \mu, 0)$  such that  $\left\{ \frac{z(L_\lambda^\mu f)'}{L_\lambda^\mu g} \right\} \in P_k(\beta)$ , for  $z \in E, 0 \leq \beta < 1$ . Let

$$\begin{aligned} \frac{z(L_\lambda^{\mu+1} f(z))'}{L_\lambda^{\mu+1} g(z)} &= (1 - \gamma)p(z) + \gamma \\ &= \left( \frac{k}{4} + \frac{1}{2} \right) \{ (1 - \gamma)p_1(z) + \gamma \} - \left( \frac{k}{4} - \frac{1}{2} \right) \{ (1 - \gamma)p_2(z) + \gamma \}, \end{aligned}$$

where  $p(0) = 1$ , and  $p(z)$  is analytic in  $E$ .

Making use of (2.2) and Theorem 2.1 with  $k = 2$ , we have

$$(2.3) \quad \left( \frac{z(L_\lambda^\mu f(z))'}{L_\lambda^\mu g(z)} - \beta \right) = \left\{ (1 - \gamma)p(z) + (\gamma - \beta) + \frac{(1 - \gamma)zp'(z)}{(1 - \alpha)q(z) + \alpha + \lambda + \mu} \right\} \in P_k,$$

and  $q \in P$ , where

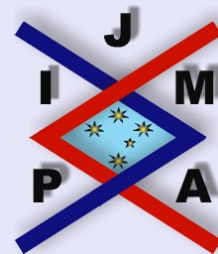
$$(1 - \alpha)q(z) + \alpha = \frac{z(L_\lambda^{\mu+1}g(z))'}{L_\lambda^{\mu+1}g(z)}, \quad z \in E.$$

Using (1.6), we form the functional  $\Phi(u, v)$  by taking  $u = u_1 + iv_2 = p_i(z)$ ,  $v = v_1 + iv_2 = zp'_i$  in (2.3) as

$$(2.4) \quad \Phi(u, v) = (1 - \gamma)u + (\gamma - \beta) + \frac{(1 - \gamma)v}{(1 - \alpha)q(z) + \alpha + \lambda + \mu}.$$

It can be easily seen that the function  $\Phi(u, v)$  defined by (2.4) satisfies the conditions (i) and (ii) of Lemma 1.1. To verify the condition (iii), we proceed, with  $q(z) = q_1 + iq_2$ , as follows:

$$\begin{aligned} & \operatorname{Re} [\Phi(iu_2, v_1)] \\ &= (\gamma - \beta) + \operatorname{Re} \left\{ \frac{(1 - \gamma)v_1}{(1 - \alpha)(q_1 + iq_2) + \alpha + \lambda + \mu} \right\} \\ &= (\gamma - \beta) + \frac{(1 - \gamma)(1 - \alpha)v_1q_1 + (1 - \gamma)(\alpha + \lambda + \mu)v_1}{[(1 - \alpha)q_1 + \alpha + \lambda + \mu]^2 + (1 - \alpha)^2q_2^2} \end{aligned}$$



On Analytic Functions Related to Certain Family of Integral Operators

Khalida Inayat Noor

Title Page

Contents

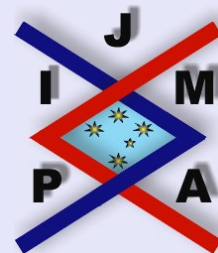


Go Back

Close

Quit

Page 12 of 14



$$\leq (\gamma - \beta) - \frac{1}{2} \frac{(1 - \gamma)(1 - \alpha)(1 + u_2^2)q_1 + (1 - \gamma)(\alpha + \lambda + \mu)(1 + u_2^2)}{[(1 - \alpha)q_1 + \alpha + \lambda + \mu]^2 + (1 - \alpha)^2 q_2^2}$$
$$\leq 0, \quad \text{for } \gamma \leq \beta < 1.$$

Therefore, applying Lemma 1.1,  $p_i \in P$ ,  $i = 1, 2$  and consequently  $p \in P_k$  and thus  $f \in T_k(\lambda, \mu + 1, \gamma, \alpha)$ .  $\square$

Using the same technique and relation (1.8) with Theorem 2.3, we have the following.

**Theorem 2.4.** For  $\lambda > -1$ ,  $\mu > 0$ ,  $\lambda + \mu > 0$ ,  $T_k^*(\lambda, \mu, \beta, 0) \subset T_k^*(\lambda, \mu + 1, \gamma, \alpha)$ , where  $\gamma$  and  $\alpha$  are as given in Theorem 2.3.

**Remark 1.** For different choices of  $k$ ,  $\lambda$  and  $\mu$ , we obtain several interesting special cases of the results proved in this paper.

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On Analytic Functions Related  
to Certain Family of Integral  
Operators

Khalida Inayat Noor

---

Title Page

Contents



Go Back

Close

Quit

Page 13 of 14

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---

On Analytic Functions Related  
to Certain Family of Integral  
Operators

Khalida Inayat Noor

---

Title Page

Contents



Go Back

Close

Quit

Page 14 of 14