# ON A CERTAIN SUBCLASS OF STARLIKE FUNCTIONS WITH NEGATIVE COEFFICIENTS

A. Y. LASHIN
Department of Mathematics
Faculty of Science
Mansoura University
Mansoura, 35516, EGYPT.
EMail: aylashin@yahoo.com

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*Abstract:* We introduce the class $\overline{H}(\alpha,\beta)$ of analytic functions with negative coefficients. In this paper we give some properties of functions in the class $\overline{H}(\alpha,\beta)$ and we obtain coefficient estimates, neighborhood and integral means inequalities for the function $f(z)$ belonging to the class $\overline{H}(\alpha,\beta)$. We also establish some results concerning the partial sums for the function $f(z)$ belonging to the class $\overline{H}(\alpha,\beta)$.

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# 1. Introduction

Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$. And let $S$ denote the subclass of $A$ consisting of univalent functions $f(z)$ in $U$.

A function $f(z)$ in $S$ is said to be starlike of order $\alpha$ if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in U),$$

for some $\alpha$ $(0 \leq \alpha < 1)$. We denote by $S^*(\alpha)$ the class of all functions in $S$ which are starlike of order $\alpha$. It is well-known that

$$S^*(\alpha) \subseteq S^*(0) \equiv S^*.$$

Further, a function $f(z)$ in $S$ is said to be convex of order $\alpha$ in $U$ if and only if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (z \in U),$$

for some $\alpha$ $(0 \leq \alpha < 1)$. We denote by $K(\alpha)$ the class of all functions in $S$ which are convex of order $\alpha$.

The classes $S^*(\alpha)$, and $K(\alpha)$ were first introduced by Robertson [8], and later were studied by Schild [10], MacGregor [4], and Pinchuk [7].

Let $T$ denote the subclass of $S$ whose elements can be expressed in the form:

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \qquad (a_k \geq 0). \tag{1.2}$$


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We denote by $T^*(\alpha)$ and $C(\alpha)$, respectively, the classes obtained by taking the intersections of $S^*(\alpha)$ and $K(\alpha)$ with $T$,

$$T^*(\alpha) = S^*(\alpha) \cap T \quad \text{and} \quad C(\alpha) = K(\alpha) \cap T.$$

The classes $T^*(\alpha)$ and $C(\alpha)$ were introduced by Silverman [11].

Let $H(\alpha, \beta)$ denote the class of functions $f(z) \in A$ which satisfy the condition

$$\operatorname{Re}\left(\frac{\alpha z^2 f''(z)}{f(z)} + \frac{z f'(z)}{f(z)}\right) > \beta$$

for some $\alpha \geq 0$, $0 \leq \beta < 1$, $\frac{f(z)}{z} \neq 0$ and $z \in U$.

The classes $H(\alpha, \beta)$ and $H(\alpha, 0)$ were introduced and studied by Obraddovic and Joshi [5], Padmanabhan [6], Li and Owa [2], Xu and Yang [14], Singh and Gupta [13], and others.

Further, we denote by $\overline{H}(\alpha, \beta)$ the class obtained by taking intersections of the class $H(\alpha, \beta)$ with $T$, that is

$$\overline{H}(\alpha, \beta) = H(\alpha, \beta) \cap T.$$

We note that

$$\overline{H}(0, \beta) = T^*(\beta) \qquad \text{(Silverman [11]).}$$


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## 2. Coefficient Estimates

**Theorem 2.1.** *A function* $f(z) \in T$ *is in the class* $\overline{H}(\alpha, \beta)$ *if and only if*

$$\sum_{k=2}^{\infty} [(k-1)(\alpha k+1) + (1-\beta)] a_k \leq 1-\beta. \tag{2.1}$$

*The result is sharp.*

*Proof.* Assume that the inequality (2.1) holds and let $|z| < 1$. Then we have

$$\left| \frac{\alpha z^2 f''(z)}{f(z)} + \frac{z f'(z)}{f(z)} - 1 \right| = \left| \frac{-\sum_{k=2}^{\infty}(k-1)(\alpha k+1) a_k z^{k-1}}{1-\sum_{k=2}^{\infty} a_k z^{k-1}} \right|$$
$$\leq \frac{\sum_{k=2}^{\infty}(k-1)(\alpha k+1) a_k}{1-\sum_{k=2}^{\infty} a_k} \leq 1-\beta.$$

This shows that the values of $\frac{\alpha z^2 f''(z)}{f(z)} + \frac{z f'(z)}{f(z)}$ lie in the circle centered at $w = 1$ whose radius is $1-\beta$. Hence $f(z)$ is in the class $\overline{H}(\alpha, \beta)$.

To prove the converse, assume that $f(z)$ defined by (1.2) is in the class $\overline{H}(\alpha, \beta)$. Then

$$\operatorname{Re}\left( \frac{\alpha z^2 f''(z)}{f(z)} + \frac{z f'(z)}{f(z)} \right) = \operatorname{Re}\left( \frac{1-\sum_{k=2}^{\infty}[\alpha k(k-1)+k] a_k z^{k-1}}{1-\sum_{k=2}^{\infty} a_k z^{k-1}} \right) > \beta \tag{2.2}$$

for $z \in U$. Choose values of $z$ on the real axis so that $\frac{\alpha z^2 f''(z)}{f(z)} + \frac{z f'(z)}{f(z)}$ is real. Upon


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clearing the denominator in (2.2) and letting $z \to 1^-$ through real values, we have

$$\beta\left(1-\sum_{k=2}^{\infty} a_k\right) \le 1-\sum_{k=2}^{\infty}[\alpha k(k-1)+k]a_k,$$

which obviously is the required result (2.1).

Finally, we note that the assertion (2.1) of Theorem 2.1 is sharp, with the extremal function being

$$f(z)=z-\frac{1-\beta}{[(k-1)(\alpha k+1)+(1-\beta)]}z^k \qquad (k\ge 2). \tag{2.3}$$

□

**Corollary 2.2.** *Let $f(z)\in T$ be in the class $\overline{H}(\alpha,\beta)$. Then we have*

$$a_k \le \frac{1-\beta}{[(k-1)(\alpha k+1)+(1-\beta)]} \qquad (k\ge 2). \tag{2.4}$$

*Equality in (2.4) holds true for the function $f(z)$ given by (2.3).*


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## 3. Some Properties of the Class $\overline{H}(\alpha, \beta)$

**Theorem 3.1.** *Let* $0 \le \alpha_1 < \alpha_2$ *and* $0 \le \beta < 1$. *Then* $\overline{H}(\alpha_2, \beta) \subset \overline{H}(\alpha_1, \beta)$.

*Proof.* It follows from Theorem 2.1. That

$$\sum_{k=2}^{\infty}[(k-1)(\alpha_1 k+1)+(1-\beta)]a_k < \sum_{k=2}^{\infty}[(k-1)(\alpha_2 k+1)+(1-\beta)]a_k \le 1-\beta$$

for $f(z) \in \overline{H}(\alpha_2, \beta)$. Hence $f(z) \in \overline{H}(\alpha_1, \beta)$. □

**Corollary 3.2.** $\overline{H}(\alpha, \beta) \subseteq T^*(\beta)$.

The proof is now immediate because $\alpha \ge 0$.


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## 4. Neighborhood Results

Following the earlier investigations of Goodman [1] and Ruscheweyh [9], we define the $\delta-$ neighborhood of function $f(z) \in T$ by:

$$N_\delta(f) = \left\{ g \in T : g(z) = z - \sum_{k=2}^{\infty} b_k z^k, \; \sum_{k=2}^{\infty} k\,|a_k - b_k| \le \delta \right\}.$$

In particular, for the identity function

$$e(z) = z,$$

we immediately have

$$N_\delta(e) = \left\{ g \in T : g(z) = z - \sum_{k=2}^{\infty} b_k z^k, \; \sum_{k=2}^{\infty} k\,|b_k| \le \delta \right\}. \tag{4.1}$$

**Theorem 4.1.** $\overline{H}(\alpha, \beta) \subseteq N_\delta(e)$, *where* $\delta = \frac{2(1-\beta)}{(2\alpha+2-\beta)}$.

*Proof.* Let $f(z) \in \overline{H}(\alpha, \beta)$. Then, in view of Theorem 2.1, since $[(k-1)(\alpha k + 1) + (1-\beta)]$ is an increasing function of $k$ $(k \ge 2)$, we have

$$(2\alpha + 2 - \beta) \sum_{k=2}^{\infty} a_k \le \sum_{k=2}^{\infty} [(k-1)(\alpha k + 1) + (1-\beta)] a_k \le 1 - \beta,$$

which immediately yields

$$\sum_{k=2}^{\infty} a_k \le \frac{1-\beta}{(2\alpha + 2 - \beta)}. \tag{4.2}$$


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On the other hand, we also find from (2.1)

$$(\alpha+1)\sum_{k=2}^{\infty} k a_k - \beta \sum_{k=2}^{\infty} a_k \le \sum_{k=2}^{\infty} [(\alpha(k-1)+1)k-\beta)] a_k$$

$$= \sum_{k=2}^{\infty} [(k-1)(\alpha k+1)+(1-\beta)] a_k \le 1-\beta. \tag{4.3}$$

From (4.3) and (4.2), we have

$$\begin{aligned}(\alpha+1)\sum_{k=2}^{\infty} k a_k &\le (1-\beta)+\beta\sum_{k=2}^{\infty} a_k \\ &\le (1-\beta)+\beta\frac{1-\beta}{(2\alpha+2-\beta)} \\ &\le \frac{2(\alpha+1)(1-\beta)}{(2\alpha+2-\beta)},\end{aligned}$$

that is,

$$\sum_{k=2}^{\infty} k a_k \le \frac{2\,(1-\beta)}{(2\alpha+2-\beta)} = \delta, \tag{4.4}$$

which in view of the definition (4.1), prove Theorem 4.1. □

Letting $\alpha = 0$, in the above theorem, we have:

**Corollary 4.2.** *$T^*(\beta) \subseteq N_\delta(e)$, where $\delta = \frac{2(1-\beta)}{(2-\beta)}$.*



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## 5. Integral Means Inequalities

We need the following lemma.

**Lemma 5.1 ([3]).** *If $f$ and $g$ are analytic in $U$ with $f \prec g$, then*

$$\int_0^{2\pi} \left|g(re^{i\theta})\right|^\delta d\theta \leq \int_0^{2\pi} \left|f(re^{i\theta})\right|^\delta d\theta,$$

*where $\delta > 0$, $z = re^{i\theta}$ and $0 < r < 1$.*

Applying Lemma 5.1, and (2.1), we prove the following theorem.

**Theorem 5.2.** *Let $\delta > 0$. If $f(z) \in \overline{H}(\alpha, \beta)$, then for $z = re^{i\theta}$, $0 < r < 1$, we have*

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^\delta d\theta \leq \int_0^{2\pi} \left|f_2(re^{i\theta})\right|^\delta d\theta,$$

*where*

$$f_2(z) = z - \frac{(1-\beta)}{(2\alpha + 2 - \beta)} z^2. \tag{5.1}$$

*Proof.* Let $f(z)$ defined by (1.2) and $f_2(z)$ be given by (5.1). We must show that

$$\int_0^{2\pi} \left|1 - \sum_{k=2}^{\infty} a_k z^{k-1}\right|^\delta d\theta \leq \int_0^{2\pi} \left|1 - \frac{(1-\beta)}{(2\alpha+2-\beta)} z\right|^\delta d\theta.$$

By Lemma 5.1, it suffices to show that

$$1 - \sum_{k=2}^{\infty} a_k z^{k-1} \prec 1 - \frac{(1-\beta)}{(2\alpha+2-\beta)} z.$$

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Setting

$$1-\sum_{k=2}^{\infty}a_k z^{k-1}=1-\frac{(1-\beta)}{(2\alpha+2-\beta)}w(z). \tag{5.2}$$

From (5.2) and (2.1), we obtain

$$\begin{aligned}|w(z)| &= \left|\sum_{k=2}^{\infty}\frac{(2\alpha+2-\beta)}{(1-\beta)}a_k z^{k-1}\right| \\ &\leq |z|\sum_{k=2}^{\infty}\frac{[(k-1)(\alpha k+1)+(1-\beta)]}{1-\beta}a_k \leq |z|\,.\end{aligned}$$

This completes the proof of the theorem. □

Letting $\alpha=0$ in the above theorem, we have:

**Corollary 5.3.** *Let* $\delta>0$. *If* $f(z)\in T^*(\beta)$, *then for* $z=re^{i\theta}$, $0<r<1$, *we have*

$$\int_0^{2\pi}\left|f(re^{i\theta})\right|^{\delta}d\theta \leq \int_0^{2\pi}\left|f_2(re^{i\theta})\right|^{\delta}d\theta,$$

*where*

$$f_2(z)=z-\frac{(1-\beta)}{(2-\beta)}z^2.$$

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## 6. Partial Sums

In this section we will examine the ratio of a function of the form (1.2) to its sequence of partial sums defined by $f_1(z) = z$ and $f_n(z) = z - \sum_{k=2}^{n} a_k z^k$ when the coefficients of $f$ are sufficiently small to satisfy the condition (2.1). We will determine sharp lower bounds for $\operatorname{Re}\left(\frac{f(z)}{f_n(z)}\right)$, $\operatorname{Re}\left(\frac{f_n(z)}{f(z)}\right)$, $\operatorname{Re}\left(\frac{f'(z)}{f_n'(z)}\right)$ and $\operatorname{Re}\left(\frac{f_n'(z)}{f'(z)}\right)$.

In what follows, we will use the well known result that

$$\operatorname{Re}\frac{1-w(z)}{1+w(z)} > 0, \qquad z \in U,$$

if and only if

$$w(z) = \sum_{k=1}^{\infty} c_k z^k$$

satisfies the inequality $|w(z)| \leq |z|$.

**Theorem 6.1.** *If $f(z) \in \overline{H}(\alpha, \beta)$, then*

$$\operatorname{Re}\frac{f(z)}{f_n(z)} \geq 1 - \frac{1}{c_{n+1}} \qquad (z \in U,\ n \in N) \tag{6.1}$$

*and*

$$\operatorname{Re}\left(\frac{f_n(z)}{f(z)}\right) \geq \frac{c_{n+1}}{1 + c_{n+1}} \qquad (z \in U,\ n \in N), \tag{6.2}$$

*where* $\left(c_k =: \frac{[(k-1)(\alpha k+1)+(1-\beta)]}{1-\beta}\right)$. *The estimates in (6.1) and (6.2) are sharp.*

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*Proof.* We employ the same technique used by Silverman [12]. The function $f(z) \in \overline{H}(\alpha, \beta)$, if and only if $\sum_{k=2}^{\infty} c_k a_k \leq 1$. It is easy to verify that $c_{k+1} > c_k > 1$. Thus,

$$\sum_{k=2}^{n} a_k + c_{n+1} \sum_{k=n+1}^{\infty} a_k \leq \sum_{k=2}^{\infty} c_k a_k \leq 1. \tag{6.3}$$

We may write

$$c_{n+1}\left\{\frac{f(z)}{f_n(z)} - \left(1 - \frac{1}{c_{n+1}}\right)\right\} = \frac{1 - \sum_{k=2}^{n} a_k z^{k-1} - c_{n+1} \sum_{k=n+1}^{\infty} a_k z^{k-1}}{1 - \sum_{k=2}^{n} a_k z^{k-1}}$$
$$= \frac{1 + D(z)}{1 + E(z)}.$$

Set

$$\frac{1 + D(z)}{1 + E(z)} = \frac{1 - w(z)}{1 + w(z)},$$

so that

$$w(z) = \frac{E(z) - D(z)}{2 + D(z) + E(z)}.$$

Then

$$w(z) = \frac{c_{n+1} \sum_{k=n+1}^{\infty} a_k z^{k-1}}{2 - 2\sum_{k=2}^{n} a_k z^{k-1} - c_{n+1} \sum_{k=n+1}^{\infty} a_k z^{k-1}}$$

and

$$|w(z)| \leq \frac{c_{n+1} \sum_{k=n+1}^{\infty} a_k}{2 - 2\sum_{k=2}^{n} a_k - c_{n+1} \sum_{k=n+1}^{\infty} a_k}.$$

Now $|w(z)| \leq 1$ if and only if

$$\sum_{k=2}^{n} a_k + c_{n+1} \sum_{k=n+1}^{\infty} a_k \leq 1,$$


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which is true by (6.3). This readily yields the assertion (6.1) of Theorem 6.1.

To see that

$$f(z) = z - \frac{z^{n+1}}{c_{n+1}} \tag{6.4}$$

gives sharp results, we observe that

$$\frac{f(z)}{f_n(z)} = 1 - \frac{z^n}{c_{n+1}}.$$

Letting $z \to 1^-$, we have

$$\frac{f(z)}{f_n(z)} = 1 - \frac{1}{c_{n+1}},$$

which shows that the bounds in (6.1) are the best possible for each $n \in N$. Similarly, we take

$$(1 + c_{n+1}) \left( \frac{f_n(z)}{f(z)} - \frac{c_{n+1}}{1 + c_{n+1}} \right) = \frac{1 - \sum_{k=2}^{n} a_k z^{k-1} + c_{n+1} \sum_{k=n+1}^{\infty} a_k z^{k-1}}{1 - \sum_{k=2}^{\infty} a_k z^{k-1}}$$

$$:= \frac{1 - w(z)}{1 + w(z)},$$

where

$$|w(z)| \le \frac{(1 + c_{n+1}) \sum_{k=n+1}^{\infty} a_k}{2 - 2\sum_{k=2}^{n} a_k + (1 - c_{n+1}) \sum_{k=n+1}^{\infty} a_k}.$$

Now $|w(z)| \le 1$ if and only if

$$\sum_{k=2}^{n} a_k + c_{n+1} \sum_{k=n+1}^{\infty} a_k \le 1,$$



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which is true by (6.3). This immediately leads to the assertion (6.2) of Theorem 6.1.

The estimate in (6.2) is sharp with the extremal function $f(z)$ given by (6.4). This completes the proof of Theorem 6.1. □

Letting $\alpha = 0$ in the above theorem, we have:

**Corollary 6.2.** *If $f(z) \in T^*(\beta)$, then*

$$\operatorname{Re} \frac{f(z)}{f_n(z)} \geq \frac{n}{(n+1-\beta)}, \qquad (z \in U)$$

*and*

$$\operatorname{Re} \frac{f_n(z)}{f(z)} \geq \frac{n+1-\beta}{(n+2-2\beta)}, \qquad (z \in U).$$

*The result is sharp for every $n$, with the extremal function*

$$f(z) = z - \frac{1-\beta}{(n+1-\beta)} z^{n+1}. \tag{6.5}$$

We now turn to the ratios involving derivatives. The proof of Theorem 6.3 below follows the pattern of that in Theorem 6.1, and so the details may be omitted.

**Theorem 6.3.** *If $f(z) \in \overline{H}(\alpha, \beta)$, then*

$$\operatorname{Re} \frac{f'(z)}{f'_n(z)} \geq 1 - \frac{n+1}{c_{n+1}} \qquad (z \in U), \tag{6.6}$$

*and*

$$\operatorname{Re} \left( \frac{f'_n(z)}{f'(z)} \right) \geq \frac{c_{n+1}}{n+1+c_{n+1}} \qquad (z \in U,\ n \in N). \tag{6.7}$$

*The estimates in (6.6) and (6.7) are sharp with the extremal function given by (6.4).*


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Letting $\alpha = 0$ in the above theorem, we have:

**Corollary 6.4.** *If $f(z) \in T^*(\beta)$, then*

$$\operatorname{Re} \frac{f'(z)}{f_n'(z)} \geq \frac{\beta n}{(n+1-\beta)}, \qquad (z \in U),$$

*and*

$$\operatorname{Re} \frac{f_n'(z)}{f'(z)} \geq \frac{n+1-\beta}{n+(1-\beta)(n+2)}, \qquad (z \in U).$$

*The result is sharp for every $n$, with the extremal function given by (6.5).*



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