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ON HADAMARD'S INEQUALITY FOR THE CONVEX MAPPINGS DEFINED ON A CONVEX DOMAIN IN THE SPACE

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ABSTRACT. In this paper we obtain some Hadamard type inequalities for triple integrals. The results generalize those obtained in (S.S. DRAGOMIR, On Hadamard's inequality for the convex mappings defined on a ball in the space and applications, *RGMIA* (preprint), 1999).

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1. INTRODUCTION

Let $f : [a, b] \to \mathbb{R}$ be a convex mapping defined on the interval $[a, b]$. The following double inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2} \tag{1.1}$$

is known in the literature as Hadamard's inequality for convex mappings.

In [1] S.S. Dragomir considered the following mapping naturally connected to Hadamard's inequality

$$H : [0,1] \to \mathbb{R}, \quad H(t) = \frac{1}{b-a}\int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right)dx$$

and proved the following properties of this function

(i) H is convex and monotonic nondecreasing.
(ii) H has the bounds

$$\sup_{t\in[0,1]} H(t) = H(1) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

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and

$$\inf_{t\in[0,1]} H(t) = H(0) = f\left(\frac{a+b}{2}\right).$$

In the recent paper [2], S.S. Dragomir gave some inequalities of Hadamard's type for convex functions defined on the ball $\overline{B}(C,R)$, where

$$C = (a,b,c) \in \mathbb{R}^3, \quad R > 0$$

and

$$\overline{B}(C,R) := \{(x,y,z) \in \mathbb{R}^3 | \ (x-a)r + (y-b)^2 + (z-c)^2 \le R^2\}$$

More precisely he proved the following theorem.

Theorem 1.1. *Let $f : \overline{B}(C,R) \to \mathbb{R}$ be a convex mapping on the ball $\overline{B}(C,R)$. Then we have the inequality*

$$f(a,b,c) \le \frac{1}{v(\overline{B}(C,R))} \iiint_{\overline{B}(C,R)} f(x,y,z)\,dxdydz \le \frac{1}{\sigma(\overline{B}(C,R))} \iint_{S(C,R)} f(x,y,z)\,d\sigma \tag{1.2}$$

where

$$S(C,R) := \{(x,y,z) \in \mathbb{R}^2 | \ (x-a)^2 + (y-b)^2 + (z-c)^2 = R^2\}$$

and

$$v(\overline{B}(C,R)) = \frac{4\pi R^3}{3}, \qquad \sigma(\overline{B}(C,R)) = 4\pi R^2.$$

In [2] S.S. Dragomir considers, for a convex mapping f defined on the ball $\overline{B}(C,R)$, the mapping $H : [0,1] \to \mathbb{R}$ given by

$$H(t) = \frac{1}{v(\overline{B}(C,R))} \iiint_{\overline{B}(C,R)} f(t(x,y,z) + (1-t)C)\,dxdydz.$$

The main properties of this mapping are contained in the following theorem.

Theorem 1.2. *With the above assumption, we have*

(i) The mapping H is convex on $[0,1]$.
(ii) H has the bounds

$$\inf_{t\in[0,1]} H(t) = H(0) = f(C) \tag{1.3}$$

and

$$\sup_{t\in[0,1]} H(t) = H(1) = \frac{1}{v(\overline{B}(C,R))} \iiint_{\overline{B}(C,R)} f(x,y,z)\,dxdydz. \tag{1.4}$$

(iii) The mapping H is monotonic nondecreasing on $[0,1]$.

In this paper we shall give a generalization of the Theorem 1.2 for a positive linear functional defined on $C(D)$, where $D \subset \mathbb{R}^m$ $(m \in \mathbb{N}^*)$ is a convex domain. We shall give also a generalization of the Theorem 1.1.

2. RESULTS

Let $D \subset \mathbb{R}^m$ be a convex domain and $A : C(D) \to \mathbb{R}$ be a given positive linear functional such that $A(e_0) = 1$, where $e_0(x) = 1,\ x \in D$. Let $x = (x_1, \dots, x_m)$ be a point from D we note by $p_i,\ i = 1, 2, \dots, m$ the function defined on D by

$$p_i(x) = x_i, \qquad i = 1, 2, \dots, m$$

and by $a_i,\ i = 1, 2, \dots, m$ the value of the functional A in p_i, i.e.

$$A(p_i) = a_i, \qquad i = 1, 2, \dots, m.$$

In addition, let f be a convex mapping on D. We consider the mapping $H : [0, 1] \to \mathbb{R}$ associated with the function f and given by

$$H(t) = A(f(tx + (1 - t)a))$$

where $a = (a_1, a_2, \dots, a_m)$ and the functional A acts analagous to the variable x.

Theorem 2.1. *With above assumption, we have*

(i) The mapping H is convex on $[0, 1]$.

(ii) The bounds of the function H are given by

$$\inf_{t \in [0,1]} H(t) = H(0) = f(a) \tag{2.1}$$

and

$$\sup_{t \in [0,1]} H(t) = H(1) = A(f). \tag{2.2}$$

(iii) The mapping H is monotonic nondecreasing on $[0, 1]$.

Proof. (i) Let $t_1, t_2 \in [0, 1]$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. Then we have

$$\begin{aligned} H(\alpha t_1 + \beta t_2) &= A[f((\alpha t_1 + \beta t_2)x + (1 - (\alpha t_1 + \beta t_2))a)] \\ &= A[f(\alpha(t_1 x + (1 - t_1)a) + \beta(t_2 x + (1 - t_2)a))] \\ &\leq \alpha A[f(t_1 x + (1 - t_1)a)] + BA[f(t_2 x + (1 - t_2)a)] \\ &= \alpha H(t_1) + \beta H(t_2) \end{aligned}$$

which proves the convexity of H on $[0, 1]$.

(ii) Let g be a convex function on D. Then there exist the real numbers $A_1, A_2, \dots, A_m$ such that

$$g(x) \geq g(a) + (x_1 - a_1)A_1 + (x_2 - a_2)A_2 + \cdots + (x_m - a_m)A_m \tag{2.3}$$

for any $x = (x_1, \dots, x_m) \in D$.

Using the fact that the functional A is linear and positive, from the inequality (2.3) we obtain the inequality

$$A(g) \geq g(a). \tag{2.4}$$

Now, for a fixed number $t,\ t \in [0, 1]$ the function $g : D \to \mathbb{R}$ defined by

$$g(x) = f(tx + (1 - t)a)$$

is a convex function. From the inequality (2.4) we obtain

$$A(f(tx + (1 - t)a)) \geq f(ta + (1 - t)a) = f(a)$$

or

$$H(t) \geq H(0)$$

for every $t \in [0, 1]$, which proves the equality (2.1).

Let $0 \le t_1 < t_2 \le 1$. By the convexity of the mapping H we have

$$\frac{H(t_2) - H(t_1)}{t_2 - t_1} \ge \frac{H(t_1) - H(0)}{t_1} \ge 0.$$

So the function H is a nondecreasing function and $H(t) \le H(1)$. The theorem is proved. □

Remark 2.1. For $m = 1,\ D = [a, b]$ and

$$A(f) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

the function H is the function which was considered in the paper [1].

Remark 2.2. For $m = 3$ and $D = \overline{B}(C, R)$ and

$$A(f) = \frac{1}{v(\overline{B}(C,R))}\iiint_{\overline{B}(C,R)} f(x,y,z)\,dxdydz$$

a being C, the function H is the functional from the Theorem 1.2.

Let D be a bounded convex domain from $\mathbb{R}^3$ with a piecewise smooth boundary S. We define the notation

$$\begin{aligned}
\sigma &:= \iint_S dS,\\
a_1 &:= \frac{1}{\sigma}\iint_S x\,dS,\\
a_2 &:= \frac{1}{\sigma}\iint_S y\,dS,\\
a_3 &:= \iint_S z\,dS,\\
v &:= \iiint_V f(x,y,z)\,dxdydz.
\end{aligned}$$

Let us assume that the surface S is oriented with the aid of the unit normal h directed to the exterior of D

$$h = (\cos\alpha, \cos\beta, \cos\gamma).$$

The following theorem is a generalization of the Theorem 1.1.

Theorem 2.2. *Let f be a convex function on D. With the above assumption we have the following inequalities*

$$v\iint_S f ds - \sigma\iint_S [(a_1 - x)\cos\alpha + (a_2 - y)\cos\beta + (a_3 - z)\cos\gamma] f(x,y,z) dS \ge 4\sigma\iiint_D f(x,y,z)\,dxdydz \tag{2.5}$$

and

$$\iiint_D f(x,y,z)\,dxdydz \ge f(x_\sigma, y_\sigma, z_\sigma)v, \tag{2.6}$$

where

$$x_\sigma = \frac{1}{v}\iiint_D x\,dxdydz, \qquad y_\sigma = \frac{1}{v}\iiint_D y\,dxdydz, \qquad z_\sigma = \frac{1}{v}\iiint_D z\,dxdydz.$$

Proof. We can suppose that the function f has the partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ and these are continuous on D.

For every point $(u,v,w) \in S$ and $(x,y,z) \in D$ the following inequality holds:

$$f(u,v,w) \geq f(x,y,z) + \frac{\partial f}{\partial x}(x,y,z)(u-x) + \frac{\partial f}{\partial y}(x,y,z)(v-y) + \frac{\partial f}{\partial z}(x,y,z)(w-z). \tag{2.7}$$

From the inequality (2.7) we have

$$\iint_S f(x,y,z)\,dS \geq f(x,y,z)\sigma + \frac{\partial f}{\partial x}(x,y,z)(a_1 - x)\sigma + \frac{\partial f}{\partial y}(x,y,z)(a_2 - y)\sigma + \frac{\partial f}{\partial z}(x,y,z)(a_3 - z)\sigma. \tag{2.8}$$

The above inequality leads us to the inequality

$$\begin{aligned} v\iint_S f(x,y,z)\,dS &\geq \sigma \iiint_D f(x,y,z)\,dxdydz \\ &+\sigma \iiint_D \left[\frac{\partial}{\partial x}((a_1-x)f(x,y,z)) + \frac{\partial}{\partial y}((a_2-y)f(x,y,z)) + \frac{\partial}{\partial z}((a_3-z)f(x,y,z))\right] dxdydz \\ &+ 3\sigma \iiint_D f(x,y,z)\,dxdydz. \end{aligned} \tag{2.9}$$

Using the Gauss-Ostrogradsky' theorem we obtain the equality

$$\begin{aligned} \iiint_D \Bigg[\frac{\partial}{\partial x}((a_1 - x)f(x,y,z) &+ \frac{\partial}{\partial y}((a_2 - y)f(x,y,z)) \\ &+ \frac{\partial}{\partial z}((a_3 - z)f(z,y,z)\Bigg] dxdydz \\ &= \iint_S [(a_1 - x)\cos\alpha + (a_2 - y)\cos\beta + (a_3 - z)\cos\gamma] f(x,y,z)\,dS. \end{aligned} \tag{2.10}$$

From the relations (2.9) and (2.10) we obtain the inequality (2.4). The inequality (2.6) is the inequality (2.4) for the functional

$$A(f) = \frac{\iiint_D f(x,y,z)\,dxdydz}{\iiint_D dxdydz}.$$

□

Remark 2.3. For $D = \overline{B}(C,R)$ we have

$$(a_1, a_2, a_3) = C$$

and

$$\cos\alpha = \frac{x - a_1}{R}, \quad \cos\beta = \frac{y - a_2}{R}, \quad \cos\gamma = \frac{z - a_3}{R}.$$

In this case the inequality (2.4) becomes

$$\sigma \iiint_{\overline{B}(C,R)} f(x,y,z)\,dxdydz \leq v \iint_{S(C,R)} f(x,y,z)\,d\sigma.$$

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