Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 2, Issue 1, Article 4, 2001

SOME ASPECTS OF CONVEX FUNCTIONS AND THEIR APPLICATIONS

J. ROOIN

INSTITUTE FOR ADVANCED STUDIES IN BASIC SCIENCES, P.O. BOX 45195-159,
GAVA ZANG, ZANJAN 45195, IRAN.
AND
FACULTY OF MATHEMATICAL SCIENCES AND COMPUTER ENGINEERING, UNIVERSITY FOR TEACHER EDUCATION, 599 TALEGHANI AVENUE, TEHRAN 15614, IRAN.
Rooin@iasbs.ic.ir

Received 13 April, 2000; accepted 04 October 2000
Communicated by N.S. Barnett

ABSTRACT. In this paper we will study some aspects of convex functions and as applications prove some interesting inequalities.

Key words and phrases: Convex Functions, Means, Lp-Spaces, Fubini's Theorem.

2000 *Mathematics Subject Classification.* 26D15, 39B62, 43A15.

1. INTRODUCTION

In [2] Sever S. Dragomir and Nicoleta M. Ionescu have studied some aspects of convex functions and obtained some interesting inequalities. In this paper we generalize the above paper to a very general case by introducing a suitable convex function of a real variable from a given convex function. Studying its properties leads to some remarkable inequalities in different abstract spaces.

2. THE MAIN RESULTS

The aim of this section is to study the properties of the function F defined below as Theorems 2.2 and 2.6.

First we mention the following simple lemma, which describes the behavior of a convex function defined on a closed interval of the real line.

Lemma 2.1. *Let F be a convex function on the closed interval $[a, b]$. Then, we have*

(i) *F takes its maximum at a or b.*
(ii) *F is bounded from below.*
(iii) *$F(a+)$ and $F(b-)$ exist (and are finite).*

ISSN (electronic): 1443-5756

The author is supported in part by the Institute for Advanced Studies in Basic Sciences.
008-00

(iv) *If the infimum of F over $[a,b]$ is less than $F(a+)$ and $F(b-)$, then F takes its minimum at a point x_0 in (a,b).*

(v) *If $a \leq x_0 < b$ (or $a < x_0 \leq b$), and $F(x_0+)$ (or $(F(x_0-))$ is the infimum of F over $[a,b]$, then F is monotone decreasing on $[a,x_0]$ (or $[a,x_0)$) and monotone increasing on $(x_0,b]$ (or $[x_0,b]$).*

Proof. See [3, 4]. ∎

Definition 2.1. Let X be a linear space, and $f : C \subseteq X \to \mathbb{R}$ be a convex mapping on a convex subset C of X. For n given elements $x_1, x_2, \cdots, x_n$ of C, we define the following mapping of real variable $F : [0,1] \to \mathbb{R}$ by

$$F(t) = \frac{\sum_{i=1}^{n} f\left(\sum_{j=1}^{n} a_{ij}(t)x_j\right)}{n},$$

where $a_{ij} : [0,1] \to \mathbb{R}^+$ $(i,j = 1,\cdots,n)$ are affine mappings, i.e., $a_{ij}(\alpha t_1 + \beta t_2) = \alpha a_{ij}(t_1) + \beta a_{ij}(t_2)$ for all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and t_1, t_2 in $[0,1]$, and for each i and j

$$\sum_{i=1}^{n} a_{ij}(t) = 1,\ \sum_{j=1}^{n} a_{ij}(t) = 1\ (0 \leq t \leq 1).$$

The next theorem contains some remarkable properties of this mapping.

Theorem 2.2. *With the above assumptions, we have:*

(i) $f\left(\frac{x_1 + \cdots + x_n}{n}\right) \leq F(t) \leq \frac{f(x_1) + \cdots + f(x_n)}{n} \quad (0 \leq t \leq 1).$

(ii) *F is convex on $[0,1]$.*

(iii) $f\left(\frac{x_1 + \cdots + x_n}{n}\right) \leq \int_0^1 F(t)dt \leq \frac{f(x_1) + \cdots + f(x_n)}{n}.$

(iv) *Let $p_i \geq 0$ with $P_n = \sum_{i=1}^{n} p_i > 0$, and t_i are in $[0,1]$ for all $i = 1,2,\cdots,n$. Then, we have the inequality:*

$$f\left(\frac{x_1 + \cdots + x_n}{n}\right) \leq F\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i t_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i F(t_i) \leq \frac{f(x_1) + \cdots + f(x_n)}{n}, \tag{2.1}$$

which is a discrete version of Hadamard's result.

Proof. (i) By the convexity of f, for all $0 \leq t \leq 1$, we have

$$\begin{aligned} F(t) &\geq f\left(\frac{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(t)x_j}{n}\right) \\ &= f\left(\frac{\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}(t)x_j}{n}\right) \\ &= f\left(\frac{\sum_{j=1}^{n} x_j}{n}\right), \end{aligned}$$

and

$$\begin{aligned} F(t) &\le \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(t) f(x_j)}{n} \\ &= \frac{\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}(t) f(x_j)}{n} \\ &= \frac{\sum_{j=1}^{n} f(x_j)}{n}. \end{aligned}$$

(ii) Let $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$ and t_1, t_2 be in $[0,1]$. Then,

$$\begin{aligned} F(\alpha t_1 + \beta t_2) &= \frac{\sum_{i=1}^{n} f\left(\sum_{j=1}^{n} a_{ij}(\alpha t_1 + \beta t_2) x_j\right)}{n} \\ &= \frac{\sum_{i=1}^{n} f\left(\alpha \sum_{j=1}^{n} a_{ij}(t_1) x_j + \beta \sum_{j=1}^{n} a_{ij}(t_2) x_j\right)}{n} \\ &\le \alpha \frac{\sum_{i=1}^{n} f\left(\sum_{j=1}^{n} a_{ij}(t_1) x_j\right)}{n} + \beta \frac{\sum_{i=1}^{n} f\left(\sum_{j=1}^{n} a_{ij}(t_2) x_j\right)}{n} \\ &= \alpha F(t_1) + \beta F(t_2). \end{aligned}$$

Thus F is convex.

(iii) F being convex on $[0,1]$, is integrable on $[0,1]$, and by (i), we get (iii).

(iv) The first and last inequalities in (2.1) are obvious from (i), and the second inequality follows from Jensen's inequality applied for the convex function F.

∎

Lemma 2.3. *The general form of an affine mapping $g : [0,1] \to \mathbb{R}$ is*

$$g(t) = (1-t)k_0 + tk_1,$$

where k_0 and k_1 are two arbitrary real numbers.

The proof follows by considering $t = (1-t)\cdot 0 + t \cdot 1$.

Lemma 2.4. *If $a_{ij} : [0,1] \to \mathbb{R}^+$ $(i, j = 1, 2, \cdots, n)$ are affine mappings such that for each t, i and j, $\sum_{i=1}^{n} a_{ij}(t) = 1$ and $\sum_{j=1}^{n} a_{ij}(t) = 1$, then there exist nonnegative numbers b_{ij} and c_{ij}, such that*

$$a_{ij}(t) = (1-t)b_{ij} + tc_{ij} \ (0 \le t \le 1;\ i, j = 1, \cdots, n), \tag{2.2}$$

and for any i and j

$$\sum_{i=1}^{n} b_{ij} = \sum_{i=1}^{n} c_{ij} = 1, \text{ and } \sum_{j=1}^{n} b_{ij} = \sum_{j=1}^{n} c_{ij} = 1.$$

Proof. The decomposition of (2.2) is immediate from Lemma 2.3, and the rest of the proof comes from below:

$$\begin{aligned} 0 &\le a_{ij}(0) = b_{ij}, \quad 0 \le a_{ij}(1) = c_{ij}, \\ \sum_{i=1}^{n} b_{ij} &= \sum_{i=1}^{n} a_{ij}(0) = 1, \quad \sum_{i=1}^{n} c_{ij} = \sum_{i=1}^{n} a_{ij}(1) = 1, \\ \sum_{j=1}^{n} b_{ij} &= \sum_{j=1}^{n} a_{ij}(0) = 1, \quad \sum_{j=1}^{n} c_{ij} = \sum_{j=1}^{n} a_{ij}(1) = 1. \end{aligned}$$

∎

Remark 2.5. A lot of simplifications occur if we take

$$b_{ij} = \delta_{ij} \text{ and } c_{ij} = \delta_{i,n+1-j} \ (i, j = 1, \cdots, n), \tag{2.3}$$

in Lemma 2.4, where δ_{ij} is the Kronecker delta.

Theorem 2.6. *With the above assumptions, if b_{ij} and c_{ij} are in the form (2.3), then we have:*

(i) *For each t in $\left[0, \frac{1}{2}\right]$, $F\left(\frac{1}{2}+t\right) = F\left(\frac{1}{2}-t\right)$.*

(ii) $\max\{F(t) : 0 \le t \le 1\} = F(0) = F(1) = \frac{1}{n}\left(f(x_1) + \cdots + f(x_n)\right)$.

(iii) $\min\{F(t) : 0 \le t \le 1\} = F\left(\frac{1}{2}\right) = \sum_{i=1}^{n} f\left(\frac{x_i + x_{n+1-i}}{2}\right) \diagup n$.

(iv) *F is monotone decreasing on $\left[0, \frac{1}{2}\right]$ and monotone increasing on $\left[\frac{1}{2}, 1\right]$.*

Proof. (i) Since $b_{ij} = \delta_{ij}$ and $c_{ij} = \delta_{i,n+1-j}$, we have

$$F(t) = \frac{\sum_{i=1}^{n} f[(1-t)x_i + t x_{n+1-i}]}{n}, \tag{2.4}$$

and therefore, for each t in $\left[0, \frac{1}{2}\right]$,

$$\begin{aligned} F\left(\frac{1}{2} - t\right) &= \frac{\sum_{i=1}^{n} f\left[\left(\frac{1}{2}+t\right)x_i + \left(\frac{1}{2}-t\right)x_{n+1-i}\right]}{n} \\ &= \frac{\sum_{i=1}^{n} f\left[\left(\frac{1}{2}+t\right)x_{n+1-i} + \left(\frac{1}{2}-t\right)x_i\right]}{n} \\ &= F\left(\frac{1}{2}+t\right). \end{aligned}$$

(ii) It is obvious from (2.4), and *(i)* of Lemma 2.1.

(iii) If $F(\frac{1}{2})$ is not the minimum of F over $[0, 1]$, then by *(i)*, there is a $0 < t \le \frac{1}{2}$, such that

$$F\left(\frac{1}{2} - t\right) = F\left(\frac{1}{2} + t\right) < F\left(\frac{1}{2}\right).$$

But, using the convexity of F over $[0, 1]$, we have

$$F\left(\frac{1}{2}\right) \le \frac{1}{2}F\left(\frac{1}{2} - t\right) + \frac{1}{2}F\left(\frac{1}{2} + t\right) < F\left(\frac{1}{2}\right),$$

a contradiction.

(iv) It is obvious from ***(iii)*** of Theorem 2.6, and ***(v)*** of Lemma 2.1.

∎

3. APPLICATIONS

Application 1. Let $x_1, x_2, \cdots, x_n$ be n nonnegative numbers. Then, with the above notations, we have

$$\sqrt[n]{x_1 x_2 \cdots x_n} \le \sqrt[n]{\prod_{i=1}^{n} \sum_{j=1}^{n} [(1-t)b_{ij} + t c_{ij}] x_j} \le \frac{x_1 + x_2 + \cdots + x_n}{n}, \tag{3.1}$$

$$\sqrt[n]{x_1 x_2 \cdots x_n} \le \sqrt[n]{\prod_{i=1}^{n} [(1-t)x_i + t x_{n+1-i}]} \le \frac{x_1 + x_2 + \cdots + x_n}{n}, \tag{3.2}$$

for all t in $[0,1]$, and

$$\sqrt[n]{x_1x_2\cdots x_n} \le e^{-1}\sqrt[n]{\prod_{i=1}^{n}\left[\frac{\left(\sum_j c_{ij}x_j\right)^{\sum_j c_{ij}x_j}}{\left(\sum_j b_{ij}x_j\right)^{\sum_j b_{ij}x_j}}\right]^{\frac{1}{\left(\sum_j c_{ij}x_j-\sum_j b_{ij}x_j\right)}}} \le \frac{x_1+x_2+\cdots+x_n}{n}. \tag{3.3}$$

In particular

$$\sqrt[n]{x_1x_2\cdots x_n} \le e^{-1}\sqrt[n]{\prod_{i=1}^{n}\left[\frac{x_{n+1-i}^{x_{n+1-i}}}{x_i^{x_i}}\right]^{\frac{1}{(x_{n+1-i}-x_i)}}} \le \frac{x_1+x_2+\cdots+x_n}{n}, \tag{3.4}$$

and

$$\sqrt{x_1x_2} \le e^{-1}\left(\frac{x_1^{x_1}}{x_2^{x_2}}\right)^{\frac{1}{x_1-x_2}} \le \frac{x_1+x_2}{2}, \tag{3.5}$$

$$\frac{2n+2}{2n+1}\left(1+\frac{1}{n}\right)^n \le e \le \sqrt{\frac{n+1}{n}}\left(1+\frac{1}{n}\right)^n. \tag{3.6}$$

Proof. If we take $f:(0,\infty)\to\mathbb{R}$, $f(x)=-\ln x$, then we have

$$F(t) = -\frac{1}{n}\sum_{i=1}^{n}\ln\left(\sum_{j=1}^{n}[(1-t)b_{ij}+tc_{ij}]x_j\right),$$

and

$$\begin{aligned}\int_0^1 F(t)dt &= -\frac{1}{n}\sum_{i=1}^{n}\int_0^1\ln\left(\sum_{j=1}^{n}[(1-t)b_{ij}+tc_{ij}]x_j\right)dt\\ &= -\frac{1}{n}\ln\prod_{i=1}^{n}\left[\frac{\left(\sum_{j=1}^n c_{ij}x_j\right)^{\sum_{j=1}^n c_{ij}x_j}}{\left(\sum_{j=1}^n b_{ij}x_j\right)^{\sum_{j=1}^n b_{ij}x_j}}\right]^{\frac{1}{\sum_{j=1}^n c_{ij}x_j-\sum_{j=1}^n b_{ij}x_j}} + 1,\end{aligned}$$

which proves (3.1) and (3.3). In particular, if we take $b_{ij}=\delta_{ij}$ and $c_{ij}=\delta_{i,n+1-j}(i,j=1,\cdots,n)$, we obtain (3.2) and (3.4) from (3.1) and (3.3) respectively. The result (3.5) is immediate from (3.4). If we take $x_1=n,\ x_2=n+1$ in (3.5), we get (3.6). ∎

Application 2. If X is a Lebesgue measurable subset of $\mathbb{R}^{\mathbf{k}}$, $p\ge 1$, and $f_1,f_2,\cdots,f_n$ belong to $L^p=L^p(X)$, then we have

$$\left\|\frac{f_1+\cdots+f_n}{n}\right\|_p^p \le \frac{\sum_{i=1}^n\left[\sum_{j=1}^n c_{ij}|f_j|;\sum_{j=1}^n b_{ij}|f_j|\right]}{n(p+1)} \le \frac{\|f_1\|_p^p+\cdots+\|f_n\|_p^p}{n}, \tag{3.7}$$

and

$$\left\|\frac{f_1+\cdots+f_n}{n}\right\|_p^p \le \frac{\sum_{i=1}^n [|f_i|\,;|f_{n+1-i}|]}{n(p+1)} \le \frac{\|f_1\|_p^p+\cdots+\|f_n\|_p^p}{n}, \tag{3.8}$$

where for each Lebesgue measurable function $g \ge 0$ and $h \ge 0$ on X,

$$[g;h] = \left\|\frac{g^{p+1}-h^{p+1}}{g-h}\right\|_1 = \int_X \frac{g^{p+1}-h^{p+1}}{g-h}dx,$$

when $g(x)=h(x)$, the integrand is understood to be $(p+1)g^p(x)$.
In particular, if p is an integer then,

$$\left\|\frac{f_1+\cdots+f_n}{n}\right\|_p^p \le \frac{\sum\limits_{i=1}^n\sum\limits_{k=0}^p \left\|f_i^k.f_{n+1-i}^{p-k}\right\|_1}{n(p+1)} \le \frac{\|f_1\|_p^p+\cdots+\|f_n\|_p^p}{n}, \tag{3.9}$$

and

$$\left\|\frac{f_1+f_2}{2}\right\|_p^p \le \frac{\sum\limits_{k=0}^p \left\|f_1^k.f_2^{p-k}\right\|_1}{p+1} \le \frac{\|f_1\|_p^p+\|f_2\|_p^p}{2}, \tag{3.10}$$

Proof. Since

$$\left\|\frac{(f_1+\cdots+f_n)}{n}\right\|_p \le \left\|\frac{(|f_1|+\cdots+|f_n|)}{n}\right\|_p$$

and the L^p-norms of f_i and $|f_i|$ are equal $(i=1,\cdots,n)$, it is sufficient to assume $f_i \ge 0$ $(i=1,\cdots,n)$. If we take $\varphi \to \|\varphi\|^p$ for the convex function $L^p \to \mathbb{R}$, then using Fubini's theorem we get

$$\begin{aligned}
\int_0^1 F(t)dt &= \frac{1}{n}\sum_{i=1}^n \int_0^1 \left\|\sum_{j=1}^n [(1-t)b_{ij}+tc_{ij}]f_j\right\|_p^p dt \\
&= \frac{1}{n}\sum_{i=1}^n \int_0^1\int_X \left(\sum_{j=1}^n [(1-t)b_{ij}+tc_{ij}]f_j(x)\right)^p dxdt \\
&= \frac{1}{n}\sum_{i=1}^n \int_X\int_0^1 \left(\sum_{j=1}^n [(1-t)b_{ij}+tc_{ij}]f_j(x)\right)^p dtdx \\
&= \frac{1}{n(p+1)}\sum_{i=1}^n \int_X \frac{\left(\sum_{j=1}^n c_{ij}f_j\right)^{p+1} - \left(\sum_{j=1}^n b_{ij}f_j\right)^{p+1}}{\sum_{j=1}^n c_{ij}f_j - \sum_{j=1}^n b_{ij}f_j}dx \\
&= \frac{\sum_{i=1}^n \left[\sum_{j=1}^n c_{ij}f_j; \sum_{j=1}^n b_{ij}f_j\right]}{n(p+1)},
\end{aligned}$$

which yields (3.7). In particular, if we set $b_{ij}=\delta_{ij}$ and $c_{ij}=\delta_{i,n+1-j}(i,j=1,\cdots,n)$, (3.8) follows from (3.7). Finally, (3.9) and (3.10) are immediate from (3.8). ∎

Remark 3.1. Let X be a Lebesgue measurable subset of $\mathbb{R}^{\mathbf{k}}$ with finite measure, and $\mathcal{M}$ be the vector space of all Lebesgue measurable functions on X with pointwise operations [1]. The set

C, consisting of all nonnegative measurable functions on X, is a convex subset of $\mathcal{M}$. Since the function $t \to \frac{t}{1+t}$ $(t \geq 0)$ is concave, the mapping $\varphi : C \to \mathbb{R}$ with

$$\varphi(f) = \int_X \frac{f}{1+f} dx \ (f \in C)$$

is concave.

Application 3. With the above notations, if $f_1, \cdots, f_n$ belong to $\mathcal{M}$, then

$$\begin{aligned}(3.11)\quad \frac{1}{n}\sum_{i=1}^{n}\int_X \frac{|f_i|}{1+|f_i|}dx &\leq m(X) - \frac{1}{n}\sum_{i=1}^{n}\int_X \frac{1}{\sum_{j=1}^{n}(c_{ij}-b_{ij})|f_j|}\ln\frac{1+\sum_{j=1}^{n}c_{ij}|f_j|}{1+\sum_{j=1}^{n}b_{ij}|f_j|}dx \\ &\leq \int_X \frac{\frac{1}{n}\sum_{i=1}^{n}|f_i|}{1+\frac{1}{n}\sum_{i=1}^{n}|f_i|}dx,\end{aligned}$$

$$\begin{aligned}(3.12)\quad \frac{1}{n}\sum_{i=1}^{n}\int_X \frac{|f_i|}{1+|f_i|}dx &\leq m(X) - \frac{1}{n}\sum_{i=1}^{n}\int_X \frac{1}{|f_{n+1-i}|-|f_i|}\ln\frac{1+|f_{n+1-i}|}{1+|f_i|}dx \\ &\leq \int_X \frac{\frac{1}{n}\sum_{i=1}^{n}|f_i|}{1+\frac{1}{n}\sum_{i=1}^{n}|f_i|}dx,\end{aligned}$$

$$\begin{aligned}(3.13)\quad \frac{1}{2}\sum_{i=1}^{2}\int_X \frac{|f_i|}{1+|f_i|}dx &\leq m(X) - \int_X \frac{1}{|f_2|-|f_1|}\ln\frac{1+|f_2|}{1+|f_1|}dx \\ &\leq \int_X \frac{\frac{1}{2}\sum_{i=1}^{2}|f_i|}{1+\frac{1}{2}\sum_{i=1}^{2}|f_i|}dx,\end{aligned}$$

in which, generally, when $a = b > 0$, the ratio $(\ln b - \ln a)/(b-a)$ is understood as $1/a$.

Proof. We can suppose that $f_i \geq 0$ $(1 \leq i \leq n)$. Since φ is concave, taking φ and ϕ instead of f and F in Theorem 2.2 respectively, we get

$$\begin{aligned}(3.14)\quad \frac{\varphi(f_1)+\cdots+\varphi(f_n)}{n} &\leq \int_0^1 \phi(t)dt \\ &\leq \varphi\left(\frac{f_1+\cdots+f_n}{n}\right).\end{aligned}$$

However, by Fubini's theorem and applying the change of variables

$$u = \sum_{j=1}^{n}[(1-t)b_{ij} + tc_{ij}]f_j(x),$$

in the following integrals, we have,

$$\begin{aligned}
\int_0^1 \phi(t)dt &= \frac{1}{n}\sum_{i=1}^{n}\int_0^1\int_X \frac{\sum_{j=1}^{n}[(1-t)b_{ij}+tc_{ij}]f_j(x)}{1+\sum_{j=1}^{n}[(1-t)b_{ij}+tc_{ij}]f_j(x)}dxdt \\
&= \frac{1}{n}\sum_{i=1}^{n}\int_X\int_0^1 \frac{\sum_{j=1}^{n}[(1-t)b_{ij}+tc_{ij}]f_j(x)}{1+\sum_{j=1}^{n}[(1-t)b_{ij}+tc_{ij}]f_j(x)}dtdx \\
&= \frac{1}{n}\sum_{i=1}^{n}\int_X \frac{1}{\sum_{j=1}^{n}(c_{ij}-b_{ij})f_j(x)}\int_{\sum_{j=1}^{n}b_{ij}f_j(x)}^{\sum_{j=1}^{n}c_{ij}f_j(x)}\left(1-\frac{1}{1+u}\right)dudx \\
&= m(X)-\frac{1}{n}\sum_{i=1}^{n}\int_X \frac{1}{\sum_{j=1}^{n}(c_{ij}-b_{ij})f_j}\ln\frac{1+\sum_{j=1}^{n}c_{ij}f_j}{1+\sum_{j=1}^{n}b_{ij}f_j}dx,
\end{aligned}$$

and after substituting this in (3.14), we obtain (3.11). The inequalities (3.12) and (3.13) are special cases of (3.11), taking $b_{ij} = \delta_{ij}$ and $c_{ij} = \delta_{i,n+1-j}$. ∎

Acknowledgement 1. I would like to express my gratitude to Professors A. R. Medghalchi and B. Mehri for their valuable comments and suggestions.

REFERENCES

[1] S. BERBERIAN, *Lectures in functional analysis and operator theory*, Springer, New York-Heidelberg-Berlin, 1974.

[2] S. S. DRAGOMIR AND N. M. IONESCU, Some remarks on convex functions, *Revue d'analyse numérique et de théorie de l'approximation*, **21** (1992), 31–36.

[3] A.W. ROBERTS AND D.E. VARBERG, *Convex functions*, Academic Press, New York and London, 1973.

[4] R. WEBSTER, *Convexity*, Oxford University Press, Oxford, New York, Tokyo, 1994.