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NECESSARY AND SUFFICIENT CONDITION FOR EXISTENCE AND UNIQUENESS OF THE SOLUTION OF CAUCHY PROBLEM FOR HOLOMORPHIC FUCHSIAN OPERATORS

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ABSTRACT. In this paper a Cauchy problem for holomorphic differential operators of Fuchsian type is investigated. Using Ovcyannikov techniques and the method of majorants, a necessary and sufficient condition for existence and uniqueness of the solution of the problem under consideration is shown.

Key words and phrases: Banach algebra, Cauchy problem, Fuchsian characteristic polynomial, Fuchsian differential operator, Fuchsian principal weight, holomorphic differentiable manifold, holomorphic hypersurface, Fuchsian principal weight, method of majorants, method of successive approximations, principal symbol, and reduced Fuchsian weight.

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1. INTRODUCTION

We introduce the method of majorants [2], [5], and [8], which plays an important role for the Cauchy problem in proving the existence of a solution. This method has been applied by many mathematicians, in particular [1], [3], and [4] to study Cauchy problems related to differential operators that are a "natural" generalization of ordinary differential operators of Fuchsian type, and to generalize the Goursat problem [8]. We also give a refinement of the method of successive approximations as in the Ovcyannikov Theorem given in [7]. Combining these two methods, we shall prove the theorem [6].

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2. NOTATIONS AND DEFINITIONS

Let us denote

$$\begin{aligned} x &= (x_0, x_1, \ldots, x_n) \equiv (x_0, x') \in \mathbb{R} \times \mathbb{R}^n, \text{ where } x' = (x_1, \ldots, x_n) \in \mathbb{R}^n, \\ \xi &= (\xi_0, \xi_1, \ldots, \xi_n) \equiv (\xi_0, \xi') \in \mathbb{R} \times \mathbb{R}^n, \text{ where } \xi' = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n, \\ \alpha &= (\alpha_0, \alpha_1, \ldots, \alpha_n) \equiv (\alpha_0, \alpha') \in \mathbb{N} \times \mathbb{N}^n, \text{ where } \alpha' = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n. \end{aligned}$$

We use Schwartz's notations

$$\begin{aligned} x^\alpha &= x_0^{\alpha_0} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \equiv x_0^{\alpha_0} (x')^{\alpha'}, |x|^\alpha = |x_0|^{\alpha_0} |x_1|^{\alpha_1} \cdots |x_n|^{\alpha_n} \\ \alpha! &= \alpha_0! \alpha_1! \cdots \alpha_n!, \ |\alpha| = \alpha_0 + \alpha_1 + \cdots + \alpha_n, \\ \beta &\leq \alpha \text{ means } \beta_j \leq \alpha_j \text{ for all } j = 0, 1, \ldots, n, \\ D^\alpha &= \frac{\partial^{|\alpha|}}{\partial_{x_0}^{\alpha_0} \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n}} \equiv D_0^{\alpha_0} D_1^{\alpha_1} \cdots D_n^{\alpha_n}, \text{ where } D_j = \frac{\partial}{\partial x_j}, 0 \leq j \leq n. \end{aligned}$$

For $k \in \mathbb{N}$, $0 \leq k \leq m$,

$$\max[0, \alpha_0 + 1 - (m-k)] \equiv [\alpha_0 + 1 - (m-k)]_+,$$

$$\begin{pmatrix} m \\ k \end{pmatrix} = \frac{m!}{(m-k)!k!}, \ C_q(j) = j(j-1)...(j-q+1),$$

by convention $C_0(j) = 1$, and the gradient of φ with respect to x will be denoted by

$$grad\varphi(x) = \left(\frac{\partial \varphi(x)}{\partial x_0}, \ldots, \frac{\partial \varphi(x)}{\partial x_n} \right).$$

We denote a linear differential operator of order m, $P(x; D)$ by $\sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$.

Definition 2.1. Let E be an $n+1$ dimensional holomorphic differentiable manifold. Let h be a holomorphic differentiable operator over E of order m_0 in a, and of order $\leq m_0$ near a. Let S be a holomorphic hypersurface of E containing a, let m be an integer $\geq m_0$, and let φ be a local equation of S in some neighborhood of a, that is, there exists an open neighborhood Ω of a such that:

$$\forall x \in \Omega, \ grad\varphi(x) \neq 0, \ x \in \Omega \cap S \Longleftrightarrow \varphi(x) = 0.$$

If $\sigma \in \mathbb{Z}$ and Y is a holomorphic function on Ω, for $x \in \Omega \backslash S$, we denote by

$$h_m^\sigma(Y)(x) = \varphi^{\sigma - m}(x) h(Y\varphi^m)(x)$$

and by $H_m^\sigma(x, \xi)$ the principal symbol of this differential operator.

(i)

$$\begin{aligned} \tau_{h,S}(a) &= \inf \Big\{ \sigma \in \mathbb{Z} : \forall Y \text{ holomorphic function in a neighborhood } \Omega \text{ of } a, \\ &\qquad \forall x \in \Omega \cap S, \lim_{x \to b, x \notin S} h_m^{\sigma+1}(Y)(x) = 0 \Big\} \end{aligned}$$

denotes the Fuchsian weight of h in a with respect to S.

(ii)

$$\tau_{h,S}^*(a) = \inf \left\{ \sigma \in \mathbb{Z} : \lim_{x \to b, x \notin S} \varphi^{-m_0}(x) H^{\sigma+1}(x; grad\varphi(x)) = 0 \right\}$$

is the Fuchsian principal weight of h in a with respect to S.

(iii)

$$\tilde{\tau}_{h,S}(a) = \inf\left\{\sigma \in \mathbb{Z} : \forall\Omega, \forall Y, \forall b \in \Omega \cap S, \lim_{x\to b, x\notin S}[h_m^{\sigma+1}(Y)(x) - Y(x)h_m^{\sigma+1}(1)(x)] = 0\right\}$$

denotes the reduced Fuchsian weight of h in a with respect to S.

A differential operator h is said to be a Fuchsian operator of weight τ in a with respect to S if the following assertions are valid:

$(H-0)$ $\tau^*_{h,S}$ is finite and constant and equal τ near $a \in S$,
$(H-1)$ $\tau_{h,S}(a) = \tau$,
$(H-2)$ $\tilde{\tau}_{h,S}(a) \leq \tau - 1$.

A Fuchsian characteristic polynomial is defined to be a polynomial $\mathcal{C}$ in λ of holomorphic coefficients in $y \in S$ by

$$\mathcal{C}(\lambda, y) = \lim_{x\to y, x\notin S} \varphi^{\tau_{h,S}(a)-\lambda}(x) h(\varphi^\lambda)(x).$$

Set

$$\mathcal{C}_1(\lambda, y) = \mathcal{C}(\lambda + \tau_{h,S}(a), y),\ \forall\lambda \in \mathbb{C},\ \forall y \in S.$$

Remark 2.1. If we choose a local card for which $\varphi(x) = x_0$ and $a = (0,\ldots,0)$, we get Baouendi-Goulaouic's definitions [1].

Remark 2.2. The number $\tau_{h,S}(a)$ is independent on m which is greater or equal to m_0.

3. MAJORANTS

The majorants play an important role in the Cauchy method to prove the existence of the solution, where the problem consists of finding a majorant function which converges.

Let α be a multi-index of $\mathbb{N}^{n+1}$ and E be a $\mathbb{C}$-Banach algebra, we define a formal series in x by

$$u(x) = \sum_{\alpha\in\mathbb{N}^{n+1}} u_\alpha \frac{x^\alpha}{\alpha!},$$

where $u_\alpha \in E$.

We denote by $E[[x]]$ the set of the formal series in x with coefficients in E.

Definition 3.1. Let $u(x)$, $v(x) \in E[[x]]$, and $\lambda \in \mathbb{C}$. We define the following operations in $E[[x]]$ by

(a) $u(x) + v(x) = u(x) = \sum_{\alpha\in\mathbb{N}^{n+1}} (u_\alpha + v_\alpha)\frac{x^\alpha}{\alpha!}$,
(b) $\lambda u(x) = \sum_{\alpha\in\mathbb{N}^{n+1}} (\lambda u_\alpha)\frac{x^\alpha}{\alpha!}$,
(c) $u(x)v(x) = \sum_{\alpha\in\mathbb{N}^{n+1}} \sum_{0\leq\beta\leq\alpha} \binom{\alpha}{\beta} u_\beta v_{\alpha-\beta}\frac{x^\alpha}{\alpha!}$.

Definition 3.2. Let

$$u(x) = \sum_{\alpha\in\mathbb{N}^{n+1}} u_\alpha \frac{x^\alpha}{\alpha!} \in E[[x]],$$

and

$$U(x) = \sum_{\alpha\in\mathbb{N}^{n+1}} U_\alpha \frac{x^\alpha}{\alpha!} \in \mathbb{R}[[x]]$$

be two formal series. We say that U majorizes u, written $U(x) \gg u(x)$, provided $U_\alpha \geq \|u_\alpha\|$ for all multi-indices α.

Definition 3.3. Let
$$u(x) = \sum_{\alpha \in \mathbb{N}^{n+1}} u_\alpha \frac{x^\alpha}{\alpha!} \in E[[x]].$$
We define the integro-differentiation of $u(x)$ by
$$D^\mu u(x) = \sum_{\alpha \geq [-\mu]_+} u_{\alpha+\mu} \frac{x^\alpha}{\alpha!},$$
for $\mu \in \mathbb{Z}^{n+1}$, and $[-\mu]_+ \equiv ([-\mu_0]_+, [-\mu_1]_+, \dots, [-\mu_n]_+)$.

(a) Let A be a finite subset of $\mathbb{Z}^{n+1}$, $P(x;D)$ is said to be a formal integro-differential operator over $E[[x]]$ if for $u(x) \in E[[x]]$,
$$P(x;D) = \sum_{\mu \in A} a_\mu(x) D^\mu u(x),$$
where $a_\mu(x) \in E[[x]]$.

(b) Let
$$\mathcal{P}(x;D) = \sum_{\mu \in \mathbb{Z}^{n+1}} A_\mu(x) D^\mu$$
and
$$P(x;D) = \sum_{\mu \in \mathbb{Z}^{n+1}} a_\mu(x) D^\mu$$
be formal integro-differential operators over $\mathbb{R}[[x]]$ and $E[[x]]$ respectively.

We say $\mathcal{P}(x;D)$ majorizes $P(x;D)$, written $\mathcal{P}(x;D) \gg P(x;D)$, provided $A_\mu(x) \gg a_\mu(x)$ for all multi-indices $\mu \in \mathbb{Z}^{n+1}$.

Definition 3.4. Consider a family $\{u^j\}_{j\in J}$, $u^j \in E[[x]]$. The family $\{u^j\}_{j\in J}$ is said to be summable if for any $\alpha \in \mathbb{N}^{n+1}$, $\mathfrak{J}_\alpha = \{j \in J : u^j_\alpha \neq 0\}$ is finite.

Theorem 3.1. *Let $v \in E[[y]], (y = (y_1, ..., y_m))$, $V \in \mathbb{R}[[y]]$ such that $V(y) \gg v(y)$. Let $u^j(x) \in E[[x]]$ for $j = 1, \dots, m$, $u^j_0 = 0$, and $U^j(x) \in \mathbb{R}[[x]]$ for $j = 1, \dots, m$, $U^j_0 = 0$ such that $U(x) \gg u(x)$ for all $j = 1, \dots, m$. Then*
$$V\left(U^1(x), \dots, U^m(x)\right) \gg v\left(u^1(x), \dots, u^m(x)\right).$$

Proof. See [7]. ☐

Definition 3.5. If $u(x) \in E[[x]]$, we denote the domain of convergence of u by
$$\mathfrak{d}(u) = \left\{ x : x \in \mathbb{C}^{n+1},\ u(x) = \sum_{\alpha \geq 0} \|u_\alpha\| \frac{|x|^\alpha}{\alpha!} < \infty \right\}.$$

Theorem 3.2. ***(majorants):*** *If $U(x) \gg u(x)$, then*
$$\mathfrak{d}(U) \subset \mathfrak{d}(u).$$

The above theorem is practical because, if the majorant series $U(x)$ converges for $|x| < r$ then $u(x)$ converges for $|x| < r$. Let us construct a majorant series through an example.

Let $r = (r_0, r_1, ..., r_n)$ and let $u(x)$ be a bounded holomorphic function on the polydisc
$$\mathcal{P}_r = \{x : x \in \mathbb{C}^{n+1}, \mid x_j \mid < r_j, \text{ for all } j = 0, 1, \dots, n\}.$$
Let $M = \sup\limits_{x \in \mathcal{P}_r} \|u(x)\|$, then it follows from Cauchy integral formula that $\|u_\alpha\| < \frac{M}{r^\alpha}\alpha!$.

If we let $U_\alpha = \frac{M}{r^\alpha}\alpha!$, then $U(x)$ majorizes $u(x)$.

Theorem 3.3. *Let $a_i, 0 \leq i \leq m$,be holomorphic functions near the origin in $\mathbb{C}^n$ that satisfy the following condition*

$$\sum_{i=0}^{m} C_i(j)a_i(0) \neq 0, \quad \forall j \in \mathbb{N}, \ (a_m = 1),$$

then there exists a holomorphic function $\mathcal{A}$ near the origin in $\mathbb{C}^n$ such that

$$\frac{1}{\sum_{i=0}^{m} C_i(j)a_i(x')} \ll \frac{\mathcal{A}(x')}{\sum_{i=0}^{m} C_i(j)}, \quad \forall j \in \mathbb{N}$$

Proof. If we write

$$\frac{\sum_{i=0}^{m} C_i(j)}{\sum_{i=0}^{m} C_i(j)a_i(x')} = \frac{\sum_{i=0}^{m} C_i(j)}{\sum_{i=0}^{m} C_i(j)a_i(0)} \cdot \frac{1}{1 - \frac{\sum_{i=0}^{m} C_i(j)(a_i(0)-a_i(x'))}{\sum_{i=0}^{m} C_i(j)a_i(0)}}$$

then we have

$$\begin{aligned}
\lim_{j\to\infty} \frac{\sum_{i=0}^{m} C_i(j)}{\sum_{i=0}^{m} C_i(j)a_i(0)} &= \lim_{j\to\infty} \frac{C_m(j)}{C_m(j)a_m(0)} \\
&= \frac{1}{a_m(0)} \\
&= 1.
\end{aligned}$$

Therefore, there exists a constant $C \geq 1$ such that

$$\left| \frac{\sum_{i=0}^{m} C_i(j)}{\sum_{i=0}^{m} C_i(j)a_i(0)} \right| \leq C, \ \forall j \in \mathbb{N}.$$

Let $\mathcal{B}(x')$ be a common majorant to $a_i(0) - a_i(x')$ for all $i = 0, 1, \ldots, m$ with $\mathcal{B}(0) = 0$, that is

$$\begin{aligned}
\frac{\sum_{i=0}^{m} C_i(j)(a_i(0) - a_i(x'))}{\sum_{i=0}^{m} C_i(j)a_i(0)} &\ll \mathcal{B}(x') \left| \frac{\sum_{i=0}^{m} C_i(j)}{\sum_{i=0}^{m} C_i(j)a_i(0)} \right| \\
&\ll C\mathcal{B}(x').
\end{aligned}$$

It follows from Theorem 3.1 that

$$\frac{1}{1 - \frac{\sum_{i=0}^{m} C_i(j)(a_i(0)-a_i(x'))}{\sum_{i=0}^{m} C_i(j)a_i(0)}} \ll \frac{1}{1 - C\mathcal{B}(x')}.$$

Choosing then $\mathcal{A}(x') = \frac{1}{1-C\mathcal{B}(x')}$, the desired conclusion easily yields. □

Corollary 3.4. *Under the conditions of Theorem 3.3, there exist two positive real numbers $M > 0$ and $r > 0$ such that*

$$\frac{1}{\mathcal{C}_1(j, x')} \ll \frac{1}{(j+1)^m} \cdot \frac{M}{1 - rt(x')}, \quad \forall j \in \mathbb{N},$$

where $t(x') = \sum_{i=0}^{m} x_i$.

Proof. The proof is similar to the proof of the Theorem 3.3, it suffices to observe that

$$\lim_{j\to\infty} \frac{\sum_{i=0}^{m} C_i(j)}{\sum_{i=0}^{m} C_i(j)a_i(0)} = 1,$$

then apply the following theorem. □

Theorem 3.5. *If $\mathcal{A}(x')$ is holomorphic near the origin in $\mathbb{C}^n$, there exist $M > 0$ and $R > 0$ such that:*

$$\mathcal{A}(x') \ll \frac{M}{R - t(x')}.$$

for all $x' \in \{x' : \sum_{i=0}^{n} \mid x_i \mid < R\}$.

Proof. See [8]. □

4. STATEMENT OF THE MAIN RESULT

Theorem 4.1 (Main Theorem)**.** *Let h be a Fuchsian holomorphic differential operator of weight $\tau_{h,S}(a)$ in a with respect to a holomorphic hypersurface S passing through a, of a holomorphic differential manifold E of dimension $n+1$, and φ a local equation of S in some neighborhood of a. Then the following assertions are equivalent:*

i) *for all $\lambda \geq \tau_{h,S}(a)$, $C(\lambda, x') \neq 0$*
ii) *for all holomorphic functions f and v in a neighborhood of a, there exists a unique holomorphic function u solving the Cauchy problem*

$$\begin{aligned} h(u) &= f \\ u - v &= \mathcal{O}(\varphi^{\tau_{h,S}(a)}). \end{aligned} \tag{4.1}$$

If we choose a local card such that $\varphi(x) = x_0$ and $a = (0, ..., 0)$, then we obtain the Baouendi-Goulaouic's Theorem [1], if $k = 0$, we obtain Cauchy-Kovalevskaya Theorem, and if $k = 1$, we obtain Hasegawa's Theorem [3].

The following theorem gives a relationship between a Fuchsian operator of arbitrary weight and a Fuchsian operator of weight zero.

Theorem 4.2. *Let h be a Fuchsian holomorphic differential operator of weight $\tau_{h,S}(a)$ in a with respect to a holomorphic hypersurface S passing through a, of a holomorphic differentiable manifold E of dimension $n+1$, and φ a local equation of S in some neighborhood of a. If we define the operator h_1 by*

$$Y \to h_1(Y) = h(Y\varphi^{\tau_{h,S}(a)}),$$

then h_1 is a Fuchsian holomorphic differential operator of weight zero in a relative to a holomorphic hypersurface S.

If $\mathcal{C}$ (respectively $\mathcal{C}_1$) denotes the Fuchsian polynomial characteristic of h (respectively h_1), then

$$\mathcal{C}_1(\lambda, y) = \mathcal{C}\left(\lambda + \tau_{h,S}(a), y\right),\ \forall \lambda \in \mathbb{C},\ \forall y \in S.$$

Proof. (1) We look for the Fuchsian weight of h_1 in a with respect to S.
Let $m \geq m_0$, where m_0 is the order of h, then

$$\begin{aligned} \varphi^{\sigma+1-m}(x) h_1(Y\varphi^m)(x) &= \varphi^{\sigma+1-m}(x) h(Y\varphi^{m+\tau_{h,S}(a)})(x) \\ &= \varphi^{(\sigma+\tau_{h,S}(a)+1)-(m+\tau_{h,S}(a))}(x) h(Y\varphi^{m+\tau_{h,S}(a)})(x), \end{aligned}$$

consequently $\tau_{h,S}(a) = 0$.

(2) We look for the principal Fuchsian weight of h_1 in a relative to S.

In the local card $\varphi(x) = x_0$ and $a = (0, \dots, 0)$, it is the exponent of x_0 in the coefficient of D_0^m of h_1. In this local card

$$\begin{aligned} P_1(u) &= P(x_0^{m_0-k}u) \\ &= x_0^k D_0^m (x_0^{m_0-k}u) + \dots \\ &= x_0^k D_0^m u + \dots \end{aligned}$$

(the points indicate the terms that have the order of differentiation with respect to x_0 less than m_0). Hence

$$\tau^*_{h,S}(a) = 0.$$

(3) If $m \geq m_0$, then

$$\begin{aligned} &\lim_{x\to b, x\notin S} \varphi^{\tau_{h_1,S}(a)-m}(x)[h_1(Y\varphi^m)(x) - Y(x)h_1(\varphi^m)(x)] \\ &\quad = \lim_{x\to b, x\notin S} \varphi^{-m}(x)[h(Y\varphi^{m+\tau_{h,S}(a)})(x) - Y(x)h(\varphi^{m+\tau_{h,S}(a)})(x)] \\ &\quad = \lim_{x\to b, x\notin S} \varphi^{\tau_{h,S}(a)-(m+\tau_{h,S}(a))}(x)[h(Y\varphi^{m+\tau_{h,S}(a)})(x) - Y(x)h(\varphi^{m+\tau_{h,S}(a)})(x)] \\ &\quad = 0, \text{ by hypothesis.} \end{aligned}$$

Finally, we have

$$\begin{aligned} \varphi^{\tau_{h_1,S}(a)-\lambda}(x)h_1(\varphi^\lambda)(x) &= \varphi^{-\lambda}(x)h(\varphi^{\lambda+\tau_{h,S}(a)})(x) \\ &= \varphi^{\tau_{h,S}(a)-(\lambda+\tau_{h,S}(a))}(x)h(\varphi^{\lambda+\tau_{h,S}(a)})(x) \end{aligned}$$

which tends to $\mathcal{C}(\lambda + \tau_{h,S}(a), y)$ as x tends to y and $x \notin S$.

This concludes the proof of the Theorem. □

5. FORMAL PROBLEM

If we choose a local card for which $\varphi(x) = x_0$ and $a = (0, \dots, 0)$ the Cauchy problem (4.1) becomes (5.1) below. We devote this section to formal calculations by looking for solutions as power series of the problem (5.1) below connected with a Fuchsian operator $P(x; D)$ of order m and weight $m - k$ with respect to x_0 at $x_0 = 0$.

We decompose this operator in the following form

$$P(x; D) = P_m(x; D_0) - Q(x; D),$$

where

$$P_m(x; D_0) = \sum_{p=0}^{k} a_{m-p}(x')x_0^{k-p}D_0^{m-p},$$

$$Q(x; D) = -\sum_{\alpha_0<m, |\alpha|\leq m} x_0^{\mu(\alpha_0)}D_0^{\alpha_0}(a_{\alpha_0,\alpha'}(x_0, x')D_{x'}^{\alpha'}),$$

with $a_m = 1$ and $\mu(\alpha_0) = [\alpha_0 + 1 - (m - k)]_+$.

Theorem 5.1. *If the coefficients of $P_m(x; D_0)$ and $Q(x; D)$ are holomorphic functions near the origin in $\mathbb{C}^{n+1}$, then the following conditions are equivalent*

i) *For all integers $\lambda \geq m - k \neq 0$,*

ii) *For any holomorphic Cauchy data* u_j, $0 \leq j \leq m-k-1$, *near the origin in* $\mathbb{C}^n$ *and for each holomorphic function* f *near the origin in* $\mathbb{C}^{n+1}$, *there exists a unique holomorphic solution* u *near the origin in* $\mathbb{C}^{n+1}$ *solving Cauchy problem*

$$\begin{aligned} P(x;D)u(x) &= f(x) \\ D_0^j u(0,x') &= u_j(x'), \quad 0 \leq j \leq m-k-1. \end{aligned} \tag{5.1}$$

Suppose that the solution $u(x_0, x')$ has the form $\sum_{j=0}^{\infty} u_j(x') \frac{x_0^j}{j!}$.

The problem is to determine $u_j(x')$ for all $j \geq 0$. It is easy to check the following statements:

If

$$u(x_0, x') = \sum_{j=0}^{\infty} u_j(x') \frac{x_0^j}{j!}$$

and

$$v(x_0, x') = \sum_{\mu=0}^{\infty} v_\mu(x') \frac{x_0^\mu}{\mu!},$$

then

$$u(x_0,x')v(x_0,x') = \sum_{j=0}^{\infty} \left[\sum_{p=0}^{j} \begin{pmatrix} j \\ p \end{pmatrix} u_{j-p}(x') v_p(x') \right] \frac{x_0^j}{j!} \tag{5.2}$$

$$D_0^p u(x_0,x') = \sum_{j=0}^{\infty} u_{j+p}(x') \frac{x_0^j}{j!} \tag{5.3}$$

$$x_0^q D_0^p u(x_0,x') = \sum_{j=0}^{\infty} [C_q(j) u_{j+p-q}(x')] \frac{x_0^j}{j!} \tag{5.4}$$

and by convention $u_k = 0$ for $k < 0$.

By using (5.2), (5.3), and (5.4), one can check easily that

$$P_m(x; D_0) u(x_0, x') = \sum_{j=0}^{\infty} [D(j,x') u_{j+m-k}(x')] \frac{x_0^j}{j!},$$

where $D(j,x') = \sum_{j=0}^{k} a_{m-q}(x') C_{k-p}(j)$ which can be written in terms of $\mathcal{C}(j,x')$ as

$$D(j,x') = \frac{\mathcal{C}(j+m-k, x')}{C_{m-k}(j+m-k)},$$

and we have

$$P_m(x; D_0) u(x_0, x') = \sum_{j=0}^{\infty} \left[\frac{\mathcal{C}(j+m-k,x')}{C_{m-k}(j+m-k)} u_{j+m-k}(x') \right] \frac{x_0^j}{j!}. \tag{5.5}$$

Similarly, if $a_\alpha(x_0, x') = \sum_{\nu=0}^{\infty} a_\alpha^\nu(x') \frac{x_0^\nu}{\nu!}$, then

$$\begin{aligned} Q(x;D)u(x_0,x') = -\sum_{j=0}^{\infty} \Bigg[& \sum_{\alpha_0 < m, |\alpha| \leq m} C_{\mu(\alpha_0)}(j) \\ & \times \sum_{p=0}^{j+\alpha_0-\mu(\alpha_0)} \begin{pmatrix} j+\alpha_0-\mu(\alpha_0) \\ p \end{pmatrix} a_\alpha^p(x') D_{x'}^{\alpha'} u_{j+\alpha_0-\mu(\alpha_0)-p}(x') \Bigg] \frac{x_0^j}{j!}. \end{aligned} \tag{5.6}$$

Finally, if $f(x_0, x') = \sum_{j=0}^{\infty} f_j(x') \frac{x_0^j}{j!}$, then by using (5.5), (5.6), and by identifying the coefficients of $P(x; D)u(x) = f(x)$ we get the following expression

$$\frac{\mathcal{C}(j+m-k, x')}{C_{m-k}(j+m-k)} u_{j+m-k}(x') = -\sum_{\alpha_0<m, |\alpha|\le m} C_{\mu(\alpha_0)}(j) \sum_{p=0}^{j+\alpha_0-\mu(\alpha_0)} \binom{j+\alpha_0-\mu(\alpha_0)}{p} a_\alpha^p(x') \times D_{x'}^{\alpha'} u_{j+\alpha_0-\mu(\alpha_0)-p}(x') + f_j(x')$$

for all $j \in \mathbb{N}$.

Lemma 5.2. *Let $P(\lambda; x') = \sum_{k=0}^{m} a_k(x')\lambda^k$, $(a_m = 1)$, be a polynomial in λ with continuous coefficients on some neighborhood $\widetilde{V}$ of the origin in $\mathbb{C}^n$.*
If $P(j; 0) \neq 0$ for all $j \in \mathbb{N}$, there exists a neighborhood V of the origin such that $P(j; x') \neq 0$ for all $x' \in V$ and all $j \in \mathbb{N}$.

Proof. We have $|P(\lambda; x')| \ge |\lambda|^m - \left|\sum_{k=0}^{m-1} a_k(x')\lambda^k\right|$. Let $M = \max_{0\le k\le m-1, x'\in\widetilde{V}} |a_k(x')|$ then

$$|P(\lambda; x')| \ge |\lambda|^m \left[1 - \frac{M}{|\lambda|} \sum_{k=0}^{m-1} \frac{1}{|\lambda|^{m-k-1}}\right].$$

If $|\lambda| > 1$, then

$$|P(\lambda; x')| > 1 - \frac{M}{|\lambda| - 1}.$$

If $x' \in \tilde{V}$ and $|\lambda| \ge 2M + 1$, then

$$|P(\lambda; x')| > \frac{1}{2}.$$

In other words, if j is an integer such that $j \ge 2M + 1$ and $x' \in \widetilde{V}$ then

$$P(j; x') \neq 0.$$

Now let $j \in \mathbb{N}$ such that $0 \le j < 2M + 1$. Since $P(j; 0) \neq 0$, then by continuity, there is a neighborhood of the origin V_j such that $P(j; x') \neq 0$ for all $x' \in V_j$.
In conclusion we choose $V = (\cap_{0\le j<2M+1} V_j) \cap \tilde{V}$, and we have

$$P(j; x') \neq 0$$

for all $x' \in V$ and all $j \in \mathbb{N}$, as required. □

Corollary 5.3. *There exists a neighborhood V of the origin such that $\mathcal{C}\,(j+m-k, x') \neq 0$ for all $x' \in V$ and all $j \in \mathbb{N}$, and the induction formula*

$$u_{j+m-k}(x') = -\frac{C_{m-k}(j+m-k)}{\mathcal{C}(j+m-k, x')} \left[\sum_{\alpha_0<m, |\alpha|\le m} C_{\mu(\alpha_0)}(j) \times \sum_{p=0}^{j+\alpha_0-\mu(\alpha_0)} \binom{j+\alpha_0-\mu(\alpha_0)}{p} a_\alpha^p(x') D_{x'}^{\alpha'} u_{j+\alpha_0-\mu(\alpha_0)-p}(x') + f_j(x')\right] \tag{5.7}$$

yields for all $x' \in V$ and all $j \in \mathbb{N}$.

Under the conditions of the Theorem 5.1, there exists a unique formal series

$$u(x_0, x') = \sum_{j=0}^{\infty} u_j(x') \frac{x_0^j}{j!}$$

solution of the problem (5.1) since all $u_j(x')$ are uniquely determined by (5.7).

6. PROOF OF THE MAIN THEOREM

Let h be a differential operator of Fuchsian type in a with respect to S of weight $\tau_{h,S}(a)$ and of order m. We want to solve (4.1) in some neighborhood of a.

Set $u - v = w$ and the problem (4.1) becomes

$$\begin{aligned} h(w) &= g, \\ w &= \mathcal{O}(\varphi^{\tau_{h,S}(a)}), \end{aligned} \tag{6.1}$$

where $g = f - h(v)$.

It follows from the second condition of (6.1) that there is a unique holomorphic function U in some neighborhood of a such that $w = \mathcal{O}(\varphi^{\tau_{h,S}(a)})$ and finding U is equivalent to finding w.

U verifies

$$h(U\varphi^{\tau_{h,S}(a)}) = g,$$

i.e. U satisfies the equation

$$h_1(U) = g,$$

where h_1 is a Fuchsian operator of weight zero in a relative to S (by Theorem 4.2).

If we choose a local card such that $\varphi(x) = x_0$ and $a = (0, ..., 0)$; in this local card the equation becomes

$$\tilde{P}(U) = Q(U) + g,$$

where

$$\begin{aligned} \tilde{P} &= \sum_{p=0}^{m} a_{m-p}(x') x_0^{m-p} D_0^{m-p}, \ (a_m = 1), \\ Q &= -\sum_{\alpha_0=0}^{m-1} x_0^{\alpha_0+1} D_0^{\alpha_0} B_{m-\alpha_0}, \end{aligned}$$

and

$$B_{m-\alpha_0} = \sum_{|\alpha'| \le m-\alpha_0} a_\alpha(x) D_{x'}^{\alpha'}.$$

Let us denote by $\mathcal{C}(j+m-k, x')$, (respectively $\mathcal{C}_1(\lambda, x')$) the Fuchsian polynomial characteristic of h, (respectively h_1). It follows from Lemma 5.2 that if $\mathcal{C}(j, 0) \neq 0$ for all $j \in \mathbb{N}$, $j \geq m - k$, then there is a neighborhood V of the origin in $\mathbb{C}^n$ for which $\mathcal{C}(j, x') \neq 0$ for all $x' \in V$ and all $j \in \mathbb{N}$, $j \geq m - k$ .Hence $\tilde{P}$ is one to one on the set of holomorphic functions at the origin.

If $u(x_0, x') = \sum_{j=0}^{\infty} u_j(x') \frac{x_0^j}{j!}$ then

$$\tilde{P}^{-1} u(x) = \sum_{j=0}^{\infty} \frac{u_j(x')}{\mathcal{C}_1(j, x')} \frac{x_0^j}{j!},$$

and the problem (6.1) is equivalent to the following problem

$$U = \left(\tilde{P}^{-1} \circ Q\right)(U) + U_0, \tag{6.2}$$

where

$$U_0 = \tilde{P}^{-1}(g).$$

As in [1], [4], [5], [7], and [8], a successive approximation method will be used in the following sense.

Let $U_{p+1} = \left(\tilde{P}^{-1} \circ Q\right)(U_p) + U_0$ for $p \geq 0$, and set $V_p = U_{p+1} - U_p$. Then

$$V_{p+1} = \left(\tilde{P}^{-1} \circ Q\right)(V_p).$$

Let $\mathcal{D} = \left\{(x_0, x') : (x_0, x') \in \mathbb{C}^{n+1}, |x_0| \leq \frac{1}{R} \text{ and } |x_1| + \cdots + |x_n| \leq \frac{1}{r}\right\}$, then V_0 is holomorphic in some open neighborhood in $\mathcal{D}$, and there is a constant denoted $\|V_0\|$ such that

$$V_p(x) \ll \|V_0\| \frac{1}{1 - x_0\xi_0} \cdot \frac{1}{1 - st(x')},$$

where $t(x') = \sum_{i=0}^{n} x_i$, $\xi_0 \geq \frac{R}{\eta_0}$, $s \geq \frac{r}{\eta_0}$, and η_0 is some given number in the open interval $(0, 1)$.

Lemma 6.1. *There exists a constant K such that*

$$V_p(x) \ll \|V_0\| \frac{K^p}{(s' - s)^{mp}} \cdot \frac{x_0^p}{1 - x_0\xi_0} \cdot \frac{1}{1 - s't(x')} \tag{6.3}$$

for all $s' > s$.

Proof. Clearly (6.3) holds for $p = 0$. Suppose (6.3) holds for p, and let us prove it for $p + 1$.

We have

$$\tilde{P}^{-1} \circ Q = \sum_{\alpha_0=0}^{m-1} \tilde{P}^{-1} \circ (x_0^{\alpha_0+1} D_0^{\alpha_0} B_{m-\alpha_0}),$$

and we want to study the action of the operators $B_{m-\alpha_0}$, $x_0^{\alpha_0+1} D_0^{\alpha_0}$, and $\tilde{P}^{-1}$on V_p.

1) We have

$$B_{m-\alpha_0}(x; D_{x'}) = \sum_{|\alpha'| \leq m-\alpha_0} a_{\alpha_0,\alpha'}(x_0, x') D_{x'}^{\alpha'}.$$

For all α such that $|\alpha| \leq m$ there is M_α for which $a_\alpha(x) \ll \frac{M_\alpha}{1-x_0R} \cdot \frac{1}{1-rt(x')}$.
Set

$$\mathcal{C}_{m-\alpha_0}(x; D_{x'}) = \frac{1}{1 - x_0R} \cdot \frac{1}{1 - rt(x')} \sum_{|\alpha'| \leq m-\alpha_0} M_{\alpha_0,\alpha'} D_{x'}^{\alpha'}.$$

By Definition 3.3, $B_{m-\alpha_0}(x;D_{x'}) \ll \mathcal{C}_{m-\alpha_0}(x;D_{x'})$. Let σ be in the open interval (s,s'), then

$$\begin{aligned}
B_{m-\alpha_0}(V_p)(x) &\ll \|V_0\| \frac{K^p}{(\sigma-s)^{mp}} \cdot \frac{1}{1-Rx_0} \cdot \frac{x_0^p}{1-\xi_0 x_0} \cdot \frac{1}{1-rt(x')} \\
&\quad \times \sum_{|\alpha'|\leq m-\alpha_0} M_{\alpha_0,\alpha'} D_{x'}^{\alpha'} \left(\frac{1}{1-\sigma t(x')} \right) \\
&\ll \|V_0\| \frac{K^p}{(\sigma-s)^{mp}} \cdot \frac{1}{1-Rx_0} \cdot \frac{x_0^p}{1-\xi_0 x_0} \cdot \frac{1}{1-rt(x')} \\
&\quad \times \sum_{|\alpha'|\leq m-\alpha_0} M_{\alpha_0,\alpha'} \frac{\sigma^{|\alpha'|}\,|\alpha'|!}{[1-\sigma t(x')]^{|\alpha'|+1}}.
\end{aligned}$$

One can check easily that

$$\begin{aligned}
\frac{1}{1-Rx_0} \cdot \frac{1}{1-\xi_0 x_0} &\ll \frac{1}{1-\frac{R}{\xi_0}} \cdot \frac{1}{1-\xi_0 x_0} \\
&\ll \frac{1}{1-\eta_0} \cdot \frac{1}{1-\xi_0 x_0}.
\end{aligned}$$

By using [8], we obtain the following majoration

$$\frac{|\alpha'|!}{[1-\sigma t(x')]^{|\alpha'|+1}} \ll \frac{(m-\alpha_0)!}{[1-\sigma t(x')]^{m-\alpha_0+1}}$$

hence

$$\begin{aligned}
B_{m-\alpha_0}(V_p)(x) \ll \|V_0\| &\frac{K^p}{(\sigma-s)^{mp}} \cdot \frac{1}{1-\eta_0} \cdot \frac{x_0^p}{1-\xi_0 x_0} \\
&\times \frac{1}{1-rt(x')} \cdot \frac{(m-\alpha_0)!}{[1-\sigma t(x')]^{m-\alpha_0+1}} \sum_{|\alpha'|\leq m-\alpha_0} M_{\alpha_0,\alpha'}\sigma^{|\alpha'|}.
\end{aligned}$$

Again by [8], there exists $\widetilde{\mathcal{C}}_{m-\alpha_0}(a,b)$ such that

$$\begin{aligned}
B_{m-\alpha_0}(V_p)(x) \ll \|V_0\| &\frac{K^p}{(\sigma-s)^{mp}} \cdot \frac{1}{1-\eta_0} R_{\alpha_0} \widetilde{\mathcal{C}}_{m-\alpha_0}(a,b) \frac{x_0^p}{1-\xi_0 x_0} \\
&\times \frac{1}{1-s't(x')} \cdot \frac{1}{1-rt(x')} \cdot \frac{(m-\alpha_0)!}{(s'-\sigma)^{m-\alpha_0}}
\end{aligned}$$

where $R_{\alpha_0} = \sum_{|\alpha'|\leq m-\alpha_0} M_{\alpha_0,\alpha'} b^{|\alpha'|}$.

If we let $a > \frac{r}{\eta_0}$, then

$$\frac{1}{1-s't(x')} \cdot \frac{1}{1-rt(x')} \ll \frac{1}{1-\eta_0} \cdot \frac{1}{1-s't(x')}.$$

Finally,

$$B_{m-\alpha_0}(V_p)(x) \ll \|V_0\| \frac{K^p}{(\sigma-s)^{mp}} \cdot \frac{R_{\alpha_0}}{(1-\eta_0)^2} \cdot \frac{\widetilde{\mathcal{C}}_{m-\alpha_0}(a,b)}{(s'-\sigma)^{m-\alpha_0}} \cdot \frac{1}{1-s't(x')} \cdot \frac{x_0^p}{1-\xi_0 x_0}.$$

2) A straightforward computation leads to the following majoration

$$x_0^{\alpha_0+1} D_0^{\alpha_0} B_{m-\alpha_0}(V_p)(x) \ll \|V_0\| \frac{K^p}{(\sigma - s)^{mp}} \cdot \frac{R_{\alpha_0}}{(1-\eta_0)^2} \cdot \frac{\widetilde{\mathcal{C}}_{m-\alpha_0}(a,b)}{(s'-\sigma)^{m-\alpha_0}} \times \left[\sum_{j=p+1}^{\infty} \xi_0^{j-p-1} C_{\alpha_0}(j-1) x_0^j\right] \frac{1}{1-s't(x')}.$$

Set

$$w_p(x) = x_0^{\alpha_0+1} D_0^{\alpha_0} B_{m-\alpha_0}(V_p)(x) = \sum_{j=0}^{\infty} \xi_0^{j-p-1} w_{p,j}(x') x_0^j,$$

hence

$$w_{p,j}(x') \ll \|V_0\| \frac{K^p}{(\sigma - s)^{mp}} \cdot \frac{R_{\alpha_0}}{(1-\eta_0)^2} \cdot \frac{\widetilde{\mathcal{C}}_{m-\alpha_0}(a,b)}{(s'-\sigma)^{m-\alpha_0}} \cdot \xi_0^{j-p-1} C_{\alpha_0}(j-1) \cdot \frac{1}{1-s't(x')}.$$

If

$$F_p(x) = \tilde{P}^{-1}(w_p)(x) = \sum_{j=0}^{\infty} F_{p,j}(x') x_0^j,$$

then

$$F_{p,j}(x') = \frac{w_{p,j}(x')}{\mathcal{C}_1(j,x')}.$$

It follows from Corollary 3.4 that

$$\begin{aligned} F_{p,j}(x') &\ll \|V_0\| \frac{K^p}{(\sigma - s)^{mp}} \cdot \frac{R_{\alpha_0}}{(1-\eta_0)^2} \cdot \frac{\widetilde{\mathcal{C}}_{m-\alpha_0}(a,b)}{(s'-\sigma)^{m-\alpha_0}} \\ &\quad \times \frac{\xi_0^{j-p-1}}{(j+1)^{m-\alpha_0}} \cdot \frac{1}{1-rt(x')} \cdot \frac{1}{1-s't(x')} \\ &\ll \|V_0\| \frac{K^p}{(\sigma - s)^{mp}} \cdot \frac{\tilde{R}_{\alpha_0}}{(s'-\sigma)^{m-\alpha_0}} \cdot \frac{\xi_0^{j-p-1}}{(p+1)^{m-\alpha_0}} \cdot \frac{1}{1-s't(x')}, \end{aligned}$$

for all $j \geq p+1$, where $\tilde{R}_{\alpha_0} = \frac{R_{\alpha_0}}{(1-\eta_0)^3} \widetilde{\mathcal{C}}_{m-\alpha_0}(a,b)$, hence

$$\begin{aligned} F_p(x) &\ll \|V_0\| \frac{K^p}{(\sigma - s)^{mp}} \cdot \frac{\tilde{R}_{\alpha_0}}{[(p+1)(s'-\sigma)]^{m-\alpha_0}} \left[\sum_{j=p+1}^{\infty} \xi_0^{j-p-1} x_0^j\right] \frac{1}{1-s't(x')} \\ &\ll \|V_0\| \frac{K^p}{(\sigma - s)^{mp}} \cdot \frac{\tilde{R}_{\alpha_0}}{[(p+1)(s'-\sigma)]^{m-\alpha_0}} \cdot \frac{x_0^{p+1}}{1-\xi_0 x_0} \cdot \frac{1}{1-s't(x')}. \end{aligned}$$

If we choose σ such that $s < \sigma < s'$ and $s' - \sigma = \frac{s'-s}{p+1}$ then the following majoration holds

$$\frac{1}{(\sigma - s)^{mp}} \cdot \frac{1}{[(p+1)(s'-\sigma)]^{m-\alpha_0}} \leq e^m \frac{(b-a)^{\alpha_0}}{(s'-s)^{m(p+1)}}$$

and consequently

$$F_p(x) \ll \|V_0\| \frac{K^p}{(s'-s)^{m(p+1)}} \cdot \tilde{R}_{\alpha_0} e^m (b-a)^{\alpha_0} \cdot \frac{x_0^{p+1}}{1-\xi_0 x_0} \cdot \frac{1}{1-s't(x')}.$$

Finally

$$V_{p+1}(x) \quad \ll \quad \|V_0\| \ \cdot \ \frac{K^p}{(s'-s)^{m(p+1)}} e^m \ \cdot \ \left[\sum_{\alpha_0=0}^{m-1} \tilde{R}_{\alpha_0}(b-a)^{\alpha_0}\right] \frac{x_0^{p+1}}{1-\xi_0 x_0} \ \cdot \ \frac{1}{1-s't(x')}.$$

Choosing then

$$K = e^m \sum_{\alpha_0=0}^{m-1} \tilde{R}_{\alpha_0}(b-a)^{\alpha_0}$$

and yields easily the lemma.

If we impose $|\xi_0 x_0| \leq \rho_0 < 1$ and $b(|x_1| + \cdots + |x_n|) \leq \rho_0 < 1$ then

$$|V_p(x)| \leq \|V_0\| \left[\frac{K\,|x_0|}{(s'-s)^m}\right]^p \frac{1}{(1-\rho_0)^2}$$

for all $p \in \mathbb{N}$ and all $s' > s$.

If $|x_0| \leq \frac{(s'-s)^m}{K'}$, where $K' > K$, then the series of general term V_p converges normally and the sequence of general term U_p converges uniformly to a holomorphic function U on some suitable choice of polydiscs centered at the origin in $\mathbb{C}^{n+1}$.

Since

$$\tilde{P}(U_{p+1}) = Q(U_p) + g,$$

then the limit U satisfies the equation

$$\tilde{P}(U) = Q(U) + g$$

therefore $h_1(U) = g$ as desired.

□

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