Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 2, Issue 3, Article 31, 2001

IMPROVEMENT OF AN OSTROWSKI TYPE INEQUALITY FOR MONOTONIC MAPPINGS AND ITS APPLICATION FOR SOME SPECIAL MEANS

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Received 10 January, 2001; accepted 23 April, 2001
Communicated by B. Mond

ABSTRACT. We first improve two Ostrowski type inequalities for monotonic functions, then provide its application for special means.

Key words and phrases: Ostrowski's inequality, Trapezoid inequality, Special means.

2000 *Mathematics Subject Classification.* 26D15, 26D10.

1. INTRODUCTION

In [1], Dragomir established the following Ostrowski's inequality for monotonic mappings.

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be a monotonic nondecreasing mapping on $[a,b]$. Then for all $x \in [a,b]$, we have the following inequality*

$$\begin{aligned}\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| &\le \frac{1}{b-a}\left\{[2x-(a+b)]f(x) + \int_a^b sgn(t-x)f(t)dt\right\} \\ &\le \frac{1}{b-a}[(x-a)(f(x)-f(a)) + (b-x)(f(b)-f(x))] \\ &\le \left[\frac{1}{2} + \frac{\left|x-\frac{a+b}{2}\right|}{b-a}\right](f(b)-f(a)). \qquad (1.1)\end{aligned}$$

ISSN (electronic): 1443-5756

Project supported by the National Natural Science Foundation of China (Grant No.10071038).
006-01

The constant $\frac{1}{2}$ is the best possible one.

In [2], Dragomir, Pečarić and Wang generalized Theorem 1.1 and proved

Theorem 1.2. *Let $f : [a, b] \to \mathbb{R}$ be a monotonic nondecreasing mapping on $[a, b]$ and t_1, t_2, $t_3 \in (a, b)$ be such that $t_1 \le t_2 \le t_3$. Then*

$$\begin{aligned}
&\left|\int_a^b f(x)dx - [(t_1 - a)f(a) + (t_3 - t_1)f(t_2) + (b - t_3)f(b)]\right| \\
&\qquad \le (b - t_3)f(b) + (2t_2 - t_1 - t_3)f(t_2) - (t_1 - a)f(a) + \int_a^b T(x)f(x)dx \\
&\qquad \le (b - t_3)(f(b) - f(t_3)) + (t_3 - t_2)(f(t_3) - f(t_2)) \\
&\qquad\qquad + (t_2 - t_1)(f(t_2) - f(t_1)) + (t_1 - a)(f(t_1) - f(a)) \\
&\qquad \le \max\{t_1 - a, t_2 - t_1, t_3 - t_2, b - t_3\}(f(b) - f(a)), \qquad (1.2)
\end{aligned}$$

where $T(x) = sgn(t_1 - x)$, for $x \in [a, t_2]$, and $T(x) = sgn(t_3 - x)$, for $x \in [t_2, b]$.

In the present paper, we firstly improve the above results, and then provide its application for some special means.

2. Main Result

We shall start with the following result.

Theorem 2.1. *Let $f : [a, b] \to \mathbb{R}$ be a monotonic nondecreasing mapping on $[a, b]$ and let t_1, t_2, $t_3 \in [a, b]$ be such that $t_1 \le t_2 \le t_3$. Then*

$$\begin{aligned}
&\left|\int_a^b f(x)dx - [(t_1 - a)f(a) + (t_3 - t_1)f(t_2) + (b - t_3)f(b)]\right| \\
&\qquad \le \max\{(b - t_3)(f(b) - f(t_3)) + (t_2 - t_1)(f(t_2) - f(t_1)), \\
&\qquad\qquad (t_3 - t_2)(f(t_3) - f(t_2)) + (t_1 - a)(f(t_1) - f(a))\} \qquad (2.1) \\
&\qquad \le \max\{t_1 - a, t_2 - t_1, t_3 - t_2, b - t_3\}(f(b) - f(a)). \qquad (2.2)
\end{aligned}$$

Proof. Since $f(x)$ is a monotonic nondecreasing mapping on $[a, b]$, we have

$$\begin{aligned}
&\left|\int_a^b f(x)dx - [(t_1 - a)f(a) + (t_3 - t_1)f(t_2) + (b - t_3)f(b)]\right| \\
&\quad = \left|\int_a^{t_1} (f(x) - f(a))dx + \int_{t_1}^{t_3} (f(x) - f(t_2))dx + \int_{t_3}^b (f(x) - f(b))dx\right| \\
&\quad = \left|\left[\int_a^{t_1} (f(x) - f(a))dx + \int_{t_2}^{t_3} (f(x) - f(t_2))dx\right]\right. \\
&\qquad \left. - \left[\int_{t_1}^{t_2} (f(t_2) - f(x))dx + \int_{t_3}^b (f(b) - f(x))dx\right]\right| \\
&\quad \le \max\{(b - t_3)(f(b) - f(t_3)) + (t_2 - t_1)(f(t_2) - f(t_1)), \\
&\qquad\qquad (t_3 - t_2)(f(t_3) - f(t_2)) + (t_1 - a)(f(t_1) - f(a))\} \\
&\quad \le \max\{t_1 - a, t_2 - t_1, t_3 - t_2, b - t_3\}(f(b) - f(a)).
\end{aligned}$$

Thus (2.1) and (2.2) is proved. □

For $t_1 = t_2 = t_3 = x$, Theorem 2.1 becomes the following corollary.

Corollary 2.2. *Let f be defined as in Theorem 2.1. Then*

$$\begin{aligned}\left|\int_a^b f(x)dx - [(x-a)f(a)+(b-x)f(b)]\right| \\ &\leq \max\{(b-x)(f(b)-f(x)),(x-a)(f(x)-f(a))\} \\ &\leq \max\{x-a,b-x\}\cdot\max\{(f(x)-f(a)),(f(b)-f(x))\} \\ &\leq \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right](f(b)-f(a)).\end{aligned}$$

For $x=\frac{a+b}{2}$, we get trapezoid inequality.

Corollary 2.3. *Let f be defined as in Theorem 2.1. Then*

$$\begin{aligned}\left|\int_a^b f(x)dx - \frac{f(a)+f(b)}{2}(b-a)\right| \\ &\leq \frac{b-a}{2}\max\left\{\left(f\left(\frac{a+b}{2}\right)-f(a)\right),\left(f(b)-f\left(\frac{a+b}{2}\right)\right)\right\} \\ &\leq \frac{1}{2}(b-a)(f(b)-f(a)).\end{aligned} \tag{2.3}$$

For $t_1=a$, $t_2=x$, $t_3=b$, we get Theorem 1.1.

3. APPLICATION FOR SPECIAL MEANS

In this section, we shall give application of Corollary 2.3. Let us recall the following means.

(1) The arithmetic mean:

$$A=A(a,b):=\frac{a+b}{2},\quad a,b\geq 0.$$

(2) The geometric mean:

$$G=G(a,b):=\sqrt{ab},\quad a,b\geq 0.$$

(3) The harmonic mean:

$$H=H(a,b):=\frac{2}{1/a+1/b},\quad a,b\geq 0.$$

(4) The logarithmic mean:

$$L=L(a,b):=\frac{b-a}{\ln b-\ln a},\quad a,b\geq 0, a\neq b;\ \text{If } a=b,\ \text{then } L(a,b)=a.$$

(5) The identric mean:

$$I=I(a,b):=\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}},\quad a,b\geq 0, a\neq b;\ \text{If } a=b,\ \text{then } I(a,b)=a.$$

(6) The p-logarithmic mean:

$$L_p=L_p(a,b):=\left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}},\quad a\neq b;\ \text{If } a=b,\ \text{then } L_p(a,b)=a,$$

where $p\neq -1,0$ and $a,b>0$.

The following simple relationships are known in the literature

$$H \leq G \leq L \leq I \leq A.$$

We are going to use inequality (2.3) in the following equivalent version:

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{f(a)+f(b)}{2}\right| \\
&\leq \frac{1}{2}\max\left\{\left(f\left(\frac{a+b}{2}\right)-f(a)\right),\left(f(b)-f\left(\frac{a+b}{2}\right)\right)\right\} \\
&\leq \frac{1}{2}(f(b)-f(a)),
\end{aligned} \tag{3.1}$$

where $f:[a,b]\to\mathbb{R}$ is monotonic nondecreasing on $[a,b]$.

3.1. **Mapping** $f(x)=x^p$.

Consider the mapping $f:[a,b]\subset(0,\infty)\to\mathbb{R},\ f(x)=x^p, p>0$. Then

$$\frac{1}{b-a}\int_a^b f(t)dt = L_p^p(a,b),$$

$$\frac{f(a)+f(b)}{2} = A(a^p,b^p),$$

$$f(b)-f(a) = p(b-a)L_{p-1}^{p-1}.$$

Then by (3.1), we get

$$\begin{aligned}
\left|L_p^p(a,b)-A(a^p,b^p)\right| &\leq \frac{1}{2}\max\left\{\left(\frac{a+b}{2}\right)^p - a^p, b^p - \left(\frac{a+b}{2}\right)^p\right\} \\
&= \frac{1}{2}\left[b^p - \left(\frac{a+b}{2}\right)^p\right] \\
&= \frac{1}{2}\left(b^p-a^p\right) - \frac{1}{2}\left(\left(\frac{a+b}{2}\right)^p - a^p\right) \\
&\leq \frac{1}{2}p(b-a)L_{p-1}^{p-1} - \frac{p(b-a)a^{p-1}}{4}.
\end{aligned} \tag{3.2}$$

Remark 3.1. The following result was proved in [2].

$$\left|L_p^p(a,b)-A(a^p,b^p)\right| \leq \frac{1}{2}p(b-a)L_{p-1}^{p-1}.$$

3.2. **Mapping** $f(x)=-1/x$.

Consider the mapping $f:[a,b]\subset(0,\infty)\to\mathbb{R},\ f(x)=-\frac{1}{x}$. Then

$$\frac{1}{b-a}\int_a^b f(t)dt = -L^{-1}(a,b),$$

$$\frac{f(a)+f(b)}{2} = -\frac{A(a,b)}{G^2(a,b)},$$

$$f(b)-f(a) = \frac{b-a}{G^2(a,b)}.$$

Then by (3.1), we get

$$\begin{aligned}\left|\frac{A(a,b)}{G^2(a,b)} - L^{-1}(a,b)\right| &\le \frac{1}{2}\max\left\{\frac{1}{a} - \frac{2}{a+b}, \frac{2}{a+b} - \frac{1}{b}\right\}\\ &= \frac{1}{2}\cdot\frac{b-a}{a(a+b)} = \frac{1}{2}\cdot\frac{b-a}{ab} - \frac{1}{2}\cdot\frac{b-a}{b(a+b)}\\ &\le \frac{1}{2}\cdot\frac{b-a}{G^2(a,b)} - \frac{1}{2}\cdot\frac{b-a}{b(a+b)}.\end{aligned}$$

Thus we get

$$0 \le AL - G^2 \le \frac{1}{2}\frac{b}{a+b}(b-a)L. \tag{3.3}$$

Remark 3.2. The following result was proved in [2].

$$0 \le AG - G^2 \le \frac{1}{2}(b-a)L.$$

3.3. **Mapping** $f(x) = \ln x$**.** Consider the mapping $f : [a,b] \subset (0,\infty) \to \mathbb{R},\ f(x) = \ln x$. Then

$$\frac{1}{b-a}\int_a^b f(t)dt = \ln I(a,b),$$

$$\frac{f(a)+f(b)}{2} = \ln G(a,b),$$

$$f(b) - f(a) = \frac{b-a}{L(a,b)}.$$

Then by (3.1), we get

$$\begin{aligned}|\ln I(a,b) - \ln G(a,b)| &\le \frac{1}{2}\max\left\{\ln\frac{a+b}{2} - \ln a, \ln b - \ln\frac{a+b}{2}\right\}\\ &= \frac{1}{2}\ln\frac{a+b}{2a} = \frac{1}{2}\frac{b-a}{L(a,b)} - \frac{1}{2}\ln\frac{2b}{a+b}.\end{aligned}$$

Thus we get

$$1 \le \frac{I}{G} \le \sqrt{\frac{a+b}{2b}}e^{\frac{1}{2}\cdot\frac{b-a}{L(a,b)}}. \tag{3.4}$$

Remark 3.3. The following result was proved in [2].

$$1 \le \frac{I}{G} \le e^{\frac{1}{2}\cdot\frac{b-a}{L(a,b)}}.$$

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