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L'HOSPITAL TYPE RULES FOR OSCILLATION, WITH APPLICATIONS

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ABSTRACT. An algorithmic description of the dependence of the oscillation pattern of the ratio $\frac{f}{g}$ of two functions f and g on the oscillation pattern of the ratio $\frac{f'}{g'}$ of their derivatives is given. This tool is then used in order to refine and extend the Yao-Iyer inequality, arising in bioequivalence studies. The convexity conjecture by Topsøe concerning information inequalities is addressed in the context of a general convexity problem. This paper continues the series of results begun by the l'Hospital type rule for monotonicity. Other applications of this rule are given elsewhere: to certain information inequalities, to monotonicity of the relative error of a Padé approximation for the complementary error function, and to probability inequalities for sums of bounded random variables.

Key words and phrases: L'Hospital's Rule, Monotonicity, Oscillation, Convexity, Yao-Iyer inequality, Bioequivalence studies, Information inequalities.

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1. L'HOSPITAL TYPE RULES FOR OSCILLATION

Let $-\infty \leq a < b \leq \infty$. Let f and g be differentiable functions defined on the interval (a, b).

Assume that g and g' are nonzero on (a, b), so that the ratios $\frac{f}{g}$ and $\frac{f'}{g'}$ are defined on (a, b). It follows that function g, being differentiable and hence continuous, does not change sign on (a, b). In other words, either $g > 0$ on the entire interval (a, b) or $g < 0$ on (a, b); assume that the same is true for g'.

The following result, which is reminiscent of the l'Hospital rule for computing limits, was stated and proved in [3].

Proposition 1.1. *Suppose that* $f(a+) = g(a+) = 0$ or $f(b-) = g(b-) = 0$.

(1) *If* $\frac{f'}{g'}$ *is increasing on* (a, b), *then* $\left(\frac{f}{g}\right)' > 0$ *on* (a, b), *and so,* $\frac{f}{g}$ *is increasing on* (a, b).

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(2) *If* $\frac{f'}{g'}$ *is decreasing on* (a,b)*, then* $\left(\frac{f}{g}\right)' < 0$ *on* (a,b)*, and so,* $\frac{f}{g}$ *is decreasing on* (a,b)*.*

Note that the conditions

(i) g' is nonzero and does not change sign on (a,b) and
(ii) $g(a+) = 0$ or $g(b-) = 0$

already imply that g is nonzero and does not change sign on (a,b); hence, one has $gg' > 0$ on (a,b) or $gg' < 0$ on (a,b).

In contrast with the l'Hospital Rule for limits, Proposition 1.1 may be generalized as follows, without requiring that f and g vanish at an endpoint of the interval.

Proposition 1.2. *Suppose that* $gg' > 0$ *on* (a,b)*,* $\limsup_{c\downarrow a} \frac{g(c)^2}{|g'(c)|}\left(\frac{f}{g}\right)'(c) \geq 0$*, and* $\frac{f'}{g'}$ *is increasing on* (a,b)*. Then* $\left(\frac{f}{g}\right)' > 0$ *on* (a,b)*.*

Proof. As in the proof of Proposition 1.1 in [3], fix any $x \in (a,b)$ and consider the function h_x defined by the formula

$$h_x(y) := f'(x)g(y) - g'(x)f(y).$$

For all $y \in (a,x)$,

$$\frac{d}{dy}h_x(y) = f'(x)g'(y) - g'(x)f'(y) = g'(x)g'(y)\left(\frac{f'(x)}{g'(x)} - \frac{f'(y)}{g'(y)}\right) > 0,$$

because g' is nonzero and does not change sign on (a,b) and $\frac{f'}{g'}$ is increasing on (a,b). Hence, the function h_x is increasing on (a,x); moreover, being continuous, h_x is increasing on $(a,x]$.

Now, fix any $c_0 \in (a,x)$. Then for all $c \in (a,c_0]$

$$\begin{aligned} f'(x)\left(g(x) - g(c)\right) - g'(x)\left(f(x) - f(c)\right) &= h_x(x) - h_x(c) \\ &\geq \varepsilon > 0, \end{aligned} \tag{1.1}$$

where

$$\varepsilon := h_x(x) - h_x(c_0). \tag{1.2}$$

Next,

$$g(x)^2\left(\frac{f}{g}\right)'(x) = f'(x)g(x) - g'(x)f(x) \tag{1.3}$$

$$= f'(x)\left(g(x) - g(c)\right) - g'(x)\left(f(x) - f(c)\right) \tag{1.4}$$

$$+ \left(\frac{f'(x)}{g'(x)} - \frac{f'(c)}{g'(c)}\right) \cdot g(c)g'(x) \tag{1.5}$$

$$+ \frac{g(c)^2}{|g'(c)|}\left(\frac{f}{g}\right)'(c) \cdot |g'(x)|\,; \tag{1.6}$$

here it is taken into account that g' is nonzero and does not change sign on (a,b), so that $\frac{g'(x)}{g'(c)} = \frac{|g'(x)|}{|g'(c)|}$.

Of the three summands in (1.4) – (1.6),

- in view of (1.1), the first summand, in (1.4), is no less than the fixed $\varepsilon > 0$ defined by (1.2), for all $c \in (a, c_0]$;

- the second summand, in (1.5), is nonnegative (and even positive) for all $c \in (a, c_0]$, because $\frac{f'}{g'}$ is increasing on (a, b) and $g(c)g'(x) > 0$; the latter inequality follows because $gg' > 0$ on (a, b) and g' does not change sign on (a, b);
- as to the last summand, in (1.6), its limit superior as $c \downarrow a$ is nonnegative, by the condition $\limsup_{c \downarrow a} \frac{g(c)^2}{|g'(c)|} \left(\frac{f}{g}\right)'(c) \geq 0$.

On the other hand, the left-hand side of (1.3), which is the sum of the three summands in (1.4) – (1.6), does not depend on c. Now the inequality $\left(\frac{f}{g}\right)'(x) \geq \varepsilon[> 0]$ follows if we let $c \downarrow a$. □

Corollary 1.3. (1) *If* $gg' > 0$ *on* (a, b)*,* $\limsup_{x \downarrow a} \frac{g(x)^2}{|g'(x)|} \left(\frac{f}{g}\right)'(x) \geq 0$*, and* $\frac{f'}{g'}$ *is increasing on* (a, b)*, then* $\left(\frac{f}{g}\right)' > 0$ *on* (a, b)*.*

(2) *If* $gg' > 0$ *on* (a, b)*,* $\liminf_{x \downarrow a} \frac{g(x)^2}{|g'(x)|} \left(\frac{f}{g}\right)'(x) \leq 0$*, and* $\frac{f'}{g'}$ *is decreasing on* (a, b)*, then* $\left(\frac{f}{g}\right)' < 0$ *on* (a, b)*.*

(3) *If* $gg' < 0$ *on* (a, b)*,* $\liminf_{x \uparrow b} \frac{g(x)^2}{|g'(x)|} \left(\frac{f}{g}\right)'(x) \leq 0$*, and* $\frac{f'}{g'}$ *is decreasing on* (a, b)*, then* $\left(\frac{f}{g}\right)' < 0$ *on* (a, b)*.*

(4) *If* $gg' < 0$ *on* (a, b)*,* $\limsup_{x \uparrow b} \frac{g(x)^2}{|g'(x)|} \left(\frac{f}{g}\right)'(x) \geq 0$*, and* $\frac{f'}{g'}$ *is increasing on* (a, b)*, then* $\left(\frac{f}{g}\right)' > 0$ *on* (a, b)*.*

Proof. Part 1 of Corollary 1.3 repeats Proposition 1.2. Part 2 can be obtained from Part 1 by replacing f by $-f$. Then Parts 3 and 4 can be obtained from Parts 1 and 2 by replacing $f(x)$ and $g(x)$ for all $x \in (a, b)$ by $f(a + b - x)$ and $g(a + b - x)$, respectively. □

Remark 1.4. *As seen from the proof of Proposition 1.2, the following variant of Corollary 1.3 holds. Fix any* $c \in (a, b)$*.*

(1) *If* $gg' > 0$ *on* (c, b)*,* $\left(\frac{f}{g}\right)'(c) \geq 0$*, and* $\frac{f'}{g'}$ *is increasing on* (c, b)*, then* $\left(\frac{f}{g}\right)' > 0$ *on* (c, b)*.*

(2) *If* $gg' > 0$ *on* (c, b)*,* $\left(\frac{f}{g}\right)'(c) \leq 0$*, and* $\frac{f'}{g'}$ *is decreasing on* (c, b)*, then* $\left(\frac{f}{g}\right)' < 0$ *on* (c, b)*.*

(3) *If* $gg' < 0$ *on* (a, c)*,* $\left(\frac{f}{g}\right)'(c) \leq 0$*, and* $\frac{f'}{g'}$ *is decreasing on* (a, c)*, then* $\left(\frac{f}{g}\right)' < 0$ *on* (a, c)*.*

(4) *If* $gg' < 0$ *on* (a, c)*,* $\left(\frac{f}{g}\right)'(c) \geq 0$*, and* $\frac{f'}{g'}$ *is increasing on* (a, c)*, then* $\left(\frac{f}{g}\right)' > 0$ *on* (a, c)*.*

Remark 1.5. It may not be immediately obvious that Proposition 1.2 — or, rather, Corollary 1.3 — is indeed a generalization of Proposition 1.1. However, the conditions

$$\limsup_{x\downarrow a} \frac{g(x)^2}{|g'(x)|} \left(\frac{f}{g}\right)'(x) \geq 0$$

and

$$\liminf_{x\uparrow b} \frac{g(x)^2}{|g'(x)|} \left(\frac{f}{g}\right)'(x) \geq 0$$

are necessary for $\frac{f}{g}$ to be increasing on (a,b), because then $\left(\frac{f}{g}\right)' \geq 0$ on (a,b). Therefore, by Part 1 (say) of Proposition 1.1, its conditions imply that

$$\limsup_{x\downarrow a} \frac{g(x)^2}{|g'(x)|} \left(\frac{f}{g}\right)'(x) \geq 0$$

and

$$\liminf_{x\uparrow b} \frac{g(x)^2}{|g'(x)|} \left(\frac{f}{g}\right)'(x) \geq 0.$$

Finally, as already mentioned, the condition of Corollary 1.3 that $gg' > 0$ on (a,b) or $gg' < 0$ on (a,b) obviously follows from the conditions of Proposition 1.1 that g' does not change sign on (a,b) and $g(a+) = 0$ or $g(b-) = 0$. Thus, Parts 1 and 4 of Corollary 1.3 generalize Part 1 of Proposition 1.1; similarly, Parts 2 and 3 of Corollary 1.3 generalize Part 2 of Proposition 1.1.

Remark 1.6. Another possible question is whether the condition $\limsup_{x\downarrow a} \frac{g(x)^2}{|g'(x)|} \left(\frac{f}{g}\right)'(x) \geq 0$ of Proposition 1.2 may be replaced by the simpler condition $\limsup_{x\downarrow a} \left(\frac{f}{g}\right)'(x) \geq 0$, which is necessary as well for $\frac{f}{g}$ to be increasing on (a,b). The answer is no; the condition $\limsup_{x\downarrow a} \left(\frac{f}{g}\right)'(x) \geq 0$, or even the condition $\left(\frac{f}{g}\right)'(a+) \geq 0$, is too weak.

A generic counter-example may be constructed as follows. Let f and G be functions defined on $\mathbb{R}$ such that

- $f(0) = 0$, $f'(0+) = f'(0) = 0$, and $f' < 0$ on $(0,\infty)$, so that $f \leq 0$ on $[0,\infty)$;
- $G > 0$, $G' < 0$, and $G'' > 0$ on $(-\infty, 0]$;

for instance, one may choose here $f(x) = -x^2\ \forall x \in \mathbb{R}$ and $G(y) = e^{-y}\ \forall y \in \mathbb{R}$. Next, define g by the formula

$$g(x) := G(f(x)), \quad x \in \mathbb{R}.$$

Then

- $g > 0$ and $g' > 0$ on $(0,\infty)$, so that $gg' > 0$ on $(0,\infty)$;
- $\left(\frac{f'}{g'}\right)'(x) = -\frac{G''(f(x))f'(x)}{G'(f(x))^2} > 0$ for $x > 0$, so that $\frac{f'}{g'}$ is increasing on $(0,\infty)$;
- $\left(\frac{f}{g}\right)'(0+) = 0$.

Thus, all the conditions of Proposition 1.2 would be satisfied for $a = 0$ and any $b > 0$ — if only the condition $\limsup_{x\downarrow a} \frac{g(x)^2}{|g'(x)|} \left(\frac{f}{g}\right)'(x) \geq 0$ were replaced by $\left(\frac{f}{g}\right)'(0+) \geq 0$.

Nonetheless, one has $\left(\frac{(f/g)'}{f'}\right)(0+) = \frac{1}{G(0)} > 0$, so that $\left(\frac{f}{g}\right)' < 0$ in a right neighborhood $(0, \delta)$ of 0, so that $\frac{f}{g}$ is *not* increasing on $(0, \delta)$.

This counter-example shows that the condition $\limsup_{x \downarrow a} \left(\frac{f}{g}\right)'(x) \geq 0$, or even the condition $\left(\frac{f}{g}\right)'(a+) \geq 0$, is just too easy to satisfy — it is enough to let $f'(a+) = 0$ and $g = G \circ f$, and then one can have $\left(\frac{f}{g}\right)'(a+) = 0$.

On the other hand, if it is required that $\left(\frac{f}{g}\right)'(0+) > 0$ — or just that $\limsup_{x \downarrow a} \left(\frac{f}{g}\right)'(x) > 0$, then the condition $\limsup_{x \downarrow a} \frac{g(x)^2}{|g'(x)|} \left(\frac{f}{g}\right)'(x) \geq 0$ obviously follows. Moreover, it is seen from the proof of Proposition 1.2 that in this case the condition that $\frac{f'}{g'}$ is increasing on (a, b) may be relaxed to the condition that $\frac{f'}{g'}$ is non-decreasing on (a, b). Thus, one has

Proposition 1.7. *If $gg' > 0$ on (a, b), $\limsup_{x \downarrow a} \left(\frac{f}{g}\right)'(x) > 0$, and $\frac{f'}{g'}$ is non-decreasing on (a, b), then $\left(\frac{f}{g}\right)' > 0$ on (a, b).*

Remark 1.8. Proposition 1.7 may also be complemented by the other three similar cases, just as Cases 2, 3, and 4 of Corollary 1.3 complement Proposition 1.2.

Similarly, the conditions that $\left(\frac{f}{g}\right)'(c) \geq 0$ and $\frac{f'}{g'}$ is increasing on (c, b) in Part 1 of Remark 1.4 may be replaced by the conditions that $\left(\frac{f}{g}\right)'(c) > 0$ and $\frac{f'}{g'}$ is non-decreasing on (c, b), with the same conclusion to hold: $\left(\frac{f}{g}\right)' > 0$ on (c, b); similar changes may be made in the other three parts of Remark 1.4.

What can be said in the absence of restrictions like $\limsup_{x \downarrow a} \frac{g(x)^2}{|g'(x)|} \left(\frac{f}{g}\right)'(x) \geq 0$? Here is an answer.

Proposition 1.9.

(1) *Suppose that $gg' > 0$ and $\frac{f'}{g'}$ is increasing on (a, b). Then there is some $c \in [a, b]$ such that $\left(\frac{f}{g}\right)' < 0$ on (a, c) and $\left(\frac{f}{g}\right)' > 0$ on (c, b). (In particular, if $c = a$, then $\left(\frac{f}{g}\right)' > 0$ on the entire interval (a, b); if $c = b$, then $\left(\frac{f}{g}\right)' < 0$ on (a, b).)*

(2) *Suppose that* $gg' > 0$ *and* $\frac{f'}{g'}$ *is decreasing on* (a,b). *Then there is some* $c \in [a,b]$ *such that* $\left(\frac{f}{g}\right)' > 0$ *on* (a,c) *and* $\left(\frac{f}{g}\right)' < 0$ *on* (c,b).

(3) *Suppose that* $gg' < 0$ *and* $\frac{f'}{g'}$ *is increasing on* (a,b). *Then there is some* $c \in [a,b]$ *such that* $\left(\frac{f}{g}\right)' > 0$ *on* (a,c) *and* $\left(\frac{f}{g}\right)' < 0$ *on* (c,b).

(4) *Suppose that* $gg' < 0$ *and* $\frac{f'}{g'}$ *is decreasing on* (a,b). *Then there is some* $c \in [a,b]$ *such that* $\left(\frac{f}{g}\right)' < 0$ *on* (a,c) *and* $\left(\frac{f}{g}\right)' > 0$ *on* (c,b).

Proof. Let us prove Part 1 of the proposition; thus, we assume here that $gg' > 0$ and $\frac{f'}{g'}$ is increasing on (a,b). Let

$$E := \left\{ x \in (a,b) : \left(\frac{f}{g}\right)'(x) \geq 0 \right\}.$$

If $E = \emptyset$, then $\left(\frac{f}{g}\right)' < 0$ on (a,b), which implies the conclusion of Part 1 of the proposition, with $c := b$.

If $E \neq \emptyset$, let $c := \inf E$, so that $c \in [a,b)$. If $c \notin E$, then there exists a sequence (c_n) in E such that $c_n \downarrow c$. Then $\left(\frac{f}{g}\right)'(c_n) \geq 0$ for all n, and so,

$$\limsup_{x \downarrow c} \frac{g(x)^2}{|g'(x)|} \left(\frac{f}{g}\right)'(x) \geq 0.$$

Therefore, according to Proposition 1.2, $\left(\frac{f}{g}\right)' > 0$ on (c,b). If $c \in E$, then $c \in (a,b)$ and $\left(\frac{f}{g}\right)'(c) \geq 0$; using now Part 1 of Remark 1.4, one comes to the same conclusion — that $\left(\frac{f}{g}\right)' > 0$ on (c,b). On the other hand, by the construction of E and c, one has $\left(\frac{f}{g}\right)' < 0$ on (a,c). This implies that the conclusion of Part 1 of the proposition holds in the case $E \neq \emptyset$, too.

The other three parts of the proposition follow from Part 1 of it; cf. the proof of Corollary 1.3. □

Theorem 1.10. *Suppose that* $gg' > 0$ *and* $\frac{f'}{g'}$ *is increasing on* (a,b).

(1) *The following statements are equivalent:*
   (a) $\left(\frac{f}{g}\right)' > 0$ *on* (a,b);
   (b) $\left(\frac{f}{g}\right)' > 0$ *in a right neighborhood of* a.

(2) *The following statements are equivalent:*
   (a) $\exists c \in (a,b)$ $\left(\frac{f}{g}\right)' < 0$ *on* (a,c) *and* $\left(\frac{f}{g}\right)' > 0$ *on* (c,b);

(b) $\left(\frac{f}{g}\right)' < 0$ *in a right neighborhood of* a *and* $\left(\frac{f}{g}\right)' > 0$ *in a left neighborhood of* b*.*

(3) *The following statements are equivalent:*

(a) $\left(\frac{f}{g}\right)' < 0$ *on* (a, b)*;*

(b) $\left(\frac{f}{g}\right)' < 0$ *in a left neighborhood of* b*.*

This theorem is immediate from Part 1 of Proposition 1.9.

Remark 1.11. If the condition that $\frac{f'}{g'}$ is increasing on (a, b) in the preamble of Theorem 1.10 is replaced by the condition that $\frac{f'}{g'}$ is decreasing on (a, b), then all of the conclusions of the theorem will hold provided that all the inequality signs in them are switched to the opposite ones. Similarly, if the condition $gg' > 0$ in the preamble of Theorem 1.10 is replaced by $gg' < 0$, then all of the conclusions of the theorem will hold provided that all the inequality signs in them are switched to the opposite ones. If both conditions in the preamble of Theorem 1.10 are switched to the opposite ones — $\frac{f'}{g'}$ is decreasing on (a, b) and $gg' < 0$, then all the three parts of Theorem 1.10 will hold without any changes. — Cf. Parts 2, 3, and 4 of Proposition 1.9.

Thus, Theorem 1.10 and Remark 1.11 provide a **complete qualitative description** of the oscillation pattern of $\frac{f}{g}$ on an interval of monotonicity of $\frac{f'}{g'}$ based on the *local* behavior of $\frac{f}{g}$ near the endpoints of the interval.

Remark 1.12. Yet, whenever possible and more convenient, Proposition 1.1 may be used instead of the more general Theorem 1.10 and Remark 1.11.

Remark 1.13. In Part 1(b) of Theorem 1.10, the condition that $\left(\frac{f}{g}\right)' > 0$ in a right neighborhood of the left endpoint a may be relaxed to the condition that $\frac{f}{g}$ is non-decreasing in a right neighborhood of a, with Part 1(a) of the theorem to hold. (Similar changes may be made in the other two parts of Theorem 1.10, as well as concerning Remark 1.11.) This too follows from Proposition 1.9, because in each of the four parts of Proposition 1.9 the conclusion implies that $\left(\frac{f}{g}\right)'$ is nonzero and does not change sign in some right neighborhood of a and the same is true for some left neighborhood of b; hence, under the conditions of Proposition 1.9 or, equivalently, under those of Theorem 1.10 and Remark 1.11, if (say) $\frac{f}{g}$ is non-decreasing in a right neighborhood of a, then $\left(\frac{f}{g}\right)' > 0$ in a right neighborhood of a.

Remark 1.14. In all the above statements, the "strict" terms "increasing" and "decreasing" and the "strict" signs " $>$" and " $<$" may be replaced, simultaneously and throughout, by their "non-strict" counterparts: "non-decreasing", "non-increasing", " $\geq$" and " $\leq$"respectively (however, it still must be assumed that g and g' are nonzero on (a, b), just for the ratios $\frac{f}{g}$ and $\frac{f'}{g'}$ to be defined on (a, b)).

In particular, it follows that the conditions $gg' > 0$ on (a, b) or $gg' < 0$ on (a, b), $\lim_{x \downarrow a} \frac{g(x)^2}{|g'(x)|} \left(\frac{f}{g}\right)'(x)$ $= 0$, and $\frac{f'}{g'}$ is constant on (a, b) imply that $\frac{f}{g}$ is constant on (a, b).

Even if the ratio $\frac{f'}{g'}$ is not monotone, something can be said on the behavior of $\frac{f}{g}$ based on that of $\frac{f'}{g'}$. Below we shall state the most general result of this work, Theorem 1.16. Toward that end, we need the following two definitions, which will be accompanied by discussion.

Definition 1.1. Let us say that a function ρ is n *waves up* on the interval (a, b), where n is a natural number, if there exist real numbers $a_0 = a < a_1 < \cdots < a_n = b$ (which we shall call the *switching points for* ρ) such that ρ is increasing on the intervals (a_{2j}, a_{2j+1}) for all $j \in \left\{0, 1, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor\right\}$ and decreasing on the intervals (a_{2j+1}, a_{2j+2}) for all $j \in \left\{0, 1, \ldots, \left\lfloor \frac{n-2}{2} \right\rfloor\right\}$; here, as usual, $\lfloor x \rfloor$ stands for the integer part of a real number x. In such a situation, one might prefer to say " n quarter-waves up" rather than " n waves up".

Definition 1.2. If, for some natural number n, a function ρ is n waves up on the interval (a, b) with the switching points $a_0, a_1, \ldots, a_n$ and r is another function defined on (a, b), let us say that *the waves of* r *on* (a, b) *follow the waves of* ρ if there exist some nonnegative integer m and real numbers $c_{-1} = a \le c_0 < c_1 < \cdots < c_m = b$ such that

(1) r is increasing on the intervals (c_{2i}, c_{2i+1}) for all $i \in \left\{0, 1, \ldots, \left\lfloor \frac{m-1}{2} \right\rfloor\right\}$ and decreasing on the intervals (c_{2i+1}, c_{2i+2}) for all $i \in \left\{-1, 0, 1, \ldots, \left\lfloor \frac{m-2}{2} \right\rfloor\right\}$;
(2) there is a strictly increasing map $\{0, 1, \ldots, m-1\} \ni k \mapsto \ell(k) \in \{0, 1, \ldots\}$ such that for all $k \in \{0, 1, \ldots, m-1\}$ one has
  (i) $\ell(k) \in \{0, 1, \ldots, n-1\}$,
  (ii) $c_k \in [a_{\ell(k)}, a_{\ell(k)+1})$, and
  (iii) $\ell(k)$ is even iff k is even.

We make standard assumptions such as $\{0, 1, \ldots, m-1\} = \emptyset$ if $m = 0$. Hence, if $m = 0$, then necessarily the map $\ell = \emptyset$ (as usual, a map is understood as a set of ordered pairs satisfying certain conditions).

Mutually interchanging the terms "increasing" and "decreasing" (concerning only ρ and r but, of course, not ℓ) in Definitions 1.1 and 1.2, one can define the notions "ρ is n waves *down* on (a, b)" and, in the latter case too, the notion "the waves of r on (a, b) follow the waves of ρ". If ρ is constant on (a, b), let us say that ρ is 0 waves up and 0 waves down on (a, b).

Note that Condition 2 of Definition 1.2 implies $m \le n$, while Condition 1 of Definition 1.2 implies that either r is m waves up on (a, b) (when $c_0 = a$) or $m + 1$ waves down on (a, b) (when $c_0 > a$).

Also, since the intervals $[a_j, a_{j+1})$ are disjoint for different j's, the Condition 2(ii) of Definition 1.2 implies that the map ℓ is uniquely determined.

Remark 1.15. Somewhat informally, the phrase "the waves of r follow the waves of ρ" may be restated this way: as one proceeds from left to right,

(i) r may switch from decrease to increase only on intervals of increase of ρ and
(ii) r may switch from increase to decrease only on intervals of decrease of ρ;

the intervals of increase/decrease of ρ are considered here to be semi-open, with the left endpoints included, except for the left-most interval, which is open.

Example 1.1. Suppose that a function ρ is increasing on (a_0, a_1) and decreasing on (a_1, a_2), where $a_0 = a < a_1 < a_2 = b$, so that ρ is n waves up on $(a, b) = (a_0, a_2)$, with $n = 2$; suppose further that the waves of a function r on (a, b) follow the waves of ρ. Then exactly one of the following five cases takes place:

(1) r is decreasing on the entire interval (a_0, a_2), which corresponds to $m = 0$, $c_0 = a_2$, and $\ell = \emptyset$ in Definition 1.2;
(2) r is increasing on the entire interval (a_0, a_2), which corresponds to $m = 1$, $c_0 = a_0$, and $\ell(0) = 0$;
(3) there is some $c_0 \in (a_0, a_1)$ such that r is decreasing on (a_0, c_0) and increasing on (c_0, a_2), which corresponds to $m = 1$, $c_0 > a_0$, and $\ell(0) = 0$;
(4) there is some $c_1 \in [a_1, a_2)$ such that r is increasing on (a_0, c_1) and decreasing on (c_1, a_2), which corresponds to $m = 2$, $c_0 = a_0$, $\ell(0) = 0$, and $\ell(1) = 1$;
(5) there are some $c_0 \in (a_0, a_1)$ and $c_1 \in [a_1, a_2)$ such that r is decreasing on (a_0, c_0), increasing on (c_0, c_1), and decreasing on (c_1, a_2); this corresponds to $m = 2$, $c_0 > a_0$, $\ell(0) = 0$, and $\ell(1) = 1$.

In particular, if ρ is 2 waves up on (a, b) and the waves of r on (a, b) follow the waves of ρ, then it follows that r is at most 2 waves up or at most 3 waves down on (a, b).

Definitions 1.1 and 1.2 are also illustrated below in Example 1.2 and, especially, in Example 1.3. The following is a further generalization of the previous results.

Theorem 1.16. *Suppose that $gg' > 0$ and $\frac{f'}{g'}$ is n waves up on (a, b), where n is a natural number. Then*

(1) *the waves of $\frac{f}{g}$ on (a, b) follow the waves of $\frac{f'}{g'}$;*
(2) *in particular, $\frac{f}{g}$ is at most n waves up or at most $n+1$ waves down on (a, b), depending on whether $\left(\frac{f}{g}\right)' > 0$ in a right neighborhood of a or not.*

In addition to this theorem, if $a_0, a_1, \ldots, a_n$ are the switching points for $\frac{f'}{g'}$, then on each of the intervals (a_{i-1}, a_i), $i = 1, \ldots, n$, of the monotonicity of $\frac{f'}{g'}$, the increase/decrease pattern of $\frac{f}{g}$ can be determined according to Theorem 1.10 and Remark 1.11 (or, alternatively, according to Proposition 1.1; cf. Remark 1.12).

Thus, Theorem 1.16, Theorem 1.10, Remark 1.11, and Proposition 1.1 provide a **complete qualitative description** of how the oscillation pattern of $\frac{f'}{g'}$ on (a, b) and the local behavior of $\frac{f}{g}$ near the endpoints of (a, b) and near the switching points of $\frac{f'}{g'}$ in (a, b) determine the oscillation pattern of $\frac{f}{g}$ on (a, b).

Proof of Theorem 1.16. Let $\rho := \frac{f'}{g'}$ and $r := \frac{f}{g}$. In view of Remark 1.15, an informal proof of the theorem is immediate from Proposition 1.9, Theorem 1.10, Remark 1.11, and Remark 1.4.

Indeed, Proposition 1.9 implies that, on any interval of increase (decrease) of ρ, only a switch from decrease to increase (respectively, from increase to decrease) of r may occur. Moreover, Remark 1.4 implies that, at the left endpoint a_{k-1} of any interval $[a_{k-1}, a_k)$ of increase (decrease) of ρ, only a switch from decrease to increase (respectively, from increase to decrease) of r may occur. Thus, one has Part 1 of the theorem. Part 2 of the theorem follows by Theorem 1.10.

The formal proof of Theorem 1.16 is conducted by induction in n, as follows.

Let us begin with $n = 1$. Then ρ is increasing on the entire interval $(a, b) = (a_0, a_1)$, and the statement of the theorem follows by Part 1 of Proposition 1.9 and Part 1 of Theorem 1.10, with $c_0 := c$; at that, $m = 1$ and $\ell(0) = 0$ if $c < b$; $m = 0$ and $\ell = \emptyset$ if $c = b$.

Let now $n \in \{2, 3, \ldots\}$. By induction, there are some $m_1 \in \{0, \ldots, n-1\}$ and $c_{-1} = a \leq c_0 < c_1 < \cdots < c_{m_1} = a_{n-1}$ such that

(1) r is increasing on the intervals (c_{2i}, c_{2i+1}) for all $i \in \left\{0, 1, \ldots, \left\lfloor \frac{m_1 - 1}{2} \right\rfloor \right\}$ and decreasing on the intervals (c_{2i+1}, c_{2i+2}) for all $i \in \left\{-1, 0, 1, \ldots, \left\lfloor \frac{m_1 - 2}{2} \right\rfloor \right\}$ and

(2) there is a strictly increasing map $\{0, 1, \ldots, m_1 - 1\} \ni k \mapsto \ell(k) \in \{0, 1, \ldots\}$ such that for all $k \in \{0, 1, \ldots, m_1 - 1\}$ one has
   (i) $\ell(k) \in \{0, 1, \ldots, n-2\}$,
   (ii) $c_k \in [a_{\ell(k)}, a_{\ell(k)+1})$, and
   (iii) $\ell(k)$ is even iff k is even.

Further, there may be four cases, depending on whether n and m_1 are even or odd.

Case 1.1. $n = 2p$, $m_1 = 2q$, both even Then $\left\lfloor \frac{n-1}{2} \right\rfloor = \left\lfloor \frac{n-2}{2} \right\rfloor = p - 1$, $\left\lfloor \frac{m_1 - 1}{2} \right\rfloor = \left\lfloor \frac{m_1 - 2}{2} \right\rfloor = q - 1$, ρ is decreasing on (a_{n-1}, a_n), and r is decreasing on $(c_{m_1 - 1}, a_{n-1})$, because $(a_{n-1}, a_n) = \big(a_{2(p-1)+1}, a_{2(p-1)+2}\big)$ and $(c_{m_1-1}, a_{n-1}) = \big(c_{2(q-1)+1}, c_{2(q-1)+2}\big)$; moreover, r is decreasing on $(c_{m_1-1}, a_{n-1}]$, since r is differentiable and hence continuous. It follows that $r'(a_{n-1}) \leq 0$. Hence, by Part 2 of Remark 1.4, r is decreasing on (a_{n-1}, a_n). Therefore, r is decreasing on (c_{m_1-1}, a_n). Let now $m := m_1$, redefine $c_{m_1} = c_m$ as a_n, and retain the map ℓ. Then, with such m, $c_0, c_1, \ldots, c_m$, and ℓ, one sees that indeed the waves of $r = \frac{f}{g}$ follow the waves of $\rho = \frac{f'}{g'}$. In particular, it is seen now from Definition 1.2 and Theorem 1.10 that $\frac{f}{g}$ is at most n waves up or at most $n+1$ waves down on (a, b), depending on whether $\left(\frac{f}{g}\right)' > 0$ in a right neighborhood of a or not.

Case 1.2. n even, m_1 odd Then ρ is decreasing on (a_{n-1}, a_n) and r is increasing on (c_{m_1-1}, a_{n-1}). Hence, by Part 2 of Proposition 1.9, there is some $c \in [a_{n-1}, a_n]$ such that r is increasing on (a_{n-1}, c) and decreasing on (c, a_n). It follows that r is increasing on (c_{m_1-1}, c). Now, if $c < a_n$, let $m := m_1 + 1$, redefine $c_{m-1} = c_{m_1}$ as c, let $c_m := a_n$ and $\ell(m-1) := n-1$, and retain the previously defined values $\ell(k)$ for all $k \in \{1, \ldots, m_1 - 1\} = \{1, \ldots, m-2\}$. If $c = a_n$, let $m := m_1$, redefine $c_{m_1} = c_m$ as a_n, and retain the map ℓ. Then the sought conclusion again follows.

The other two cases, **Case 2.1. n odd, m_1 even** and **Case 2.2. n odd, m_1 odd**, are quite similar. Namely, Case 2.1 is similar to Case 1.2, and Case 2.2 is similar to Case 1.1. □

Remark 1.17. Theorem 1.16 holds if the terms "up" and "down" are mutually interchanged everywhere in the statement. The effect of replacing of $gg' > 0$ by $gg' < 0$ is that either in the assumption regarding the waves of $\frac{f'}{g'}$ or in the conclusion regarding the waves of $\frac{f}{g}$ (but not in both) the terms "up" and "down" must be mutually interchanged; cf. Remark 1.11.

As Theorem 1.16 shows, there is a relation between the functions $r = \frac{f}{g}$ and $\rho = \frac{f'}{g'}$. Next, we shall look at their relation from another viewpoint. Let us write

$$r' = \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} = \frac{\rho - r}{g/g'}.$$

If we now let

$$h := \frac{g'}{g}, \tag{1.7}$$

then

$$\rho = r + \frac{r'}{h}; \tag{1.8}$$

moreover, since each of the functions g and g' does not change sign on the interval, one sees that

$$h \text{ does not change sign} \tag{1.9}$$

on the interval. Vice versa, if (1.9) is true, then solving (1.7) for g yields

$$|g(x)| = \exp \int h(x)\,dx \quad \text{and} \quad f(x) = r(x)g(x), \tag{1.10}$$

so that f and g can be restored (up to a nonzero constant factor) based on r and h, where r is an arbitrary differentiable function and h is any function which is nowhere zero and does not change sign (on the interval). Thus, r and h can serve as a kind of "free parameters" to represent all admissible pairs $r = \frac{f}{g}$ and $\rho = \frac{f'}{g'}$, via (1.8).

Another helpful observation is immediate from (1.8) and (1.7) (as usual, it is assumed that $\operatorname{sign} u = 1$ if $u > 0$, $\operatorname{sign} u = -1$ if $u < 0$, and $\operatorname{sign} u = 0$ if $u = 0$):

Proposition 1.18.

(1) *Let* $gg' > 0$ *on* (a, b). *Then* $\operatorname{sign} r' = \operatorname{sign}(\rho - r)$ *on* (a, b).
(2) *Let* $gg' < 0$ *on* (a, b). *Then* $\operatorname{sign} r' = -\operatorname{sign}(\rho - r)$ *on* (a, b).

This proposition quite agrees with the intuition. Indeed, if one is only interested in local behavior of ρ and r in a neighborhood of a point $x \in (a, b)$, then one may re-define f and g in a right neighborhood of a — without changing values of f and g in the neighborhood of point x — in such a way that $f(a+) = g(a+) = 0$. Let us interpret (a, b) as a time interval. Then, for y in the neighborhood of the interior point x of (a, b), the ratio

$$r(y) = \frac{f(y)}{g(y)} = \frac{f(y) - f(a+)}{g(y) - g(a+)} \tag{1.11}$$

may be interpreted as the *average* rate of change of f relative to g over the time interval (a, y), while

$$\rho(y) = \frac{f'(y)}{g'(y)}$$

is interpreted as the *instantaneous* rate of change of f relative to g at the time moment y. Intuitively, it is clear that, if at some time moment y the instantaneous relative rate ρ exceeds (say) the average relative rate r, then the latter must be increasing at that point of time, and vice versa. A corroboration of this comes from the generalized mean value theorem, which implies, in view of (1.11), that $r(y) = \rho(z)$ for some $z \in (a, y)$, whence $\rho(y) > r(y)$ provided that ρ is increasing on $(a, y]$.

Now we are ready to complement Theorem 1.16 by

Proposition 1.19. *For ρ and r as in Theorem 1.16 or Remark 1.17, an equality of the form*

$$c_k = a_{\ell(k)}, \text{ for some } k = 0, 1, \ldots, m-1 \tag{1.12}$$

(which is admissible according to Definition 1.2, Part 2(ii)) in fact is only possible if c_k is the left endpoint a of the interval (a, b); that is, only if $k = \ell(k) = 0$.

Proof. Assume the contrary — that under the conditions of Theorem 1.16 (say), $c := c_k$ is an interior point of (a, b) such that (1.12) takes place. Then it follows from Definitions 1.1 and 1.2 that one of the following two cases must take place: either

(i) there is some $\delta > 0$ such that ρ and r are both decreasing on $(c - \delta, c)$ and are both increasing on $(c, c + \delta)$ or
(ii) there is some $\delta > 0$ such that ρ and r are both increasing on $(c - \delta, c)$ and are both decreasing on $(c, c + \delta)$.

Consider case (i). Since ρ is decreasing on $(c - \delta, c)$, the limit $\rho(c-)$ exists; moreover, in view of the generalized Mean Value Theorem,

$$\rho(c) = \frac{f'(c)}{g'(c)} = \lim_{x \uparrow c} \frac{f(x) - f(c)}{g(x) - g(c)} = \rho(c-).$$

Further, in view of Remark 1.13 and Proposition 1.18, there is some $\delta > 0$ such that $r'(x) < 0$ and $r(x) > \rho(x) > \rho(c-) = \rho(c)$ $\forall x \in (c - \delta, c)$; also, $\rho(c) = r(c)$, since c is the point of a local minimum for r, and so, $r'(c) = 0$. Hence, in view of (1.8) and (1.7),

$$\frac{d}{dx} \ln |g(x)| = \frac{g'(x)}{g(x)} = \frac{-r'(x)}{r(x) - \rho(x)} > \frac{-r'(x)}{r(x) - \rho(c)} = \frac{-r'(x)}{r(x) - r(c)} = -\frac{d}{dx} \ln (r(x) - r(c))$$

for all $x \in (c - \delta, c)$. Integration of this inequality yields

$$\frac{g(x_2)}{g(x_1)} > \frac{r(x_1) - r(c)}{r(x_2) - r(c)} \tag{1.13}$$

whenever $c - \delta < x_1 < x_2 < c$. Let now $x_2 \uparrow c$ (while x_1 is fixed). Then (by the continuity of r and g) the right-hand side of (1.13) tends to ∞ while its left-hand side tends to the finite limit $\frac{g(c)}{g(x_1)}$. This contradiction show that case (i) is impossible.

Quite similarly, one shows that case (ii) is impossible. (Note that the assumption that ρ is increasing on $(c, c + \delta)$ was not even used here.) □

On the other hand, examples when $c_0 = a_0$ takes place are many and very easy to construct; to obtain a simplest one, let $a := 0$, $b := \infty$, $f(x) := x^2$, and $g(x) := x$.

Remark 1.20. If g were allowed to be discontinuous at some point(s) of (a, b) and one were only concerned with the possibility that both r and ρ could be extended by continuity to the entire interval (a, b), then the conclusion of Proposition 1.19 — and even that of Theorem 1.16 — would not hold. For example, let $a := -2/3$, $b := \infty$; $f(x) := -x^2 e^{-2/x}$ and $g(x) := e^{-2/x}$ for $x \neq 0$. Then $r(x) = -x^2$ and $\rho(x) = -x^2 - x^3$, so that both r and ρ can obviously be extended by continuity to the entire interval $(a, b) = (-2/3, \infty)$. Here, ρ is n waves down on (a, b), with $n = 2$ and the switching points $a_0 = -2/3$, $a_1 = 0$, and $a_2 = \infty$. If it were true that

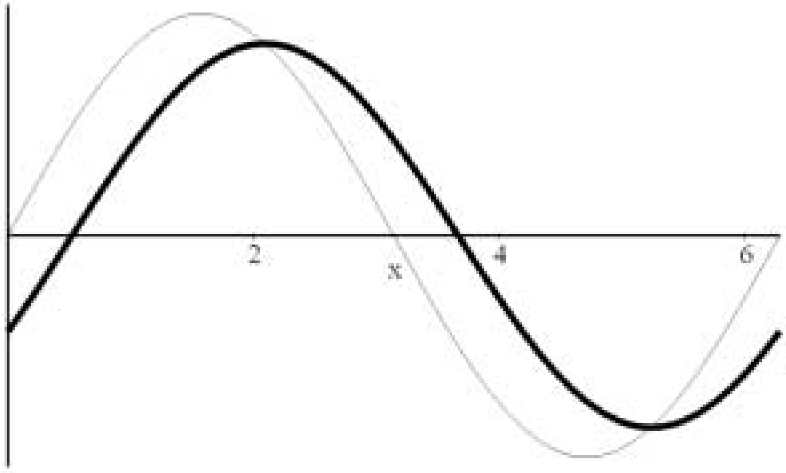

Figure 1.1: $\rho(x) = \dfrac{2}{\sqrt{3}} \sin x$, *thin line;* $r(x) = \sin\left(x - \dfrac{\pi}{6}\right)$, *thick line*

the waves of r on (a, b) follow (in the sense of Definition 1.2) the waves of ρ, then one would necessarily have here $c_{-1} = -2/3$, $c_0 = 0$, $c_1 = \infty$, and $m = 1$, whence

$$c_0 = a_1 = 0 \tag{1.14}$$

would be an interior point of the interval (a, b), in contrast with the conclusion of Proposition 1.19. Even the conclusion of Theorem 1.16 does not hold in this situation; indeed, if the conclusion of Theorem 1.16 were true here, then (1.14) would imply $\ell(0) = 1$, which would contradict the requirement that $\ell(k)$ be even whenever k is even.

Remark 1.21. Formula (1.8) provides yet another insight into the relation between ρ and r. Indeed, at any point x_0 of local extrema of r, one must have $r'(x_0) = 0$, which implies $\rho\,(x_0) = r\,(x_0)$, so that ρ attains all the extreme values of r inside the interval, and then may even exceed them. It follows that the amplitude of the oscillations of ρ is no less than that of r. Together with Theorem 1.16, this means that the waves of r may be thought of as obtained from the waves of ρ by a certain kind of delaying and smoothing down procedure.

Here are two examples to illustrate above results and observations.

Example 1.2. Let $a := 0$, $b := 2\pi$, $f(x) := e^{x\sqrt{3}} \sin\left(x - \dfrac{\pi}{6}\right)$, and $g(x) := e^{x\sqrt{3}}$. This corresponds to the choice of $r(x) = \sin\left(x - \dfrac{\pi}{6}\right)$ and $h = \sqrt{3}$ in (1.7) and (1.8), so that

$$\rho(x) = \sin\left(x - \frac{\pi}{6}\right) + \frac{\cos\left(x - \frac{\pi}{6}\right)}{\sqrt{3}} = \frac{2}{\sqrt{3}} \sin x.$$

Figure 1.1 above shows that indeed the waves of r are of a smaller amplitude and are delayed (by the constant shift $\dfrac{\pi}{6}$) relative to the waves of ρ. It is also seen that the waves of ρ and r are interwoven; more exactly, the graphs of ρ and r intersect each other at the points of extrema of r.

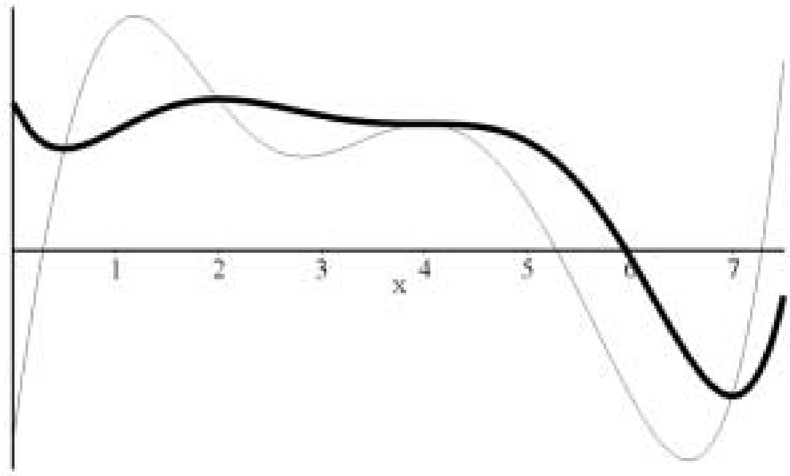

Figure 1.2: ρ, thin line; r, thick line

Example 1.3. Now, let $a := 0$, $b := 7.5$,

$$r(x) := 75 + \int_0^x (u - 1/2)(u - 2)(u - 4)^2(u - 7)\,du$$
$$= \frac{1}{6}x^6 - \frac{7}{2}x^5 + \frac{221}{8}x^4 - \frac{307}{3}x^3 + 176x^2 - 112x + 75 \quad \text{and}$$
$$h(x) := \frac{(x-4)^2 + x^2 + 10}{40}, \quad \text{which corresponds to}$$

$$g(x) = C \cdot \exp \int_0^x h(u)\,du$$
$$= C \cdot \exp \frac{x^3 - 6x^2 + 39x}{60}, \quad \text{where } C \text{ is any nonzero constant,}$$
$$f(x) = r(x)g(x)$$
$$= C \cdot \left(\frac{1}{6}x^6 - \frac{7}{2}x^5 + \frac{221}{8}x^4 - \frac{307}{3}x^3 + 176x^2 - 112x + 75\right) \exp \frac{x^3 - 6x^2 + 39x}{60},$$
$$r'(x) = (x - 1/2)(x - 2)(x - 4)^2(x - 7), \quad \text{and}$$
$$\rho(x) = \frac{1}{6}x^6 - \frac{7}{2}x^5 + \frac{221}{8}x^4 - \frac{307}{3}x^3 + 176x^2 - 112x + 75$$
$$+ 40\frac{(x - 1/2)(x - 2)(x - 4)^2(x - 7)}{(x-4)^2 + x^2 + 10}.$$

The graphs of ρ and r are demonstrated by Figure 1.2. The points of change from increase to decrease or vice versa for r plus the endpoints of the interval $(a, b) = (0, 7.5)$ are given by the table

c_{-1}	c_0	c_1	c_2	c_3
0	0.5	2	7	7.5

so that $m = 3$; here and in what follows we use the notation of Definitions 1.1 and 1.2. The points of change from increase to decrease or vice versa for ρ plus the endpoints of the interval $(a, b) = (0, 7.5)$ are given by the table

a_0	a_1	a_2	a_3	a_4	a_5
0	$1.18\dots$	$2.82\dots$	4	$6.57\dots$	7.5

so that $n = 5$. The map ℓ in Definition 1.2 is given by the table

k	0	1	2
$\ell(k)$	0	1	4

One can see that indeed $c_k \in \left(a_{\ell(k)}, a_{\ell(k)+1}\right)$ for $k \in \{0, 1, \dots, m-1\}$, and $\ell(k)$ is even iff k is even.

As in Example 1.2, one can see here that the waves of r are of smaller amplitude and delayed relative to the waves of ρ. Again, the waves of ρ and r are interwoven in the sense described in Example 1.2.

On the interval $(0, 0.5)$, the instantaneous relative rate ρ is less than the average relative rate r; this is the same as r' being negative on $(0, 0.5)$, which one can see too.

On the next interval, $(0.5, 2)$, one has $\rho > r$, which is the same as $r' > 0$.

Further to the right, on the interval $(2, 7)$, one has $\rho < r$ and $r' < 0$ (except that at $x = 4$ one has $\rho = r$ and $r' = 0$), so that r is decreasing everywhere on $(2, 7)$; the graphs of ρ and r look as if r "feels" to some extent the up and down (quarter-)waves of ρ near $x = 4$, and yet, r "misses" these (quarter-)waves of ρ.

Finally, on the interval $(7, 7.5)$, one has $\rho > r$ and $r' > 0$.

The delay-and-flatten manner of the waves of r to follow the waves of ρ is especially manifest to the right of $x = 5$.

2. APPLICATIONS

In the first subsection of this section, we shall apply the results above to obtain a refinement of an inequality for the normal family of probability distributions due to Yao and Iyer; this inequality arises in bioequivalence studies; we shall also obtain an extension to the case of the Cáuchy family of distributions. In the second subsection below, the convexity conjecture by Topsøe [6] concerning information inequalities is addressed in the context of a general convexity problem.

Other applications of l'Hospital type rules are given: in [3], to certain information inequalities; in [4], to monotonicity of the relative error of a Padé approximation for the complementary error function; in [5], to probability inequalities for sums of bounded random variables.

2.1. **Refinement and Extension of the Yao-Iyer Inequality.**

2.1.1. *The normal case.* Consider the ratio

$$r(z) := \frac{\mathsf{P}(|X| < z)}{\mathsf{P}(|Z| < z)}, \quad z > 0, \tag{2.1}$$

of the cumulative probability distribution functions

$$F(z) := \mathsf{P}(|X| < z) \quad \text{and} \quad G(z) := \mathsf{P}(|Z| < z) \tag{2.2}$$

of random variables $|X|$ and $|Z|$, where $Z \sim N(0, 1)$, $X \sim N(\mu, \sigma^2)$, $\mu \in \mathbb{R}$, and $\sigma > 0$.

Consider also the ratio

$$\rho(z) := \frac{F'(z)}{G'(z)} = \frac{1}{2\sigma} \cdot \frac{\varphi\left(\frac{z-\mu}{\sigma}\right)}{\varphi(z)} + \frac{1}{2\sigma} \cdot \frac{\varphi\left(\frac{z+\mu}{\sigma}\right)}{\varphi(z)}, \quad z > 0.$$

In [2], a simple proof was given of the Yao-Iyer [1] inequality

$$r(z) > \min(r(0+), r(\infty)) \quad \forall z \in (0,\infty) \quad \forall(\mu,\sigma) \neq (0,1), \tag{2.3}$$

which arises in bioequivalence studies; here $r(\infty) := \lim_{z\to\infty} r(z) = 1$ and, by the usual l'Hospital Rule for limits,

$$r(0+) = \rho(0+) = \frac{\varphi\left(\frac{\mu}{\sigma}\right)}{\sigma\varphi(0)} = \frac{1}{\sigma}\exp\left(-\frac{\mu^2}{2\sigma^2}\right),$$

where φ is the standard normal density. Of course, in the trivial case $(\mu,\sigma) = (0,1)$, one has $r(z) = 1 \quad \forall z > 0$.

The proof of the Yao-Iyer inequality given in [2] was based on the following lemma.

Lemma 2.1. *For all $\mu \in \mathbb{R}$ and $\sigma > 0$, there exists some $b \in [0,\infty]$ such that ρ is increasing on $(0,b)$ and decreasing on (b,∞). (In other words, ρ is either 1 wave down or at most 2 waves up on $(0,\infty)$).*

Based on this lemma and results of the previous section, we shall obtain the following refinement of the Yao-Iyer inequality, from which the inequality is immediate.

Theorem 2.2. *Let $\mu \in \mathbb{R}$ and $\sigma > 0$.*

1. *If $\sigma < 1$ and $\sigma^2 + \left(\frac{\mu}{\sigma}\right)^2 \leq 1$, then r is decreasing on $(0,\infty)$ from $r(0+)$ to $r(\infty) = 1$.*
2. *If $\sigma < 1$ and $\sigma^2 + \left(\frac{\mu}{\sigma}\right)^2 > 1$, then there exists some $c \in (0,\infty)$ such that r is increasing on $(0,c]$ from $r(0+)$ to $r(c)$ and decreasing on $[c,\infty)$ from $r(c)$ to 1. Moreover, $c \geq b$, where b is defined by Lemma 2.1.*
3. *If $\sigma = 1$ and $\mu = 0$, then $r = 1$ everywhere on $(0,\infty)$.*
4. *If $\sigma = 1$ and $\mu \neq 0$, then r is increasing on $(0,\infty)$ from $r(0+)$ to 1.*
5. *If $\sigma > 1$, then r is increasing on $(0,\infty)$ from $r(0+)$ to 1.*

Proof. On $(0,\infty)$, one has

$$\frac{r'}{2\varphi} = \frac{Q}{G^2}, \tag{2.4}$$

where

$$Q := \rho G - F \tag{2.5}$$

and F and G are the distribution functions defined above. Further, on $(0,\infty)$,

$$Q' = \rho' G \tag{2.6}$$

and

$$\rho'(z) = \left(z - \frac{z-\mu}{\sigma^2}\right)\frac{\varphi\left(\frac{z-\mu}{\sigma}\right)}{2\sigma\varphi(z)} + \left(z - \frac{z+\mu}{\sigma^2}\right)\frac{\varphi\left(\frac{z+\mu}{\sigma}\right)}{2\sigma\varphi(z)}, \quad z > 0, \tag{2.7}$$

so that $\rho'(0+) = 0$. Now, by the usual l'Hospital Rule and in view of (2.6),

$$\lim_{z\downarrow 0}\frac{Q(z)}{G(z)^2} = \lim_{z\downarrow 0}\frac{Q'(z)}{2G(z)\varphi(z)} = \frac{\rho'(0+)}{2\varphi(0)} = 0;$$

this and (2.4) imply

$$r'(0+) = 0. \tag{2.8}$$

Therefore, using Lemma 2.1, Theorem 1.16, and Remark 1.17, one sees that r is either 1 wave down on $(0,\infty)$ or at most 2 waves up on $(0,\infty)$. To discriminate between these cases, it suffices to consider the sign of r' in a right neighborhood of 0 and that in a left neighborhood of ∞.

By (2.4) and (2.6), one has $\operatorname{sign} r' = \operatorname{sign} Q$ and $\operatorname{sign} Q' = \operatorname{sign} \rho'$ on $(0,\infty)$. Also, by (2.5), $Q(0+) = 0$. It follows that, in a right neighborhood of 0, $\operatorname{sign} Q = \operatorname{sign} Q'$, and so,

$$\operatorname{sign} r' = \operatorname{sign} \rho', \tag{2.9}$$

provided that ρ' does not change sign in such a neighborhood. But, as we saw, $\rho'(0+) = 0$. Hence, (2.9) implies that, in a right neighborhood of 0,

$$\operatorname{sign} r' = \operatorname{sign} \rho'', \tag{2.10}$$

provided that ρ'' does not change sign in such a neighborhood.

Further, one has the identity

$$\begin{aligned} 2\sigma\rho''(z) = {}& \left[\left(1-\frac{1}{\sigma^2}\right)+\left(z-\frac{z-\mu}{\sigma^2}\right)^2\right]\frac{\varphi\left(\frac{z-\mu}{\sigma}\right)}{\varphi(z)} \\ &+\left[\left(1-\frac{1}{\sigma^2}\right)+\left(z-\frac{z+\mu}{\sigma^2}\right)^2\right]\frac{\varphi\left(\frac{z+\mu}{\sigma}\right)}{\varphi(z)}, \quad z>0. \end{aligned}$$

In particular,

$$\rho''(0+) = \left[\sigma^2+\left(\frac{\mu}{\sigma}\right)^2-1\right]\frac{\varphi\left(\frac{\mu}{\sigma}\right)}{\sigma^3\varphi(0)}. \tag{2.11}$$

By (2.10) and (2.11), in a right neighborhood of 0,

$$\operatorname{sign} r' = \operatorname{sign}\left[\sigma^2+\left(\frac{\mu}{\sigma}\right)^2-1\right] \quad \text{if} \quad \sigma^2+\left(\frac{\mu}{\sigma}\right)^2 \neq 1. \tag{2.12}$$

It is not difficult to see that $\rho'''(0+) = 0$. By (2.11), in the case $\sigma^2+\left(\frac{\mu}{\sigma}\right)^2 = 1$, one has $\rho''(0+) = 0$; also in this case, one can see that

$$\rho^{IV}(0+) = -2\left(\frac{\mu}{\sigma^2}\right)^4\frac{\varphi\left(\frac{\mu}{\sigma}\right)}{\sigma\varphi(0)} < 0 \quad \text{if} \quad \mu \neq 0.$$

It follows now from (2.10) that, in a right neighborhood of 0,

$$r' < 0 \quad \text{if} \quad \sigma^2+\left(\frac{\mu}{\sigma}\right)^2 = 1 \quad \text{and} \quad \mu \neq 0. \tag{2.13}$$

Of course, if $\sigma^2+\left(\frac{\mu}{\sigma}\right)^2 = 1 \quad \text{and} \quad \mu = 0$, then $\sigma = 1$, so that this is the trivial case, in which $r = 1$ everywhere on $(0,\infty)$. Thus, (2.12) and (2.13) provide a complete description of the sign of r' in a right neighborhood of 0.

Let us now consider the sign of r' in a left neighborhood of ∞. Let $z \to \infty$; then (2.7) implies that $\rho'(z) \to 0$ if $\sigma < 1$ and $\rho'(z) \to \infty$ if $\sigma > 1$ or $\sigma = 1$ and $\mu \neq 0$. Therefore, in view of (2.4) and (2.5), in a left neighborhood of ∞,

$$\operatorname{sign} r' = \operatorname{sign}(\sigma - 1) \quad \text{if} \quad \sigma \neq 1; \tag{2.14}$$

$$r' > 0 \quad \text{if} \quad \sigma = 1 \quad \text{and} \quad \mu \neq 0. \tag{2.15}$$

Recall that r is either 1 wave down on $(0, \infty)$ or at most 2 waves up on $(0, \infty)$.

Consider now Part 1 of the theorem, when $\sigma < 1$ and $\sigma^2 + \left(\frac{\mu}{\sigma}\right)^2 \leq 1$. Then (2.12) and (2.13) imply $r' < 0$ in a right neighborhood of 0. Hence, r is decreasing in a right neighborhood of 0, and so, any waves-up pattern is impossible for r. Therefore, r is 1 wave down on $(0, \infty)$, that is, in this case r is decreasing everywhere on $(0, \infty)$. Thus, Part 1 of the theorem is completely proved.

Assume next that $\sigma < 1$ and $\sigma^2 + \left(\frac{\mu}{\sigma}\right)^2 > 1$, as in Part 2 of the theorem. Then (2.12) and (2.14) imply, respectively, that $r' > 0$ in a right neighborhood of 0 and $r' < 0$ in a left neighborhood of ∞. Now Part 2 of the theorem follows using Theorem 1.16.

Part 3 of the theorem is trivial, and only serves the purpose of completeness.

If $\sigma = 1$ and $\mu \neq 0$, as in Part 4 of the theorem, then (2.12) and (2.15) imply that $r' > 0$ in a right neighborhood of 0, as well as in a left neighborhood of ∞. Since r may have at most 2 waves up on $(0, \infty)$, Part 4 of the theorem now follows.

The proof of Part 5 of the theorem is quite similar to that of Part 4; the difference is that in this case one uses (2.14) instead of (2.15). □

2.1.2. *The Cáuchy case.* In this subsection, r is still assumed to have the form defined by (2.1) and (2.2), but X and Z are now assumed to have, respectively, the Cáuchy distribution with arbitrary parameters $a \in \mathbb{R}$ and $b > 0$ and the standard Cáuchy distribution, with the densities

$$p_{a,b}(z) := \frac{1}{\pi b} \cdot \frac{1}{1 + \left(\frac{z-a}{b}\right)^2} \quad \text{and} \quad p_{0,1}(z) := \frac{1}{\pi} \cdot \frac{1}{1 + z^2}.$$

We shall show that the analogue

$$r(z) > \min(r(0+), r(\infty)) \quad \forall z \in (0, \infty) \quad \forall (a, b) \neq (0, 1)$$

of the Yao-Iyer [1] inequality (2.3) takes place in this case too; note that a and b are the location and scale parameters, respectively, of the Cáuchy distribution, just as μ and σ are those of the normal distribution. Here, it is easy to see that

$$r(0+) = \frac{b}{a^2 + b^2} \quad \text{and} \quad r(\infty) = 1.$$

Moreover, we shall show that the following analogue of Theorem 2.2 takes place in this Cáuchy distribution setting.

Theorem 2.3. *Let $a \in \mathbb{R}$ and $b > 0$.*

(1) *If $b^4 - b^2 + a^2(a^2 + 2b^2 + 3) < 0$, then r is decreasing on $(0, \infty)$ from $r(0+)$ to $r(\infty) = 1$.*
(2) *If $b^4 - b^2 + a^2(a^2 + 2b^2 + 3) = 0$ and $a \neq 0$, then r is decreasing on $(0, \infty)$ from $r(0+)$ to $r(\infty) = 1$.*
(3) *If $b^4 - b^2 + a^2(a^2 + 2b^2 + 3) = 0$ and $a = 0$, then $b = 1$, and $r = 1$ everywhere on $(0, \infty)$.*

(4) *If* $b^4 - b^2 + a^2(a^2 + 2b^2 + 3) > 0$, *then there exists some* $c \in (0, \infty)$ *such that* r *is increasing on* $(0, c]$ *from* $r(0+)$ *to* $r(c)$ *and decreasing on* $[c, \infty)$ *from* $r(c)$ *to* 1.

Proof. Consider the ratio

$$\rho(z) := \frac{F'(z)}{G'(z)} = b \cdot R(y), \tag{2.16}$$

where

$$\begin{aligned} R(y) &:= \frac{f(y)}{g(y)}, \\ y &:= z^2, \\ f(y) &:= y^2 + \left(a^2 + b^2 + 1\right) y + \left(a^2 + b^2\right), \\ g(y) &:= y^2 + 2\left(b^2 - a^2\right) y + \left(a^2 + b^2\right)^2 \\ &= \left((z-a)^2 + b^2\right)\left((z+a)^2 + b^2\right), \end{aligned}$$

so that $g > 0$ on $(0, \infty)$.

One has

$$\frac{g(y)^2}{2bz}\rho'(z) = g(y)^2 R'(y) = \left(f'g - fg'\right)(y), \quad z > 0. \tag{2.17}$$

It follows that $\rho'(0+) = 0$, whence (2.8) holds in this case too; cf. (2.4)–(2.6).

Next,

$$\left(\frac{g'}{f'}\right)'(y) = \frac{2\left(1 + 3a^2 - b^2\right)}{\left(2y + a^2 + b^2 + 1\right)^2}, \quad y > 0.$$

Therefore, $\frac{g'}{f'}$ is at most 1 wave up or down on $(0, \infty)$, and so, by Theorem 1.16, $\frac{g}{f}$ is at most 2 waves up or down on $(0, \infty)$, and then so are $R = \frac{f}{g}$ and ρ (recall (2.16)); again by Theorem 1.16, this and (2.8) imply that r too is at most 2 waves up or down on $(0, \infty)$.

Further, since f and g are polynomials of the same degree, it follows from (2.17) that $\rho'(z) \to 0$ as $z \to \infty$. Hence (cf. (2.4) and (2.5)), $r' < 0$ in a left neighborhood of ∞. This and the fact that r is at most 2 waves up or down on $(0, \infty)$ imply that either

(i) r is constant on $(0, \infty)$ or
(ii) r is decreasing everywhere on $(0, \infty)$ or
(iii) there exists some $c \in (0, \infty)$ such that r is increasing on $(0, c]$ and decreasing on $[c, \infty)$.

To discriminate between these three cases, it suffices to know the sign of r' in a right neighborhood of 0. Since (2.9) holds in this case too, (2.17) implies

$$\operatorname{sign} r'(z) = \operatorname{sign}(f'g - fg')(y) \tag{2.18}$$

for z in a right neighborhood of 0; remember that, by definition, $y = z^2$. Further,

$$(f'g - fg')(0+) = \left(a^2 + b^2\right)\left[b^4 - b^2 + a^2\left(a^2 + 2b^2 + 3\right)\right].$$

Hence, in a right neighborhood of 0,

$$\operatorname{sign} r' = \operatorname{sign}\left[b^4 - b^2 + a^2\left(a^2 + 2b^2 + 3\right)\right] \quad \text{if} \quad b^4 - b^2 + a^2\left(a^2 + 2b^2 + 3\right) \neq 0.$$

Now Parts 1 and 4 of the theorem follow.

In the remaining case, when $b^4 - b^2 + a^2(a^2 + 2b^2 + 3) = 0$, one has $(f'g - fg')(0+) = 0$; hence, by (2.18), for z in a right neighborhood of 0,

$$\operatorname{sign} r'(z) = \operatorname{sign}(f'g - fg')(y) = \operatorname{sign}(f'g - fg')'(y) = \operatorname{sign}(f''g - fg'')(y).$$

However,

$$(f''g - fg'')(0+) = 2(g-f)(0+) = b^4 - b^2 + a^2\left(a^2 + 2b^2 - 1\right) = -4a^2 < 0$$

provided that $b^4 - b^2 + a^2(a^2 + 2b^2 + 3) = 0$ and $a \neq 0$, and then we see that $r' < 0$ in a right neighborhood of 0. This yields Part 2 of the theorem.

Part 3 is trivial.

The theorem is completely proved. □

2.2. **Application: the convexity problem.** Let us consider here the problem of the convexity of the ratio $\frac{f}{g}$ of two sufficiently smooth functions f and g. Suppose first that the derivatives f' and g' are rational functions. One has

$$\left(\frac{f}{g}\right)' = \frac{f_1}{g_1}, \quad \frac{f_1'}{g_1'} = \frac{f_2}{g_2}, \quad \text{and} \quad \frac{f_2'}{g_2'} = \frac{f_3}{g_3}, \tag{2.19}$$

where

$$\begin{aligned} f_1 &:= \frac{f'}{g'}g - f, \quad g_1 := \frac{g^2}{g'}; \\ f_2 &:= \frac{f''}{g''}g' - f', \quad g_2 := 2\frac{g'^2}{g''} - g; \\ f_3 &:= f'''g'' - f''g''', \quad g_3 := 3g''^2 - 2g'g'''. \end{aligned} \tag{2.20}$$

Here, at each of the two steps, from $\frac{f_1}{g_1}$ to $\frac{f_2}{g_2}$ and from $\frac{f_2}{g_2}$ to $\frac{f_3}{g_3}$, we proceed to isolate the presumably most complicated expression in the numerator or denominator (or both) and get instead its derivative, which is presumably a simpler (in this case, rational) expression.

In these two steps, one gets to $\frac{f_3}{g_3}$, which is a rational function.

Note also that the convexity/concavity of $\frac{f}{g}$ corresponds to the increase/decrease of $\frac{f_1}{g_1}$.

Thus, the l'Hospital type results of Section 1 allow one to determine the intervals of convexity of $\frac{f}{g}$ in a completely algorithmic manner by reduction of the convexity problem to that of the oscillation pattern of the rational function, $\frac{f_3}{g_3}$, and then going the same steps *back* to $\frac{f}{g}$ and at that studying the signs of the derivatives of the ratios $\frac{f_s}{g_s}$ *locally*, near the switching points (from increase to decrease or vice versa) of each of the ratios $\frac{f_{s+1}}{g_{s+1}}$, starting from $s = 2$ to 1 to 0, where $\frac{f_0}{g_0} := \frac{f}{g}$. In some cases, though, such as the following Example 2.1, the situation may clear up before getting all the way to $\frac{f_3}{g_3}$.

Example 2.1. In [6], a simple proof (due to Arjomand, Bahrangiri, and Rouhani) of the monotonicity of the function $(0, 1/2) \ni p \mapsto \frac{H(p,q)}{\ln p \cdot \ln q}$ is given; here $q := 1 - p$ and $H(p,q) := -p\ln p - q\ln q$ is the entropy function; that proof is based on the identity

$$\frac{H(p,q)}{\ln p \cdot \ln q} = -\frac{p}{\ln q} - \frac{q}{\ln p}$$

and the convexity of $\frac{q}{\ln p}$ in $p \in (0,1)$. See [3] for another simple proof of the monotonicity of $\frac{H(p,q)}{\ln p \cdot \ln q}$ — based on the l'Hospital type rule, stated above as Proposition 1.1.

The convexity of $\frac{q}{\ln p}$ is also easy to prove using our results. Indeed, here $f(p) = q$, $g(p) = \ln p$, $f_1(p) = -p \ln p - q$, $g_1(p) = p \ln^2 p$, $f_2(p) = 1$, and $g_2(p) = -2 - \ln p$. Moreover, $f_1(1-) = g_1(1-) = 0$ and $f_1'(p) = -\ln p > 0$ for $p \in (0,1)$, so that $f_1 < 0$ on $(0,1)$. Since $\frac{g_2}{f_2}$ is obviously decreasing on $(0,1)$, Proposition 1.1 implies that so is $\frac{g_1}{f_1}$; hence, $\frac{f_1}{g_1}$ is increasing on $(0,1)$ (note that $f_1 < 0$ and $g_1 > 0$ on $(0,1)$). By (2.19), the convexity of $\frac{q}{\ln p}$ now follows.

Let us now illustrate the proposed approach with a more involved example.

Example 2.2. Let $f(x) = \ln(1+x^2)$ and $g(x) = \operatorname{arccot} x$. We shall use results of Section 1 to analyze the convexity properties of $\frac{f}{g}$ on $(a,b) = (-\infty, \infty)$. One has, for all real x,

$$f_1(x) = -2x \operatorname{arccot} x - \ln\left(1+x^2\right) \quad \text{and} \quad g_1(x) = -(1+x^2)\operatorname{arccot}^2 x$$

and, for all $x \neq 0$,

$$\begin{aligned} f_2(x) &= -\frac{1}{x}; \qquad & g_2(x) &= \frac{1}{x} - \operatorname{arccot} x; \\ f_2'(x) &= \frac{1}{x^2}; \qquad & g_2'(x) &= -\frac{1}{x^2\left(1+x^2\right)}. \end{aligned}$$

Hence, for all $x \neq 0$,

$$\frac{f_2'(x)}{g_2'(x)} = \frac{f_3(x)}{g_3(x)} = -\left(1+x^2\right),$$

which is increasing in $x \in (-\infty, 0)$ and decreasing in $x \in (0, \infty)$.

Now let us analyze the signs of g_2 and g_2' on the intervals $(-\infty,0)$ and $(0,\infty)$ and the local behavior of $\frac{f_2}{g_2}$ near the switching points $-\infty$, 0, and ∞ of $\frac{f_3}{g_3}$, thus making the first step back, from $\frac{f_3}{g_3}$ to $\frac{f_2}{g_2}$. Since $g_2(\infty) = 0$ and $g_2'(x) < 0$ for all $x \neq 0$, one has $g_2 > 0$ on $(0,\infty)$; it is obvious that $g_2 < 0$ on $(-\infty, 0)$; thus, $\frac{f_2}{g_2}$ is defined on each of the intervals $(-\infty,0)$ and $(0,\infty)$. In addition, $f_2(\infty) = g_2(\infty) = 0$. Hence, by Proposition 1.1, $\frac{f_2}{g_2}$ is decreasing on $(0,\infty)$. To analyze the signs of $\left(\frac{f_2}{g_2}\right)'$ near $-\infty$ and 0, note that

$$x^2 g_2^2(x) \left(\frac{f_2}{g_2}\right)'(x) = \frac{x}{1+x^2} - \operatorname{arccot} x \to \begin{cases} -\pi < 0 & \text{as} \quad x \to -\infty, \\ -\pi/2 < 0 & \text{as} \quad x \to 0. \end{cases}$$

Now Remark 1.11 implies that $\frac{f_2}{g_2}$ is decreasing on the interval $(-\infty,0)$; thus, $\frac{f_2}{g_2}$ is decreasing on each of the intervals $(-\infty, 0)$ and $(0,\infty)$.

Further, let us analyze the signs of g_1 and g_1' on the intervals $(-\infty,0)$ and $(0,\infty)$ and the local behavior of $\frac{f_1}{g_1}$ near the switching points $-\infty$, 0, and ∞ of $\frac{f_2}{g_2}$, thus making the second (and final) step back, from $\frac{f_2}{g_2}$ to $\frac{f_1}{g_1} = \left(\frac{f}{g}\right)'$. For all $x \neq 0$, one has $g_1(x) = -(1+x^2)\operatorname{arccot}^2 x < 0$ and $g_1'(x) = 2xg_2(x)\operatorname{arccot} x > 0$; thus, $\frac{f_1}{g_1}$ is defined on each of the intervals $(-\infty, 0)$ and $(0, \infty)$. It remains to determine the sign of $\left(\frac{f_1}{g_1}\right)'$ near the endpoints: $-\infty$, 0, and ∞:

$$\left(\frac{f_1}{g_1}\right)'(x) \sim -\frac{2}{\pi x^2} < 0 \quad \text{as} \quad x \to -\infty,$$
$$\left(\frac{f_1}{g_1}\right)'(0) = \frac{4}{\pi} > 0, \quad \text{and}$$
$$\left(\frac{f_1}{g_1}\right)'(x) \sim \frac{2}{x} > 0 \quad \text{as} \quad x \to \infty.$$

Therefore, $\left(\frac{f}{g}\right)' = \frac{f_1}{g_1}$ switches once from decrease to increase on $(-\infty, 0)$ and then is increasing on $(0, \infty)$.

We conclude that there is some $c \in (-\infty, 0)$ such that $\frac{f}{g}$ is concave on $(-\infty, c)$ and convex on (c, ∞); in fact, $c = -0.751\ldots$.

Of course, the steps like the ones described by (2.19)-(2.20) will work not only in the case when both f' and g' are rational functions but also in other cases when f' and g' are given by simpler expressions than f and g.

Topsøe [6] conjectured that

$$\frac{\ln\left(\frac{H(p)}{\ln 2}\right)}{\ln(4pq)} \tag{2.21}$$

is convex in $p \in (0, 1)$, where again $q := 1 - p$ and $H(p) := H(p, q) := -p\ln p - q\ln q$. Here the derivative f' of $f := \ln\left(\frac{H}{\ln 2}\right)$ is not a rational function. However, one can still use the same kind of algorithm as the one demonstrated in (2.19)-(2.20), because $f' = \frac{H'}{H}$ and $H''(p) = -\frac{1}{pq}$ is rational; then the problem reduces again to that of the oscillation pattern of a rational function plus local analysis near the switching points. In this sense, the problem can be solved in a rather algorithmic manner. Indeed, one can write here

$$\frac{f_3}{g_3} = \frac{P_m}{Q_n}, \tag{2.22}$$

where $P_m(p)$ and $Q_n(p)$ are polynomials in $H(p)$ of some degrees m and n over the field $\mathcal{R}(p, H'(p))$ of all rational expressions in p and $H'(p)$; in fact, in (2.22), one has $m = 2$ and $n = 3$; moreover, here $Q_n(p) = H(p)^3$.

Consider such a rational expression $\frac{P_m}{Q_n}$ over the field $\mathcal{R}(p, H'(p))$. Let us call the sum $m+n$ of the degrees of the numerator and denominator the height of the rational expression $\frac{P_m}{Q_n}$; if

$P_m = 0$, define the height as -1. If the height of the rational expression $\frac{P_m}{Q_n}$ is greater than 0 (as we have it in (2.22) for $\frac{f}{g}$ given by (2.21)), let us rewrite $\frac{P_m}{Q_n}$ so that the leading coefficient of either the numerator or denominator is 1 (as we have it in case $\frac{f}{g}$ is given by (2.21) for the denominator $Q_n(p) = H(p)^3$); then, without loss of generality, one may assume that the leading coefficient of either P_m or Q_n is already 1. Then $\frac{P'_m}{Q'_n}$ too is a rational expression over the field $\mathcal{R}(p, H'(p))$, but its height is at most $m + n - 1$ vs. the height $m + n$ of $\frac{P_m}{Q_n}$ (the derivatives P'_m and Q'_n of P_m and Q_n are taken here of course in p); indeed, since H'' is rational in p, the derivative $P'_m(p)$ in p of any polynomial

$$P_m(p) = R_m(p, H'(p)) \cdot H(p)^m + R_{m-1}(p, H'(p)) \cdot H(p)^{m-1} + \cdots + R_0(p, H'(p))$$

in $H(p)$ — with the coefficients $R_m(p, H'(p)), \ldots, R_0(p, H'(p))$ being rational expressions in p and $H'(p)$ — is again a polynomial in $H(p)$ of degree at most m over the field $\mathcal{R}(p, H'(p))$; moreover, the degree of $P'_m(p)$ is at most $m - 1$ over the field $\mathcal{R}(p, H'(p))$ in case the leading coefficient $R_m(p, H'(p))$ is 1 (or any other nonzero constant).

Therefore, the basic step, from $\frac{P_m}{Q_n}$ to $\frac{P'_m}{Q'_n}$, reduces the height at least by 1. Repeating such basic steps, one comes to an expression of height at most 0, which then itself belongs to the field $\mathcal{R}(p, H'(p))$ of all rational expressions in p and $H'(p)$ only — rather than in p, $H'(p)$, and $H(p)$. Thus, $H(p)$ will be eliminated.

An analogous (even if very long) series of steps afterwards will eliminate $H'(p)$, and then one will have just to consider the monotonicity of a ratio of polynomials in p with certain real coefficients.

(As in all statements of Section 1, when considering the relation between $\frac{f}{g}$ and $\frac{f'}{g'}$ one needs also to control the sign of gg'. In particular, if either P_m or Q_n is constant and the leading coefficient of the other one of these two is also constant, then the basic step needs to be modified; yet, such an exception would be only easier to deal with.)

After all these, say N, steps have been done, one has of course to go the same N steps *back* to $\frac{f}{g}$, studying at that the signs of the derivative of each of the ratios $\frac{f_s}{g_s}$ *locally*, near the switching points (from increase to decrease or vice versa) of the ratio $\frac{f_{s+1}}{g_{s+1}}$, for the integer values of s going down from N to 0, where $\frac{f_0}{g_0} := \frac{f}{g}$. Here, at each of the switching points, one might need to use repeatedly the usual l'Hospital Rule for limits, eliminating $H(p)$ and then $H'(p)$ in the same manner as described above. Numerical approximations might also be needed. Remark 1.12 may be useful at some of these steps.

However, all this long process is essentially algorithmic and will necessarily come to an end after a *finite* number of steps, and then the oscillation pattern of the ratio $\frac{f_1}{g_1} = \left(\frac{f}{g}\right)'$ will be completely determined. Thus, the convexity pattern of $\frac{f}{g}$ will be completely determined.

Of course, the number N of the basic steps in this case will be many more than two (in contrast with above Examples 2.1 and 2.2), and the volume of calculations will be enormous. For this reason, we shall not pursue this problem further at this point.

In contrast with this convexity problem, the proof of the monotonicity of the ratio (2.21) based on a l'Hospital type rule is very simple; see [3].

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