Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 3, Issue 1, Article 11, 2002

ON GRÜSS TYPE INTEGRAL INEQUALITIES

B.G. PACHPATTE

57, SHRI NIKETEN COLONY
AURANGABAD - 431 001,
(MAHARASHTRA) INDIA.
bgpachpatte@hotmail.com

Received 29 June, 2001; accepted 18 October, 2001.
Communicated by J. Sándor

ABSTRACT. In this paper we establish two new integral inequalities similar to that of the Grüss inequality by using a fairly elementary analysis.

Key words and phrases: Grüss type, Integral Inequalities, Absolutely Continuous, Integral Identity.

2000 *Mathematics Subject Classification.* 26D15, 26D20.

1. INTRODUCTION

In 1935 (see [4, p. 296]), G. Grüss proved the following integral inequality which gives an estimation for the integral of a product in terms of the product of integrals:

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx \cdot \frac{1}{b-a}\int_a^b g(x)\,dx\right| \leq \frac{1}{4}(M-m)(N-n),$$

provided that f and g are two integrable functions on $[a,b]$ and satisfying the condition

$$m \leq f(x) \leq M, \quad n \leq g(x) \leq N,$$

for all $x \in [a,b]$, where m, M, n, N are given real constants.

A great deal of attention has been given to the above inequality and many papers dealing with various generalizations, extensions and variants have appeared in the literature, see [1] – [6] and the references cited therein. The main purpose of the present paper is to establish two new integral inequalities similar to that of the Grüss inequality involving functions and their higher order derivatives. The analysis used in the proof is elementary and our results provide new estimates on inequalities of this type.

ISSN (electronic): 1443-5756

070-01

2. Statement of Results

In this section, we state our results to be proved in this paper. In what follows, we denote by $\mathbb{R}$, the set of real numbers and $[a,b] \subset \mathbb{R}$, $a<b$.

Our main results are given in the following theorems.

Theorem 2.1. *Let $f,g:[a,b]\to\mathbb{R}$ be functions such that $f^{(n-1)}, g^{(n-1)}$ are absolutely continuous on $[a,b]$ and $f^{(n)}, g^{(n)} \in L_\infty[a,b]$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right) - \frac{1}{2(b-a)^2}\int_a^b\left[\left(\sum_{k=1}^{n-1}F_k(x)\right)g(x) + \left(\sum_{k=1}^{n-1}G_k(x)\right)f(x)\right]dx\right| \le \frac{1}{2(b-a)^2}\int_a^b\left(|g(x)|\left\|f^{(n)}\right\|_\infty + |f(x)|\left\|g^{(n)}\right\|_\infty\right)A_n(x)\,dx, \tag{2.1}$$

where

$$F_k(x) = \left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x), \tag{2.2}$$

$$G_k(x) = \left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]g^{(k)}(x), \tag{2.3}$$

$$A_n(x) = \int_a^b |K_n(x,t)|\,dt, \tag{2.4}$$

in which $K_n:[a,b]^2\to\mathbb{R}$ is given by

$$K_n(x,t) = \begin{cases} \dfrac{(t-a)^n}{n!} & \textit{if} \quad t\in[a,x] \\[2ex] \dfrac{(t-b)^n}{n!} & \textit{if} \quad t\in(x,b] \end{cases} \tag{2.5}$$

and

$$\begin{aligned}\left\|f^{(n)}\right\|_\infty &= \sup_{t\in[a,b]}\left|f^{(n)}(t)\right| < \infty,\\ \left\|g^{(n)}\right\|_\infty &= \sup_{t\in[a,b]}\left|g^{(n)}(t)\right| < \infty,\end{aligned}$$

for $x\in[a,b]$ and $n\ge 1$ a natural number.

Theorem 2.2. *Let $p,q:[a,b]\to\mathbb{R}$ be functions such that $p^{(n-1)}, q^{(n-1)}$ are absolutely continuous on $[a,b]$ and $p^{(n)}, q^{(n)}\in L_\infty[a,b]$. Then*

$$\left|\frac{1}{b-a}\int_a^b p(x)\,q(x)\,dx - n\left(\frac{1}{b-a}\int_a^b p(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b q(x)\,dx\right) + \frac{1}{2(b-a)}\int_a^b\left[\left(\sum_{k=1}^{n-1}P_k(x)\right)q(x) + \left(\sum_{k=1}^{n-1}Q_k(x)\right)p(x)\right]dx\right| \le \frac{1}{2(n-1)!\,(b-a)^2}\int_a^b\left(|q(x)|\left\|p^{(n)}\right\|_\infty + |p(x)|\left\|q^{(n)}\right\|_\infty\right)B_n(x)\,dx, \tag{2.6}$$

where

$$P_k(x) = \frac{(n-k)}{k!} \cdot \frac{p^{(k-1)}(a)(x-a)^k - p^{(k-1)}(b)(x-b)^k}{b-a}, \tag{2.7}$$

$$Q_k(x) = \frac{(n-k)}{k!} \cdot \frac{q^{(k-1)}(a)(x-a)^k - q^{(k-1)}(b)(x-b)^k}{b-a}, \tag{2.8}$$

$$B_n(x) = \int_a^b \left|(x-t)^{n-1} r(t,x)\right| dt, \tag{2.9}$$

in which

$$r(t,x) = \begin{cases} t-a, & \textit{if} \quad a \le t \le x \le b, \\ t-b, & \textit{if} \quad a \le x < t \le b, \end{cases}$$

and

$$\begin{aligned} \left\|p^{(n)}\right\|_\infty &= \sup_{t\in[a,b]} \left|p^{(n)}(t)\right| < \infty, \\ \left\|q^{(n)}\right\|_\infty &= \sup_{t\in[a,b]} \left|q^{(n)}(t)\right| < \infty, \end{aligned}$$

for $x \in [a,b]$ *and* $n \ge 1$ *is a natural number.*

3. PROOF OF THEOREM 2.1

From the hypotheses on f, we have the following integral identity (see [1, p. 52]):

$$\int_a^b f(t)\,dt = \sum_{k=0}^{n-1} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x) + (-1)^n \int_a^b K_n(x,t) f^{(n)}(t)\,dt, \tag{3.1}$$

for $x \in [a,b]$. In [1] the identity (3.1) is proved by mathematical induction. For a different proof, see [6]. The identity (3.1) can be rewritten as

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{b-a}\sum_{k=1}^{n-1} F_k(x) - \frac{(-1)^n}{b-a}\int_a^b K_n(x,t) f^{(n)}(t)\,dt. \tag{3.2}$$

Similarly, from the hypotheses on g we have the identity

$$g(x) = \frac{1}{b-a}\int_a^b g(t)\,dt - \frac{1}{b-a}\sum_{k=1}^{n-1} G_k(x) - \frac{(-1)^n}{b-a}\int_a^b K_n(x,t) g^{(n)}(t)\,dt. \tag{3.3}$$

Multiplying (3.2) by $g(x)$ and (3.3) by $f(x)$ and summing the resulting identities and integrating from a to b and rewriting we have

$$\begin{aligned}(3.4)\quad \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx &= \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)\\ &\quad -\frac{1}{2(b-a)^2}\int_a^b\left[\left(\sum_{k=1}^{n-1}F_k(x)\right)g(x)+\left(\sum_{k=1}^{n-1}G_k(x)\right)f(x)\right]dx\\ &\quad -\frac{1}{2(b-a)^2}\left[\int_a^b g(x)\left\{\frac{(-1)^n}{b-a}\int_a^b K_n(x,t)\,f^{(n)}(t)\,dt\right\}dx\right.\\ &\qquad\left.+\int_a^b f(x)\left\{\frac{(-1)^n}{b-a}\int_a^b K_n(x,t)\,g^{(n)}(t)\,dt\right\}dx\right].\end{aligned}$$

From (3.4) we observe that

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)\right.\\ &\qquad\left.-\frac{1}{2(b-a)^2}\int_a^b\left[\left(\sum_{k=1}^{n-1}F_k(x)\right)g(x)+\left(\sum_{k=1}^{n-1}G_k(x)\right)f(x)\right]dx\right|\\ &\leq \frac{1}{2(b-a)^2}\int_a^b\left(|g(x)|\left(\int_a^b|K_n(x,t)|\left|f^{(n)}(t)\right|dt\right)\right.\\ &\qquad\left.+|f(x)|\left(\int_a^b|K_n(x,t)|\left|g^{(n)}(t)\right|dt\right)\right)dx\\ &\leq \frac{1}{2(b-a)^2}\int_a^b\left(|g(x)|\left\|f^{(n)}\right\|_\infty+|f(x)|\left\|g^{(n)}\right\|_\infty\right)A_n(x),\end{aligned}$$

which is the required inequality in (2.1). The proof is complete.

4. Proof of Theorem 2.2

From the hypotheses on p we have the following integral identity (see [2, p. 291]):

$$\begin{aligned}(4.1)\quad \frac{1}{n}\left(p(x)+\sum_{k=1}^{n-1}P_k(x)\right)-\frac{1}{b-a}\int_a^b p(y)\,dy\\ =\frac{1}{n!\,(b-a)}\int_a^b(x-t)^{n-1}\,r(t,x)\,p^{(n)}(t)\,dt,\end{aligned}$$

for $x\in[a,b]$. The identity (4.1) can be rewritten as

$$\begin{aligned}(4.2)\quad p(x)=\frac{n}{b-a}\int_a^b p(x)\,dx-\sum_{k=1}^{n-1}P_k(x)\\ +\frac{1}{(n-1)!\,(b-a)}\int_a^b(x-t)^{n-1}\,r(t,x)\,p^{(n)}(t)\,dt.\end{aligned}$$

Similarly, from the hypotheses on q we have the identity

$$q(x) = \frac{n}{b-a}\int_a^b q(x)\,dx - \sum_{k=1}^{n-1} Q_k(x) + \frac{1}{(n-1)!\,(b-a)}\int_a^b (x-t)^{n-1} r(t,x)\, q^{(n)}(t)\,dt. \tag{4.3}$$

Multiplying (4.2) by $q(x)$ and (4.3) by $p(x)$ and summing the resulting identities and integrating from a to b and rewriting we have

$$\begin{aligned} \frac{1}{b-a}\int_a^b p(x)\,q(x)\,dx &= n\left(\frac{1}{b-a}\int_a^b p(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b q(x)\,dx\right) \\ &\quad - \frac{1}{2(b-a)}\int_a^b \left[\left(\sum_{k=1}^{n-1} P_k(x)\right) q(x) + \left(\sum_{k=1}^{n-1} Q_k(x)\right) p(x)\right] dx \\ &\quad + \frac{1}{2(n-1)!\,(b-a)^2}\left[\int_a^b q(x)\left\{\int_a^b (x-t)^{n-1} r(t,x)\, p^{(n)}(t)\,dt\right\} dx \right. \\ &\quad \left. + \int_a^b p(x)\left\{\int_a^b (x-t)^{n-1} r(t,x)\, q^{(n)}(t)\,dt\right\} dx\right]. \end{aligned} \tag{4.4}$$

From (4.4) and following the similar arguments as in the last part of the proof of Theorem 2.1, we get the desired inequality in (2.6). The proof is complete.

REFERENCES

[1] P. CERONE, S.S. DRAGOMIR AND J. ROUMELIOTIS, Some Ostrowski type inequalities for $n-$time differentiable mappings and applications, *Demonstratio Mathematica*, **32**(2) (1999), 679–712. *RGMIA Research Report Collection*, **1**(1) (1998), 51–66. ONLINE: `http://rgmia.vu.edu.au/v1n1.html`

[2] A.M. FINK, Bounds on the deviation of a function from its averages, *Chechslovak Math. Jour.*, **42** (1992), 289–310.

[3] G.V. MILOVANOVIĆ AND J.E. PEČARIĆ, On generalization of the inequality of A. Ostrowski and some related applications, *Univ. Beograd. Publ. Elek. Fak. Ser. Mat. Fiz.*, **544–576** (1976), 155–158.

[4] D.S MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.

[5] B.G. PACHPATTE, A note on some inequalities analogous to Grüss inequality, *Octogon Math. Magazine*, **5** (1997), 62–66.

[6] J. SÁNDOR, On the Jensen-Hadamard inequality, *Studia Univ. Babes-Bolyai, Math.*, **36**(1) (1991), 9–15.