Journal of Inequalities in Pure and Applied Mathematics

ON GENERALIZATIONS OF L. YANG'S INEQUALITY

CHANG-JIAN ZHAO AND LOKENATH DEBNATH

Department of Mathematics
Shanghai University
Shanghai 200436
The People's Republic of China;

and

Department of Mathematics
Binzhou Teachers College
Binzhou, Shandong 256604
The People's Republic of China.
EMail: chjzhao@163.com

Department of Mathematics
University of Texas-Pan American
Edinburg, Texas 78539-2999, U.S.A.
EMail: Debnathl@panam.edu

volume 3, issue 4, article 56, 2002.

Received 03 December, 2001; accepted 02 June, 2002.

Communicated by: Feng Qi

ISSN (electronic): 1443-5756
085-01

Abstract

A geometric inequality due to L. Yang involving cosine and sine functions is generalized.

2000 Mathematics Subject Classification: 26D15
Key words: L. Yang's inequality, Theory of distribution of functional values, Jensen inequality

Contents

1. Introduction

The well-known inequality due to L. Yang [1, pp. 116–118] can be stated as follows:

If $A > 0$, $B > 0$, $A + B \leq \pi$ and $0 \leq \lambda \leq 1$, then

$$\cos^2 \lambda A + \cos^2 \lambda B - 2\cos \lambda A \cdot \cos \lambda B \cdot \cos \lambda \pi \geq \sin^2 \lambda \pi. \tag{1.1}$$

The equality in (1.1) holds if and only if $\lambda = 0$ or $A + B = \pi$.

L. Yang's inequality plays an important role in the theory of distribution of values of functions. See [1].

In this paper we will generalize inequality (1.1).

2. Main Results

In this paper, we use the following notations and abbreviations:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!};$$

$$x_{ij}^{[1]} = \frac{\lambda}{2}(\pi + A_i + A_j), \quad x_{ij}^{[2]} = \frac{\lambda}{2}(\pi - A_i - A_j);$$

$$x_{ij}^{[3]} = \frac{\lambda}{2}(\pi + A_i - A_j), \quad x_{ij}^{[4]} = \frac{\lambda}{2}(\pi - A_i + A_j);$$

$$H_{ij} = \cos^2 \lambda A_i + \cos^2 \lambda A_j - 2\cos \lambda A_i \cdot \cos \lambda A_j \cdot \cos \lambda\pi.$$

Lemma 2.1 ([2]). *If* $A_i > 0$, $A_j > 0$, $A_i + A_j \le \pi$ *for* $1 \le i, j \le n$, *and* $-1 \le \lambda \le 1$, *then*

$$H_{ij} - \sin^2 \lambda\pi = 4 \prod_{k=1}^{4} \sin x_{ij}^{[k]}.$$

Proof. By using the following two identities

$$\cos \lambda (A_i + A_j) \cos \lambda (A_i - A_j) = \cos^2 \lambda A_i + \cos^2 \lambda A_j - 1,$$

$$\cos \lambda (A_i + A_j) + \cos \lambda (A_i - A_j) = 2 \cos \lambda A_i \cos \lambda A_j,$$

it is easy to observe that

$$\begin{aligned} &H_{ij} - \sin^2 \lambda\pi \\ &\quad = \cos^2 \lambda A_i + \cos^2 \lambda A_j - 1 - 2\cos \lambda A_i \cos \lambda A_j \cos \lambda\pi + \cos^2 \lambda\pi \\ &\quad = \cos^2 \lambda\pi + \cos \lambda (A_i + A_j) \cos \lambda (A_i - A_j) \\ &\qquad - [\cos \lambda (A_i + A_j) + \cos \lambda (A_i - A_j)] \cos \lambda\pi \end{aligned}$$

$$= [\cos\lambda\pi - \cos\lambda(A_i + A_j)][\cos\lambda\pi - \cos\lambda(A_i - A_j)]$$
$$= 4\sin\frac{\lambda}{2}(\pi + A_i + A_j)\sin\frac{\lambda}{2}(\pi - A_i - A_j)$$
$$\times \sin\frac{\lambda}{2}(\pi + A_i - A_j)\sin\frac{\lambda}{2}(\pi - A_i + A_j).$$

The proof is complete. □

Theorem 2.2. *If* $A_i > 0$, $\lambda_k > 0$ $(i = 1, 2, \ldots, n;\ k = 1, 2, 3, 4)$, $\sum_{i=1}^{n} A_i \le \pi$, $n \ge 2$ *being a natural number, and* $-1 \le \lambda \le 1$, *then*

$$\binom{n}{2}\sin^2\lambda\pi \le (n - 1 + \cos\lambda\pi)\sum_{k=1}^{n}\cos^2\lambda A_k - \cos\lambda\pi\left[\sum_{i=1}^{n}\cos\lambda A_i\right]^2 \le \binom{n}{2}\sin^2\lambda\pi + \frac{\left(\sum_{k=1}^{4}\lambda_k\right)^4}{64\prod_{k=1}^{4}\lambda_k}\sum_{1\le i<j\le n}\sin^4\theta_{ij}, \tag{2.1}$$

where

$$\theta_{ij} = \frac{\sum_{k=1}^{4}\lambda_k x_{ij}^{[k]}}{\sum_{k=1}^{4}\lambda_k}. \tag{2.2}$$

The equalities in (2.1) hold if and only if $\lambda = 0$.

Proof. Since $y = \sin x$ is a continuous and convex (or concave, resp.) function on $[-\pi, 0]$ (or $[0, \pi]$, resp.), and $x_{ij}^{[k]} \in [-\pi, 0]$ (or $[0, \pi]$, resp.) for $-1 \le \lambda \le 0$

(or $0 \le \lambda \le 1$, resp.), and using Jensen's inequality (see [3]), we observe that

$$\sin\theta_{ij} \le (\text{or } \ge \text{, resp.}) \frac{\sum_{k=1}^{4} \lambda_k \sin x_{ij}^{[k]}}{\sum_{k=1}^{4} \lambda_k}.$$

Consequently

$$\left(\sum_{k=1}^{4} \lambda_k\right)^4 (\sin\theta_{ij})^4 \ge \left(\sum_{k=1}^{4} \lambda_k \sin x_{ij}^{[k]}\right)^4. \tag{2.3}$$

On the other hand, since $\lambda_k \sin\left(-x_{ij}^{[k]}\right) \ge 0$ $\left(\text{or } \lambda_k \sin\left(x_{ij}^{[k]}\right) \ge 0\text{, resp.}\right)$, then

$$\begin{aligned} &\frac{1}{4}\sum_{k=1}^{4} \lambda_k \sin\left(-x_{ij}^{[k]}\right) \ge \sqrt[4]{\prod_{k=1}^{4} \lambda_k \sin x_{ij}^{[k]}}. \\ &\left(\text{or } \frac{1}{4}\sum_{k=1}^{4} \lambda_k \sin\left(x_{ij}^{[k]}\right) \ge \sqrt[4]{\prod_{k=1}^{4} \lambda_k \sin x_{ij}^{[k]}}\text{, resp.}\right). \end{aligned} \tag{2.4}$$

From (2.3) and (2.4), we obtain

$$\prod_{k=1}^{4} \sin x_{ij}^{[k]} \le \frac{\left(\sum_{k=1}^{4} \lambda_k\right)^4}{4^4 \prod_{k=1}^{4} \lambda_k} \sin^4\theta_{ij}. \tag{2.5}$$

By using Lemma 2.1, we have

$$\sin^2\lambda\pi \le H_{ij} \le \frac{\left(\sum_{k=1}^{4} \lambda_k\right)^4}{64 \prod_{k=1}^{4} \lambda_k} \sin^4\theta_{ij} + \sin^2\lambda\pi. \tag{2.6}$$

Summing both sides of (2.6) for $1 \le i < j \le n$ yields

$$\sum_{1\le i<j\le n} \sin^2 \lambda\pi \le \sum_{1\le i<j\le n} H_{ij} \le \sum_{1\le i<j\le n} \left(\frac{\left(\sum_{k=1}^{4} \lambda_k\right)^4}{64 \prod_{k=1}^{4} \lambda_k} \sin^4 \theta_{ij} + \sin^2 \lambda\pi \right). \tag{2.7}$$

It is not difficult to see that

$$\sum_{1\le i<j\le n} \sin^2 \lambda\pi = \binom{n}{2} \sin^2 \lambda\pi. \tag{2.8}$$

Direct computing yields

$$\sum_{1\le i<j\le n} \left(\frac{\left(\sum_{k=1}^{4} \lambda_k\right)^4}{64 \prod_{k=1}^{4} \lambda_k} \sin^4 \theta_{ij} + \sin^2 \lambda\pi \right) = \frac{\left(\sum_{k=1}^{4} \lambda_k\right)^4}{64 \prod_{k=1}^{4} \lambda_k} \sum_{1\le i<j\le n} \sin^4 \theta_{ij} + \binom{n}{2} \sin^2 \lambda\pi, \tag{2.9}$$

and

$$\sum_{1\le i<j\le n} H_{ij} = \sum_{1\le i<j\le n} \left(\cos^2 \lambda A_i + \cos^2 \lambda A_j - 2\cos \lambda A_i \cos \lambda A_j \cos \lambda\pi\right) \tag{2.10}$$

$$= \sum_{1\le i<j\le n} (\cos^2 \lambda A_i + \cos^2 \lambda A_j) - \sum_{1\le i<j\le n} 2\cos\lambda A_i \cos\lambda A_j \cos\lambda\pi$$

$$= (n-1)\sum_{k=1}^{n} \cos^2 \lambda A_k - \cos\lambda\pi \left[\left(\sum_{i=1}^{n} \cos\lambda A_i \right)^2 - \sum_{i=1}^{n} \cos^2 \lambda A_i \right]$$

$$= (n-1+\cos\lambda\pi)\sum_{k=1}^{n} \cos^2 \lambda A_k - \cos\lambda\pi \left(\sum_{i=1}^{n} \cos\lambda A_i \right)^2 .$$

Putting (2.8), (2.9) and (2.10) into (2.7), inequality (2.7) reduces to inequality (2.1). The proof is complete. □

Remark 2.1. *If taking* $\lambda_i = \frac{1}{4}$ $(i = 1,2,3,4)$ *in the right-hand of* (2.7), *we find that*

$$\sum_{1\le i<j\le n} \left(\frac{\left(\sum_{k=1}^{4} \lambda_k\right)^4}{64\prod_{k=1}^{4} \lambda_k} \sin^4 \theta_{ij} + \sin^2 \lambda\pi \right) \tag{2.11}$$

$$= \sum_{1\le i<j\le n} \left(\frac{\left(\sum_{k=1}^{4} \lambda_k\right)^4}{64\prod_{k=1}^{4} \lambda_k} \left(\sin \frac{\sum_{k=1}^{4} \lambda_k x_{ij}^{[k]}}{\sum_{k=1}^{4} \lambda_k} \right)^4 + \sin^2 \lambda\pi \right)$$

$$= \sum_{1\le i<j\le n} \left(4\left(\sin\left(\frac{1}{4}\sum_{k=1}^{4} x_{ij}^{[k]} \right) \right)^4 + \sin^2 \lambda\pi \right)$$

$$= \sum_{1\le i<j\le n} \left(4\sin^4 \frac{\lambda}{2}\pi + \sin^2 \lambda\pi \right) = 4\binom{n}{2} \sin^2 \frac{\lambda\pi}{2}.$$

Therefore, inequality (2.1) reduces to the following inequality

$$\binom{n}{2}\sin^2\lambda\pi \le (n-1+\cos\lambda\pi)\sum_{k=1}^{n}\cos^2\lambda A_k - \cos\lambda\pi\left(\sum_{i=1}^{n}\cos\lambda A_i\right)^2 \le 4\binom{n}{2}\sin^2\frac{\lambda}{2}\pi. \tag{2.12}$$

The equalities in (2.12) hold if and only if $\lambda = 0$.

Letting $n = 2$ *in (2.12), we have the following result: If* $A > 0$, $B > 0$, $A + B \le \pi$, *and* $-1 \le \lambda \le 1$, *then*

$$\sin^2\lambda\pi \le \cos^2\lambda A + \cos^2\lambda B - 2\cos\lambda A\cos\lambda B\cos\lambda\pi \le 4\sin^2\frac{\lambda}{2}\pi. \tag{2.13}$$

The equalities in (2.13) holds if and only if $\lambda = 0$.

It is obvious that inequality (2.13) is the same as L. Yang's inequality (1.1).

Theorem 2.3. *If* $A_i > 0$ *and* $\lambda_k > 0$ *for* $i = 1, 2, \ldots, n$ *and* $k = 1, 2, 3, 4$, $\sum_{i=1}^{n} A_i \le \pi$, $n \ge 2$, $-1 \le \lambda \le 1$, *then*

$$0 \le (n-1)\sum_{k=1}^{n}\cos^2\lambda A_k - \binom{n}{2}\sin^2\lambda\pi - \cos\lambda\pi\sum_{\substack{1\le i,j\le n\\ i\ne j}}\cos\lambda A_i\cos\lambda A_j \tag{2.14}$$

$$\leq \frac{\left(\sum_{k=1}^{4}\lambda_k\right)^4}{128\prod_{k=1}^{4}\lambda_k}\sum_{\substack{1\leq i,j\leq n\\ i\neq j}}\sin^4\theta_{ij},$$

where

$$\theta_{ij} = \frac{\sum_{k=1}^{4}\lambda_k x_{ij}^{[k]}}{\sum_{k=1}^{4}\lambda_k}.$$

The equalities in (2.14) hold if and only if $\lambda = 0$.

Proof. It follows from inequality (2.6) that

$$0 \leq H_{ij} - \sin^2\lambda\pi \leq \frac{\left(\sum_{k=1}^{4}\lambda_k\right)^4}{64\prod_{k=1}^{4}\lambda_k}\sin^4\theta_{ij}. \tag{2.15}$$

Summing both sides of (2.15) for $i \neq j$, first over j from 1 to n and then over i from 1 to n of the resulting inequality, then

$$0 \leq \sum_{\substack{1\leq i,j\leq n\\ i\neq j}} H_{ij} - 2\binom{n}{2}\sin^2\lambda\pi \leq \frac{\left(\sum_{k=1}^{4}\lambda_k\right)^4}{64\prod_{k=1}^{4}\lambda_k}\sum_{\substack{1\leq i,j\leq n\\ i\neq j}}\sin^4\theta_{ij}. \tag{2.16}$$

On the other hand

$$\sum_{\substack{1\leq i,j\leq n\\ i\neq j}} H_{ij} = 2(n-1)\sum_{k=1}^{n}\cos^2\lambda A_k - 2\cos\lambda\pi\sum_{\substack{1\leq i,j\leq n\\ i\neq j}}\cos\lambda A_i\cos\lambda A_j. \tag{2.17}$$

Putting (2.17) into (2.16), we get inequality (2.14). ☐

Remark 2.2. *In* (2.14), *if* $\lambda_k = \frac{1}{4}$ *for* $k = 1, 2, 3, 4$, *then*

$$0 \le (n-1)\sum_{k=1}^{n}\cos^2\lambda A_k - \binom{n}{2}\sin^2\lambda\pi - \cos\lambda\pi \sum_{\substack{1\le i,j\le n \\ i\ne j}} \cos\lambda A_i \cos\lambda A_j \le \binom{n}{2}\sin^4\frac{\lambda}{2}\pi.$$

The equalities in (2.18) *hold if and only if* $\lambda = 0$.

Letting $n = 2$, *then* (2.18) *reduces to inequality* (2.13).

References

[1] L. YANG, *Distribution of Values and New Research*, Science Press, Beijing, 1982. (Chinese).

[2] Ch.-J. ZHAO, *Further research of L. Yang's inequality*, J. Binzhou Teachers College, **12**(4) (1996), 32–34. (Chinese).

[3] D. S. MITRINOVIĆ, *Analytic Inequalities*, Springer-Verlag, 1970.