

# Journal of Inequalities in Pure and Applied Mathematics

## ON GENERALIZATIONS OF L. YANG'S INEQUALITY

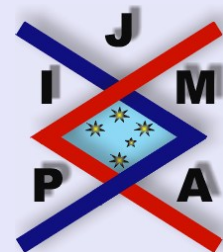
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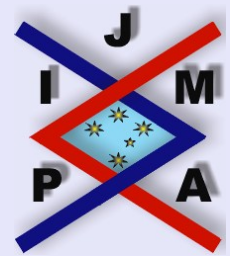
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## Abstract

A geometric inequality due to L. Yang involving cosine and sine functions is generalized.

*2000 Mathematics Subject Classification:* 26D15

*Key words:* L. Yang's inequality, Theory of distribution of functional values, Jensen inequality

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# 1. Introduction

The well-known inequality due to L. Yang [1, pp. 116–118] can be stated as follows:

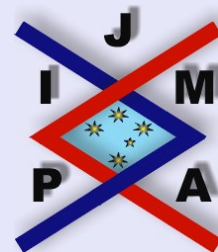
If  $A > 0$ ,  $B > 0$ ,  $A + B \leq \pi$  and  $0 \leq \lambda \leq 1$ , then

$$(1.1) \quad \cos^2 \lambda A + \cos^2 \lambda B - 2 \cos \lambda A \cdot \cos \lambda B \cdot \cos \lambda \pi \geq \sin^2 \lambda \pi.$$

The equality in (1.1) holds if and only if  $\lambda = 0$  or  $A + B = \pi$ .

L. Yang's inequality plays an important role in the theory of distribution of values of functions. See [1].

In this paper we will generalize inequality (1.1).



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## 2. Main Results

In this paper, we use the following notations and abbreviations:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!};$$

$$x_{ij}^{[1]} = \frac{\lambda}{2}(\pi + A_i + A_j), \quad x_{ij}^{[2]} = \frac{\lambda}{2}(\pi - A_i - A_j);$$

$$x_{ij}^{[3]} = \frac{\lambda}{2}(\pi + A_i - A_j), \quad x_{ij}^{[4]} = \frac{\lambda}{2}(\pi - A_i + A_j);$$

$$H_{ij} = \cos^2 \lambda A_i + \cos^2 \lambda A_j - 2 \cos \lambda A_i \cdot \cos \lambda A_j \cdot \cos \lambda \pi.$$

**Lemma 2.1 ([2]).** *If  $A_i > 0$ ,  $A_j > 0$ ,  $A_i + A_j \leq \pi$  for  $1 \leq i, j \leq n$ , and  $-1 \leq \lambda \leq 1$ , then*

$$H_{ij} - \sin^2 \lambda \pi = 4 \prod_{k=1}^4 \sin x_{ij}^{[k]}.$$

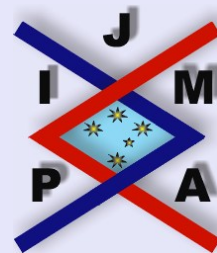
*Proof.* By using the following two identities

$$\cos \lambda (A_i + A_j) \cos \lambda (A_i - A_j) = \cos^2 \lambda A_i + \cos^2 \lambda A_j - 1,$$

$$\cos \lambda (A_i + A_j) + \cos \lambda (A_i - A_j) = 2 \cos \lambda A_i \cos \lambda A_j,$$

it is easy to observe that

$$\begin{aligned} H_{ij} - \sin^2 \lambda \pi &= \cos^2 \lambda A_i + \cos^2 \lambda A_j - 1 - 2 \cos \lambda A_i \cos \lambda A_j \cos \lambda \pi + \cos^2 \lambda \pi \\ &= \cos^2 \lambda \pi + \cos \lambda (A_i + A_j) \cos \lambda (A_i - A_j) \\ &\quad - [\cos \lambda (A_i + A_j) + \cos \lambda (A_i - A_j)] \cos \lambda \pi \end{aligned}$$



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$$\begin{aligned}
&= [\cos \lambda \pi - \cos \lambda (A_i + A_j)] [\cos \lambda \pi - \cos \lambda (A_i - A_j)] \\
&= 4 \sin \frac{\lambda}{2} (\pi + A_i + A_j) \sin \frac{\lambda}{2} (\pi - A_i - A_j) \\
&\quad \times \sin \frac{\lambda}{2} (\pi + A_i - A_j) \sin \frac{\lambda}{2} (\pi - A_i + A_j).
\end{aligned}$$

The proof is complete.  $\square$

**Theorem 2.2.** *If  $A_i > 0$ ,  $\lambda_k > 0$  ( $i = 1, 2, \dots, n$ ;  $k = 1, 2, 3, 4$ ),  $\sum_{i=1}^n A_i \leq \pi$ ,  $n \geq 2$  being a natural number, and  $-1 \leq \lambda \leq 1$ , then*

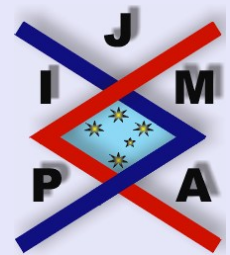
$$\begin{aligned}
(2.1) \quad &\binom{n}{2} \sin^2 \lambda \pi \\
&\leq (n - 1 + \cos \lambda \pi) \sum_{k=1}^n \cos^2 \lambda A_k - \cos \lambda \pi \left[ \sum_{i=1}^n \cos \lambda A_i \right]^2 \\
&\leq \binom{n}{2} \sin^2 \lambda \pi + \frac{(\sum_{k=1}^4 \lambda_k)^4}{64 \prod_{k=1}^4 \lambda_k} \sum_{1 \leq i < j \leq n} \sin^4 \theta_{ij},
\end{aligned}$$

where

$$(2.2) \quad \theta_{ij} = \frac{\sum_{k=1}^4 \lambda_k x_{ij}^{[k]}}{\sum_{k=1}^4 \lambda_k}.$$

The equalities in (2.1) hold if and only if  $\lambda = 0$ .

*Proof.* Since  $y = \sin x$  is a continuous and convex (or concave, resp.) function on  $[-\pi, 0]$  (or  $[0, \pi]$ , resp.), and  $x_{ij}^{[k]} \in [-\pi, 0]$  (or  $[0, \pi]$ , resp.) for  $-1 \leq \lambda \leq 0$



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(or  $0 \leq \lambda \leq 1$ , resp.), and using Jensen's inequality (see [3]), we observe that

$$\sin \theta_{ij} \leq (\text{or } \geq, \text{ resp.}) \frac{\sum_{k=1}^4 \lambda_k \sin x_{ij}^{[k]}}{\sum_{k=1}^4 \lambda_k}.$$

Consequently

$$(2.3) \quad \left( \sum_{k=1}^4 \lambda_k \right)^4 (\sin \theta_{ij})^4 \geq \left( \sum_{k=1}^4 \lambda_k \sin x_{ij}^{[k]} \right)^4.$$

On the other hand, since  $\lambda_k \sin(-x_{ij}^{[k]}) \geq 0$  (or  $\lambda_k \sin(x_{ij}^{[k]}) \geq 0$ , resp.), then

$$(2.4) \quad \frac{1}{4} \sum_{k=1}^4 \lambda_k \sin(-x_{ij}^{[k]}) \geq \sqrt[4]{\prod_{k=1}^4 \lambda_k \sin x_{ij}^{[k]}}.$$

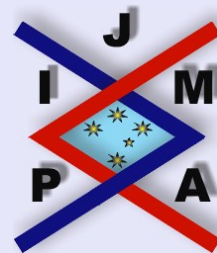
$$\left( \text{or } \frac{1}{4} \sum_{k=1}^4 \lambda_k \sin(x_{ij}^{[k]}) \geq \sqrt[4]{\prod_{k=1}^4 \lambda_k \sin x_{ij}^{[k]}}, \text{ resp.} \right).$$

From (2.3) and (2.4), we obtain

$$(2.5) \quad \prod_{k=1}^4 \sin x_{ij}^{[k]} \leq \frac{(\sum_{k=1}^4 \lambda_k)^4}{4^4 \prod_{k=1}^4 \lambda_k} \sin^4 \theta_{ij}.$$

By using Lemma 2.1, we have

$$(2.6) \quad \sin^2 \lambda \pi \leq H_{ij} \leq \frac{(\sum_{k=1}^4 \lambda_k)^4}{64 \prod_{k=1}^4 \lambda_k} \sin^4 \theta_{ij} + \sin^2 \lambda \pi.$$



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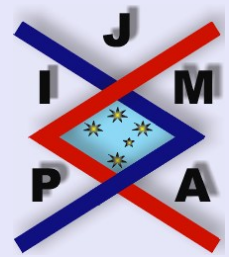


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Summing both sides of (2.6) for  $1 \leq i < j \leq n$  yields

$$(2.7) \quad \sum_{1 \leq i < j \leq n} \sin^2 \lambda \pi \leq \sum_{1 \leq i < j \leq n} H_{ij} \\ \leq \sum_{1 \leq i < j \leq n} \left( \frac{(\sum_{k=1}^4 \lambda_k)^4}{64 \prod_{k=1}^4 \lambda_k} \sin^4 \theta_{ij} + \sin^2 \lambda \pi \right).$$

It is not difficult to see that

$$(2.8) \quad \sum_{1 \leq i < j \leq n} \sin^2 \lambda \pi = \binom{n}{2} \sin^2 \lambda \pi.$$

Direct computing yields

$$(2.9) \quad \sum_{1 \leq i < j \leq n} \left( \frac{(\sum_{k=1}^4 \lambda_k)^4}{64 \prod_{k=1}^4 \lambda_k} \sin^4 \theta_{ij} + \sin^2 \lambda \pi \right) \\ = \frac{(\sum_{k=1}^4 \lambda_k)^4}{64 \prod_{k=1}^4 \lambda_k} \sum_{1 \leq i < j \leq n} \sin^4 \theta_{ij} + \binom{n}{2} \sin^2 \lambda \pi,$$

and

$$(2.10) \quad \sum_{1 \leq i < j \leq n} H_{ij} \\ = \sum_{1 \leq i < j \leq n} (\cos^2 \lambda A_i + \cos^2 \lambda A_j - 2 \cos \lambda A_i \cos \lambda A_j \cos \lambda \pi)$$

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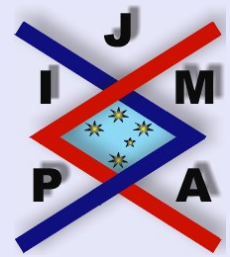


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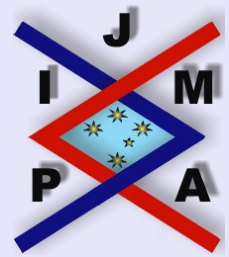
$$\begin{aligned}
 &= \sum_{1 \leq i < j \leq n} (\cos^2 \lambda A_i + \cos^2 \lambda A_j) - \sum_{1 \leq i < j \leq n} 2 \cos \lambda A_i \cos \lambda A_j \cos \lambda \pi \\
 &= (n-1) \sum_{k=1}^n \cos^2 \lambda A_k - \cos \lambda \pi \left[ \left( \sum_{i=1}^n \cos \lambda A_i \right)^2 - \sum_{i=1}^n \cos^2 \lambda A_i \right] \\
 &= (n-1 + \cos \lambda \pi) \sum_{k=1}^n \cos^2 \lambda A_k - \cos \lambda \pi \left( \sum_{i=1}^n \cos \lambda A_i \right)^2.
 \end{aligned}$$

Putting (2.8), (2.9) and (2.10) into (2.7), inequality (2.7) reduces to inequality (2.1). The proof is complete.  $\square$

**Remark 2.1.** If taking  $\lambda_i = \frac{1}{4}$  ( $i = 1, 2, 3, 4$ ) in the right-hand of (2.7), we find that

$$\begin{aligned}
 (2.11) \quad &\sum_{1 \leq i < j \leq n} \left( \frac{(\sum_{k=1}^4 \lambda_k)^4}{64 \prod_{k=1}^4 \lambda_k} \sin^4 \theta_{ij} + \sin^2 \lambda \pi \right) \\
 &= \sum_{1 \leq i < j \leq n} \left( \frac{(\sum_{k=1}^4 \lambda_k)^4}{64 \prod_{k=1}^4 \lambda_k} \left( \sin \frac{\sum_{k=1}^4 \lambda_k x_{ij}^{[k]}}{\sum_{k=1}^4 \lambda_k} \right)^4 + \sin^2 \lambda \pi \right) \\
 &= \sum_{1 \leq i < j \leq n} \left( 4 \left( \sin \left( \frac{1}{4} \sum_{k=1}^4 x_{ij}^{[k]} \right) \right)^4 + \sin^2 \lambda \pi \right) \\
 &= \sum_{1 \leq i < j \leq n} \left( 4 \sin^4 \frac{\lambda}{2} \pi + \sin^2 \lambda \pi \right) = 4 \binom{n}{2} \sin^2 \frac{\lambda \pi}{2}.
 \end{aligned}$$





Therefore, inequality (2.1) reduces to the following inequality

$$\begin{aligned}
 (2.12) \quad & \binom{n}{2} \sin^2 \lambda \pi \\
 & \leq (n-1 + \cos \lambda \pi) \sum_{k=1}^n \cos^2 \lambda A_k - \cos \lambda \pi \left( \sum_{i=1}^n \cos \lambda A_i \right)^2 \\
 & \leq 4 \binom{n}{2} \sin^2 \frac{\lambda}{2} \pi.
 \end{aligned}$$

The equalities in (2.12) hold if and only if  $\lambda = 0$ .

Letting  $n = 2$  in (2.12), we have the following result: If  $A > 0$ ,  $B > 0$ ,  $A + B \leq \pi$ , and  $-1 \leq \lambda \leq 1$ , then

$$(2.13) \quad \sin^2 \lambda \pi \leq \cos^2 \lambda A + \cos^2 \lambda B - 2 \cos \lambda A \cos \lambda B \cos \lambda \pi \leq 4 \sin^2 \frac{\lambda}{2} \pi.$$

The equalities in (2.13) holds if and only if  $\lambda = 0$ .

It is obvious that inequality (2.13) is the same as L. Yang's inequality (1.1).

**Theorem 2.3.** If  $A_i > 0$  and  $\lambda_k > 0$  for  $i = 1, 2, \dots, n$  and  $k = 1, 2, 3, 4$ ,  $\sum_{i=1}^n A_i \leq \pi$ ,  $n \geq 2$ ,  $-1 \leq \lambda \leq 1$ , then

$$\begin{aligned}
 (2.14) \quad & 0 \leq (n-1) \sum_{k=1}^n \cos^2 \lambda A_k - \binom{n}{2} \sin^2 \lambda \pi \\
 & \quad - \cos \lambda \pi \sum_{\substack{1 \leq i, j \leq n \\ i \neq j}} \cos \lambda A_i \cos \lambda A_j
 \end{aligned}$$

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$$\leq \frac{(\sum_{k=1}^4 \lambda_k)^4}{128 \prod_{k=1}^4 \lambda_k} \sum_{\substack{1 \leq i, j \leq n \\ i \neq j}} \sin^4 \theta_{ij},$$

where

$$\theta_{ij} = \frac{\sum_{k=1}^4 \lambda_k x_{ij}^{[k]}}{\sum_{k=1}^4 \lambda_k}.$$

The equalities in (2.14) hold if and only if  $\lambda = 0$ .

*Proof.* It follows from inequality (2.6) that

$$(2.15) \quad 0 \leq H_{ij} - \sin^2 \lambda \pi \leq \frac{(\sum_{k=1}^4 \lambda_k)^4}{64 \prod_{k=1}^4 \lambda_k} \sin^4 \theta_{ij}.$$

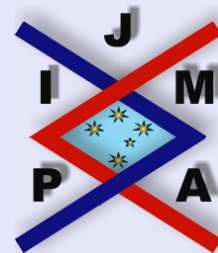
Summing both sides of (2.15) for  $i \neq j$ , first over  $j$  from 1 to  $n$  and then over  $i$  from 1 to  $n$  of the resulting inequality, then

$$(2.16) \quad 0 \leq \sum_{\substack{1 \leq i, j \leq n \\ i \neq j}} H_{ij} - 2 \binom{n}{2} \sin^2 \lambda \pi \leq \frac{(\sum_{k=1}^4 \lambda_k)^4}{64 \prod_{k=1}^4 \lambda_k} \sum_{\substack{1 \leq i, j \leq n \\ i \neq j}} \sin^4 \theta_{ij}.$$

On the other hand

$$(2.17) \quad \sum_{\substack{1 \leq i, j \leq n \\ i \neq j}} H_{ij} = 2(n-1) \sum_{k=1}^n \cos^2 \lambda A_k - 2 \cos \lambda \pi \sum_{\substack{1 \leq i, j \leq n \\ i \neq j}} \cos \lambda A_i \cos \lambda A_j.$$

Putting (2.17) into (2.16), we get inequality (2.14).  $\square$



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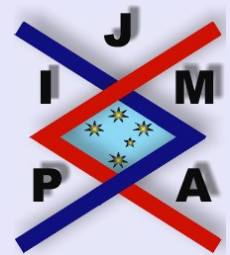
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**Remark 2.2.** In (2.14), if  $\lambda_k = \frac{1}{4}$  for  $k = 1, 2, 3, 4$ , then

$$\begin{aligned}
 0 &\leq (n-1) \sum_{k=1}^n \cos^2 \lambda A_k - \binom{n}{2} \sin^2 \lambda \pi \\
 &\quad - \cos \lambda \pi \sum_{\substack{1 \leq i, j \leq n \\ i \neq j}} \cos \lambda A_i \cos \lambda A_j \\
 &\leq \binom{n}{2} \sin^4 \frac{\lambda}{2} \pi.
 \end{aligned}$$

The equalities in (2.18) hold if and only if  $\lambda = 0$ .

Letting  $n = 2$ , then (2.18) reduces to inequality (2.13).




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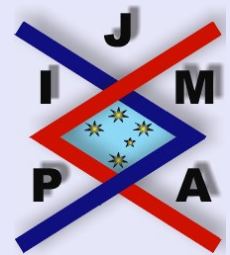
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