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A HÖLDER-TYPE INEQUALITY FOR POSITIVE FUNCTIONALS ON Φ-ALGEBRAS

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ABSTRACT. The main purpose of this paper is to establish with a constructive proof the following Hölder-type inequality: let A be a uniformly complete Φ-algebra, T be a positive linear functional, and p, q be rational numbers such that $p^{-1} + q^{-1} = 1$. Then the inequality

$$T(|fg|) \leq (T(|f|^p))^{1/p} (T(|g|^q))^{1/q}$$

holds for all $f, g \in A$.

Key words and phrases: Hölder inequality, Positive linear functional, f-algebra, Uniformly complete ϕ-algebra.

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The classical Hölder inequalities are often obtained using classical real analysis and convexity. Moreover, these inequalities involve exponents in the field $\mathbb{R}$ of real numbers. The inequalities, when suitably interpreted, make sense in the general context of Φ-algebras, that is, archimedean f-algebras with an identity element. In this more general context the tools of classical real analysis (for instance, convexity of natural logarithm) are not available. In spite of that, surprisingly, we offer a purely algebraic proof of a Hölder-type inequality for positive (linear) functionals on a uniformly complete Φ-algebra. However, although one can define the exponents in $\mathbb{R}$ of elements in a uniformly complete Φ-algebra *via* Krivine's approach (see [9]), which relies heavily on representation theory and then on the Axiom of Choice (i.e., Zorn's Lemma), we reduce our general study to the situation of rational exponents, avoid any use of the representation tools, and keep our proofs intrinsic, constructive and elementary.

In this paper, we use the classical monographs [10] by Luxemburg and Zaanen, and [11] by Zaanen as a starting point, and we refer to these works for unexplained terminology and notations.

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The (relatively) uniform topology on vector lattices (or Riesz spaces) plays a key role in the context of this work. Let us therefore recall the definition and some elementary properties of this topology. By $\mathbb{N}$ we mean the set $\{1,2,...\}$. Let L be an archimedean vector lattice and V be a nonempty subset of L. A sequence $(f_n)_{n\in\mathbb{N}}$ of elements of L is said to *converge V-uniformly* to $f\in L$ if there exists $v\in V$ so that for each real number $\varepsilon>0$ there exists $n_0\in\mathbb{N}$ such that $|f-f_n|\leq\varepsilon|v|$ whenever $n\geq n_0$. In this case, f is called the *V-limit* of $(f_n)_{n\in\mathbb{N}}$ (which is unique because L is assumed to be archimedean). A non empty subset D of L is said to be *V-closed* if D contains all of the V-limits of its V-uniformly convergent sequences. We define thus the closed sets of the so-called *V-topology* on L. An alternative definition as well as elementary properties of the V-topology are presented in [6, 1.2, p. 526]. The well-known *uniform topology* on L is precisely the L-topology. The sequence $(f_n)_{n\in\mathbb{N}}$ in L is called a *uniform Cauchy sequence* if there exists $f\in L$ so that for each real number $\varepsilon>0$ there exists $n_0\in\mathbb{N}$ such that $|f_m-f_n|\leq\varepsilon|f|$ whenever $n,m\geq n_0$. The vector lattice L is said to be *uniformly complete* if every uniform Cauchy sequence in A has a (unique) uniform limit in A. For more background on uniform topology on vector lattices we refer the reader to [10, Sections 16 and 63].

The next paragraph deals with the notion of Φ-algebras. A vector lattice A is called a *lattice-ordered algebra* if there exists an associative multiplication in A with the usual algebraic properties such that $fg\in A^+$ for all $f,g\in A^+$. The lattice-ordered algebra A is said to be an *f-algebra* if $f\wedge g=0$ and $h\geq 0$ in A imply $f\wedge(hg)=f\wedge(gh)=0$. The f-algebras received their name from Birkhoff and Pierce in [3]. The most classical example of an f-algebra is the algebra $C(X)$ of all real-valued continuous functions on a topological space X. The squares in an f-algebra are positive. Using the Axiom of Choice, Birkhoff and Pierce in [3] proved that any archimedean f-algebra is commutative. However, a Zorn's Lemma free proof of this important result, due to Zaanen, can be found in [11, Theorem 140.10]. Besides, Chapter 20 in [11] is devoted to the elementary theory of f-algebras. In [7], Henriksen and Johnson have called an archimedean f-algebra with an identity element a *Φ-algebra.* Let A be a uniformly complete Φ-algebra (Henriksen and Johson used *uniformly closed* instead of *uniformly complete*). It was proven by Beukers and Huijsmans (see [2, Corollary 6]) that for every $f\in A^+$ and $n\in\mathbb{N}$, there exists a unique $g\in A^+$ such that $g^n=f$ (this fact can be also deduced directly from [5, Corollary 4.11] by Buskes, de Pagter and van Rooij). This element g is called the nth-*root* of f and it is denoted by $f^{1/n}$. It follows easily that for every $f\in A$, the p-power f^p is well defined for every nonnegative rational number p. Of course, f^p also is defined for a negative p provided that f has an inverse in A. The uniqueness of nth-roots in A together with the commutativity of A guarantees the validity of classical products rules such as $f^pf^q=f^{p+q},(f^p)^q=f^{pq},(fg)^p=f^pg^p,....$ All of the aforementioned results will be used below without further mention. The reader can consult Section 3 in [7] for more information about uniformly complete Φ-algebras.

Throughout this paper, A stands for a uniformly complete Φ-algebra. The multiplicative identity of A will be denoted by e.

We plunge into the matter by the following basic lemma, which turns out to be useful for later purposes.

Lemma 1. *Let m and n be natural numbers such that $m\leq n$. Then the inequality*

$$m(e-f^n)\leq n(e-f^m)$$

holds for all $f\in A^+$.

Proof. The result is obvious for $m = 0$ or $m = n$. We assume therefore that $0 < m < n$. In particular $n \geq 2$. Consider now the polynomial

$$P(X) = mX^n - nX^m + n - m \in \mathbb{R}[X].$$

It is easily seen that $P(X)$ is divisible by $(X-1)^2$ and that the quotient is

$$Q(X) = \sum_{k=0}^{m-1} (n-m)(k+1) X^k + \sum_{k=m}^{n-2} m(n-(k+1)) X^k$$

(with the second summation equal 0 if $m = n-1$). Clearly all of the coefficients of $Q(X)$ are nonnegative, so $Q(X) \in \mathbb{R}^+[X]$. Accordingly, if $f \in A^+$ then $Q(f) \in A^+$ and therefore

$$P(f) = (f-e)^2 Q(f) \in A^+,$$

that is,

$$mf^n - nf^m + (n-m)e \in A^+.$$

This completes the proof of the lemma. □

The next result is deduced from the lemma above by classical means. The details follow.

Lemma 2. *Let $\alpha \in [0,1]$ be a rational number. Then the inequality*

$$f^\alpha \leq \alpha f + (1-\alpha) e$$

holds for all $f \in A^+$.

Proof. Since the cases $\alpha = 0$ and $\alpha = 1$ are trivial, we suppose that $\alpha \in (0,1)$. Choose natural numbers m and n such that $0 < m < n$ and $\alpha = m/n$. Instead of f in the inequality proved in Lemma 1, take $f^{1/n}$. We get that

$$m\left(e - \left(f^{1/n}\right)^n\right) \leq n\left(e - \left(f^{1/n}\right)^m\right)$$

and therefore

$$f^\alpha = \left(f^{1/n}\right)^m \leq \frac{m}{n} f + \frac{n-m}{n} e = \alpha f + (1-\alpha) e,$$

as required. □

From now on, p and q are rational numbers such that $1 < p, q$ and $p^{-1} + q^{-1} = 1$. Before stating the next lemma, we point out that if $g \in A^+$ has an inverse g^{-1} in A then $g^{-1} \in A^+$ (see Theorem 142.2 in [11]).

Lemma 3. *The inequality*

$$fg \leq p^{-1} f^p + q^{-1} g^q$$

holds for all $f, g \in A^+$.

Proof. Let $f, g \in A^+$ and suppose, at first, that g has an inverse g^{-1} in A^+. It follows from Lemma 2 that

$$\left(fg^{-1}\right)^{1/p} \leq p^{-1} f g^{-1} + q^{-1} e.$$

Multiplying both sides by g, we obtain that

$$f^{1/p} g^{1/q} \leq p^{-1} f + q^{-1} g.$$

Now, let $g \in A^+$ be arbitrary. For each $n \in \mathbb{N}$, we put $g_n = g + n^{-1} e \in A^+$. Since g_n has an inverse in A^+ (see [7, 3.3, p. 84] or [11, Theorem 146.3]), we can apply the result of the first case to g_n. Consequently,

$$f^{1/p} g_n^{1/q} \leq p^{-1} f + q^{-1} g_n \leq p^{-1} f + q^{-1} g + q^{-1} n^{-1} e. \tag{1}$$

On the other hand, it follows from Lemma 2 that if $0 \leq h \leq e$ in A and $0 \leq \alpha \leq 1$ is a rational number then

$$h^{\alpha} \leq \alpha h + (1-\alpha)\, e \leq \alpha e + (1-\alpha)\, e = e.$$

Now the substitution $h = g g_n^{-1}$ yields

$$f^{1/p} g^{1/q} \leq f^{1/p} g_n^{1/q}. \tag{2}$$

Combining (1) and (2), we get

$$f^{1/p} g^{1/q} \leq p^{-1} f + q^{-1} g + q^{-1} n^{-1} e,$$

that is

$$f^{1/p} g^{1/q} - p^{-1} f - q^{-1} g \leq q^{-1} n^{-1} e.$$

Since A is archimedean, we derive

$$f^{1/p} g^{1/q} \leq p^{-1} f + q^{-1} g.$$

Taking in the last inequality f^p and g^q instead of f and g, respectively, we obtain the desired result. □

Let T be a *functional* on A, that is, a linear map from A into $\mathbb{R}$. We say that T is *positive* if $T(f) \geq 0$ for all $f \in A^+$. Next we present some equivalent properties of positive functionals on Φ-algebras.

Theorem 4. *Let T be a positive functional on A and $f \in A^+$. Then the following are equivalent:*

(i) $T(f) = 0$.
(ii) $T(fg) = 0$ *for all* $g \in A$.
(iii) $T(f^m) = 0$ *for some* $m \in \mathbb{N}$.

Proof. The proof we present here was suggested by a referee and it is much more elegant and simpler than the initial one.

(i) ⇒ (ii) We can assume $g \in A^+$. Then from

$$0 \leq g - g \wedge ne \leq n^{-1} g^2 \qquad (n \in \mathbb{N})$$

(see [11, Theorem 142.7]) we derive

$$0 \leq T(fg) - T(fg \wedge nf) \leq n^{-1} T(fg^2) \qquad (n \in \mathbb{N}).$$

But

$$0 \leq T(fg \wedge nf) \leq nT(f) = 0 \qquad (n \in \mathbb{N}).$$

Hence from archimedeanity it follows that $T(fg) = 0$.

(ii) ⇒ (iii) Obvious.

(iii) ⇒ (i) The result is trivial for $m = 1$, so assume that $m \geq 2$. The proof proceeds by induction on m. If $m = 2$ then $T(f^2) = 0$ and therefore

$$0 \leq T\left((nf - e)^2\right) = T(e) - 2nT(f) \qquad (n \in \mathbb{N}).$$

Consequently,

$$0 \leq 2nT(f) \leq T(e) \qquad (n \in \mathbb{N})$$

and then $T(f) = 0$. Now, let $m \geq 3$ such that $T(f^m) = 0$ and assume that the result holds for all m' with $2 \leq m' < m$. Let $k = 0$ or $k = 1$ so that $m + k = 2m'$. In view of '(i) ⇒ (ii)', and since

$$T\left(f^{m+k}\right) = T\left(f^m f^k\right) \quad \text{and} \quad T(f^m) = 0,$$

we get

$$T\left(\left(f^{m'}\right)^2\right) = T\left(f^{2m'}\right) = T\left(f^{m+k}\right) = 0.$$

The induction hypothesis yields that $T\left(f^{m'}\right)=0$ then $T(f)=0$ and we are done. □

We are now in position to prove the main result of the present work.

Theorem 5. *Let T be a positive functional of A. Then the Hölder-type inequality*

$$T(|fg|)\leq\left(T\left(|f|^{p}\right)\right)^{1/p}\left(T\left(|g|^{q}\right)\right)^{1/q}$$

holds for all $f,g\in A$.

Proof. Since $|fg|=|f|\,|g|$, it suffices to show the inequality for $f,g\in A^{+}$. Let $\mu=\left(T\left(f^{p}\right)\right)^{1/p}$ and $\eta=\left(T\left(g^{q}\right)\right)^{1/q}$. First, assume that $\mu\eta\neq 0$. Applying Lemma 3 to $\mu^{-1}f$ and $\eta^{-1}g$, we get

$$\mu^{-1}\eta^{-1}fg\leq p^{-1}\left(\mu^{-1}f\right)^{p}+q^{-1}\left(\eta^{-1}g\right)^{q}=p^{-1}\mu^{-p}f^{p}+q^{-1}\eta^{-q}g^{q}.$$

and therefore

$$\mu^{-1}\eta^{-1}T(fg)=T\left(\mu^{-1}\eta^{-1}fg\right)\leq p^{-1}\mu^{-p}T\left(f^{p}\right)+q^{-1}\eta^{-q}T\left(g^{q}\right)=1.$$

Hence

$$T(fg)\leq\mu\eta=\left(T\left(f^{p}\right)\right)^{1/p}\left(T\left(g^{q}\right)\right)^{1/q}$$

Now, suppose that $\mu\eta=0$. Take for instance $\mu=0$, that is, $T\left(f^{p}\right)=0$. If $r=[p]+1-p$ then

$$T\left(f^{[p]+1}\right)=T\left(f^{p}f^{r}\right)=0,$$

and thus $T(fg)=0$ (by Theorem 4). This completes the proof of the theorem. □

At last, we extend the inequality above to the more general setting of positive linear maps between two uniformly complete Φ-algebras. To this end, we have to recall some definitions. A linear map T between two vector lattices L and V is said to be *positive* if $T(f)\in V^{+}$ whenever $f\in L^{+}$ (the reader is encouraged to consult [1] for the theory of positive linear maps on vector lattices). Let L be an archimedean vector lattice. We call L *slender* after Buskes and van Rooij in [6, 1.2, p. 526] if L contains a countable $\mathbb{Q}$-linear sublattice V such that L is the V-closure of V (here $\mathbb{Q}$ is the field of rational numbers). Finally, recall that a positive element e in a vector lattice L is called a *strong order unit* in L if for each $f\in L$ there exists a real number λ such that $|f|\leq\lambda e$.

At this point, we give our extension result.

Corollary 6. *Let A and B be uniformly complete Φ-algebras and assume that the multiplicative identity of B is a strong order unit in B. If $T:A\to B$ is a positive linear map then the Hölder-type inequality*

$$T(|fg|)\leq\left(T\left(|f|^{p}\right)\right)^{1/p}\left(T\left(|g|^{q}\right)\right)^{1/q}$$

holds for all $f,g\in A$.

Proof. As usual, we can assume that $f,g\in A^{+}$. Consider A_0 the uniformly complete Φ-subalgebra of A generated by f,g and e. In view of Lemma 2.6 in [5], A is countably generated as a vector lattice. Hence by [6, 1.2 (ii)], A_0 is a slender vector sublattice of A. On the other hand, it follows from Lemma 1.3 in [6] that T maps A_0 into a slender vector sublattice L of B. Let V denote the vector sublattice of B generated by L and the unit element e_B of B. By [6, 1.2 (iii)], V is slender. Consider at this point the V-closure B_0 of V in B. Since B is uniformly complete, so is B_0 (by Lemma 1.1 (iii) in [6]). Now, e_B is a strong order unit in B and therefore e_B is a strong order unit in B_0. In summary, B_0 is a uniformly complete slender vector lattice with e_B as strong order unit. We infer that B_0 is supplied with a multiplication $*$ in such a manner that B_0 is a Φ-algebra with e_B as multiplicative identity (see [8, p. 166] for a constructive proof of the existence of $*$). But in view of Proposition 3.6 in [4], $*$ coincides with the multiplication in B and B_0 is thus a uniformly complete slender Φ-subalgebra of B. Consequently, T can be seen as a positive linear map between two uniformly complete slender

Φ-algebras, so we may suppose without loss of generality that both A and B are slender. Take now an arbitrary multiplicative positive functional ω on B. Applying Theorem 5 to the positive functional ωT on A, we derive

$$\omega T(fg) \leq (\omega T(f^p))^{1/p} (\omega T(g^q))^{1/q}.$$

Since ω is multiplicative, we get

$$\omega T(fg) \leq \omega \left[(T(f^p))^{1/p} (T(g^q))^{1/q} \right]$$

and thus

$$\omega \left[(T(f^p))^{1/p} (T(g^q))^{1/q} - T(fg) \right] \geq 0.$$

The last inequality holds for every multiplicative positive functional ω on B and, by Corollary 2.5 (i) in [5], for any real-valued lattice homomorphism ω on B (recall that a real-valued *lattice homomorphism* ω on B is a linear functional ω on B such that $\omega(|f|) = |\omega(f)|$ for all $f \in B$). Since B is assumed to be slender, the set $\mathcal{H}(B)$ of all real-valued lattice homomorphisms on B separates the points of B, that is, if $f \in B$ and $\omega(f) = 0$ for all $\omega \in \mathcal{H}(B)$ then $f = 0$ (by Theorem 2.2 in [6]). In particular, if $f \in B$ and $\omega(f) \geq 0$ for all $\omega \in \mathcal{H}(B)$ then $f \geq 0$. It follows *via* the inequality above that

$$(T(f^p))^{1/p} (T(g^q))^{1/q} - T(fg) \geq 0,$$

which is the desired inequality. □

Comment. The key step in the proof of Corollary 6 above is the construction of the Φ-algebra B_0 such that $\mathcal{H}(B_0)$ separates the points of B_0. But the Φ-algebra B already has this separation property. Indeed, since B is a uniformly complete Φ-algebra the multiplicative identity of which is a strong order unit, B is isomorphic as a Φ-algebra to a $C(X)$ for some compact Hausdorff topological space X (see, for instance, [7, 3.2, p. 84]). The construction of B_0 seems thus to be superfluous. However, such a construction allows us to avoid any use of Axiom of Choice, which is our wish in this paper. Notice that the representation of B by $C(X)$ relies heavily on Zorn's Lemma.

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