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A NEW INEQUALITY SIMILAR TO HILBERT'S INEQUALITY

BICHENG YANG

Department of Mathematics
Guangdong Education College
Guangzhou
Guangdong 510303
PEOPLE'S REPUBLIC OF CHINA
EMail: bcyang@pub.guangzhou.gd.cn

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Abstract

In this paper, we build a new inequality similar to Hilbert's inequality with a best constant factor. As an application, we consider its equivalent form.

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Contents

1. Introduction

If $0 < \sum_{n=0}^{\infty} a_n^2 < \infty$ and $0 < \sum_{n=0}^{\infty} b_n^2 < \infty$, then the famous Hilbert's inequality (see Hardy et al. [1]) is given by

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{m+n+1} < \pi\left(\sum_{n=0}^{\infty} a_n^2 \sum_{n=0}^{\infty} b_n^2\right)^{\frac{1}{2}}, \tag{1.1}$$

where the constant factor π is the best possible. Recently, Yang and Debnath [2, 3] and Yang [4, 5] gave (1.1) some extensions and improvements, and Kuang and Debnath [6] considered its strengthened versions and generalizations.

The major objective of this paper is to build a new inequality similar to (1.1), which relates to the double series form as

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\ln m + \ln n + 1} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\ln emn}. \tag{1.2}$$

For this, we must estimate the following weight coefficient

$$\omega(n) = \sum_{m=1}^{\infty}\frac{1}{m \ln emn}\left(\frac{\ln\sqrt{en}}{\ln\sqrt{em}}\right)^{\frac{1}{2}} \quad (n \in N), \tag{1.3}$$

and do some preparatory works.

2. Some Lemmas

Let f have its first four derivatives on $[1,\infty)$ and $(-1)^n f^{(n)}(x) > 0$ $(n = 0,\dots,4)$, and $f(x),\ f'(x) \longrightarrow 0$ $(x \to \infty)$, then (see [6, (2.1)])

$$\sum_{k=1}^{\infty} f(k) < \int_1^{\infty} f(x)dx + \frac{1}{2}f(1) - \frac{1}{12}f'(1). \tag{2.1}$$

Lemma 2.1. *For $n \in N$, define $R(n)$ as*

$$R(n) = \frac{1}{(2\ln\sqrt{en})^{\frac{1}{2}}}\int_0^{\frac{1}{2\ln\sqrt{en}}} \frac{1}{(1+u)u^{\frac{1}{2}}}du - \frac{2}{3\ln en} - \frac{1}{12(\ln en)^2}. \tag{2.2}$$

Then we have $R(n) > 0 (n \in N)$.

Proof. Integrating by parts, we have

$$\begin{aligned}
\int_0^{\frac{1}{2\ln\sqrt{en}}} \frac{1}{(1+u)u^{\frac{1}{2}}}du &= 2\int_0^{\frac{1}{2\ln\sqrt{en}}} \frac{1}{(1+u)}du^{\frac{1}{2}} \\
&= (2\ln\sqrt{en})^{\frac{1}{2}}\frac{1}{\ln en} + 2\int_0^{\frac{1}{2\ln\sqrt{en}}} u^{\frac{1}{2}}\frac{1}{(1+u)^2}du \\
&= (2\ln\sqrt{en})^{\frac{1}{2}}\frac{1}{\ln en} + \frac{4}{3}\int_0^{\frac{1}{2\ln\sqrt{en}}} \frac{1}{(1+u)^2}du^{3/2} \\
&= (2\ln\sqrt{en})^{\frac{1}{2}}\frac{1}{\ln en} + \frac{1}{3}(2\ln\sqrt{en})^{\frac{1}{2}}\frac{1}{(\ln en)^2} \\
&\quad + \frac{8}{3}\int_0^{\frac{1}{2\ln\sqrt{en}}} u^{3/2}\frac{1}{(1+u)^3}du
\end{aligned}$$

$$> (2\ln\sqrt{en})^{\frac{1}{2}}\frac{1}{\ln en}+\frac{1}{3}(2\ln\sqrt{en})^{\frac{1}{2}}\frac{1}{(\ln en)^2}.$$

Hence by (2.2), we have

$$R(n) > \frac{1}{\ln en}+\frac{1}{3(\ln en)^2}-\frac{2}{3\ln en}-\frac{1}{12(\ln en)^2}=\frac{1}{3\ln en}+\frac{1}{4(\ln en)^2}>0.$$

The lemma is thus proved. □

Lemma 2.2. *If $\omega(n)$ is defined by (1.3), then $\omega(n)<\pi$, for $n\in N$.*

Proof. For fixed $n\in N$, setting

$$f_n(x)=\frac{1}{x\ln enx}\left(\frac{\ln\sqrt{en}}{\ln\sqrt{ex}}\right)^{\frac{1}{2}},\quad x\in[1,\infty),$$

we find $f_n(1)=\frac{1}{\ln en}(2\ln\sqrt{en})^{\frac{1}{2}}$, and

$$f_n'(x)=-\frac{1}{x^2\ln enx}\left(\frac{\ln\sqrt{en}}{\ln\sqrt{ex}}\right)^{\frac{1}{2}}-\frac{1}{x^2\ln^2 enx}\left(\frac{\ln\sqrt{en}}{\ln\sqrt{ex}}\right)^{\frac{1}{2}}$$
$$-\frac{1}{2x^2\ln enx}\cdot\frac{(\ln\sqrt{en})^{\frac{1}{2}}}{(\ln\sqrt{ex})^{\frac{3}{2}}},$$

$$f_n'(1)=-\left(\frac{2}{\ln en}+\frac{1}{\ln^2 en}\right)(2\ln\sqrt{en})^{\frac{1}{2}}.$$

Setting $u = \frac{\ln\sqrt{e}x}{\ln\sqrt{e}n}$ in the following integral, we obtain

$$\int_1^\infty f_n(x)dx = \int_{\frac{1}{2\ln\sqrt{e}n}}^\infty \frac{1}{1+u}\left(\frac{1}{u}\right)^{\frac{1}{2}} du = \pi - \int_0^{\frac{1}{2\ln\sqrt{e}n}} \frac{1}{1+u}\left(\frac{1}{u}\right)^{\frac{1}{2}} du.$$

Hence by (2.1), (2.2) and Lemma 2.1, we have

$$\begin{aligned}
\omega(n) &= \sum_{m=1}^{\infty} f_n(m) \\
&< \int_1^\infty f_n(x)dx + \frac{1}{2}f_n(1) - \frac{1}{12}f_n'(1) \\
&= \pi - \int_0^{1/(2\ln\sqrt{e}n)} \frac{1}{1+u}\left(\frac{1}{u}\right)^{\frac{1}{2}} du + \left(\frac{2}{3\ln en} + \frac{1}{12\ln^2 en}\right)(2\ln\sqrt{e}n)^{\frac{1}{2}} \\
&= \pi - (2\ln\sqrt{e}n)^{\frac{1}{2}} R(n) < \pi.
\end{aligned}$$

The lemma is proved. □

Lemma 2.3. *For* $0 < \epsilon < 1$, *we have*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{mn\ln emn}\left(\frac{1}{\ln\sqrt{e}m\ln\sqrt{e}n}\right)^{\frac{1+\epsilon}{2}} > \frac{1}{\epsilon}(\pi + o(1)) \quad (\epsilon \to 0^+). \tag{2.3}$$

Proof. Setting $u = \frac{\ln\sqrt{e}x}{\ln\sqrt{e}y}$ in the following integral, we find

$$\int_{\sqrt{e}}^\infty \frac{1}{x\ln exy}\left(\frac{1}{\ln\sqrt{e}x}\right)^{\frac{1+\epsilon}{2}} dx$$

$$
\begin{aligned}
&= \left(\frac{1}{\ln\sqrt{ey}}\right)^{\frac{1+\epsilon}{2}} \int_{\frac{1}{\ln\sqrt{ey}}}^{\infty} \frac{1}{1+u}\left(\frac{1}{u}\right)^{\frac{1+\epsilon}{2}} du \\
&= \left(\frac{1}{\ln\sqrt{ey}}\right)^{\frac{1+\epsilon}{2}} \int_{0}^{\infty} \frac{1}{1+u}\left(\frac{1}{u}\right)^{\frac{1+\epsilon}{2}} du \\
&\qquad - \left(\frac{1}{\ln\sqrt{ey}}\right)^{\frac{1+\epsilon}{2}} \int_{0}^{\frac{1}{\ln\sqrt{ey}}} \frac{1}{1+u}\left(\frac{1}{u}\right)^{\frac{1+\epsilon}{2}} du \\
&> \left(\frac{1}{\ln\sqrt{ey}}\right)^{\frac{1+\epsilon}{2}} \int_{0}^{\infty} \frac{1}{1+u}\left(\frac{1}{u}\right)^{\frac{1+\epsilon}{2}} du \\
&\qquad - \left(\frac{1}{\ln\sqrt{ey}}\right)^{\frac{1+\epsilon}{2}} \int_{0}^{\frac{1}{\ln\sqrt{ey}}} \left(\frac{1}{u}\right)^{\frac{1+\epsilon}{2}} du \\
&= \left(\frac{1}{\ln\sqrt{ey}}\right)^{\frac{1+\epsilon}{2}} (\pi + o(1)) - \frac{2}{1-\epsilon}\left(\frac{1}{\ln\sqrt{ey}}\right) \quad (\epsilon \longrightarrow 0^+).
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{mn\ln emn}\left(\frac{1}{\ln\sqrt{em}\ln\sqrt{en}}\right)^{\frac{1+\epsilon}{2}} \\
&\qquad > \int_{\sqrt{e}}^{\infty}\int_{\sqrt{e}}^{\infty} \frac{1}{xy\ln exy}\left(\frac{1}{\ln\sqrt{ex}\ln\sqrt{ey}}\right)^{\frac{1+\epsilon}{2}} dxdy \\
&\qquad = \int_{\sqrt{e}}^{\infty} \frac{1}{y}\left(\frac{1}{\ln\sqrt{ey}}\right)^{\frac{1+\epsilon}{2}} \left[\int_{\sqrt{e}}^{\infty} \frac{1}{x\ln exy}\left(\frac{1}{\ln\sqrt{ex}}\right)^{\frac{1+\epsilon}{2}} dx\right] dy
\end{aligned}
$$

$$
\begin{aligned}
&> (\pi + o(1)) \int_{\sqrt{e}}^{\infty} \frac{1}{y} \left(\frac{1}{\ln \sqrt{e}y} \right)^{1+\epsilon} dy - \frac{2}{1-\epsilon} \int_{\sqrt{e}}^{\infty} \frac{1}{y} \left(\frac{1}{\ln \sqrt{e}y} \right)^{\frac{1+\epsilon}{2}+1} dy \\
&= (\pi + o(1)) \frac{1}{\epsilon} - \frac{4}{1-\epsilon^2} \\
&= \frac{1}{\epsilon}(\pi + o(1)) \quad (\epsilon \to 0^+).
\end{aligned}
$$

The lemma is proved. □

3. Main Result and an Application

Theorem 3.1. *If* $0 < \sum_{n=1}^{\infty} na_n^2 < \infty$ *and* $0 < \sum_{n=1}^{\infty} nb_n^2 < \infty$, *then*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\ln emn} < \pi\left(\sum_{n=1}^{\infty} na_n^2 \sum_{n=1}^{\infty} nb_n^2\right)^{\frac{1}{2}}, \tag{3.1}$$

where the constant factor π *is the best possible.*

Proof. By Cauchy's inequality and (1.3), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\ln emn} \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\frac{a_m}{(\ln emn)^{\frac{1}{2}}}\left(\frac{\ln\sqrt{e}m}{\ln\sqrt{e}n}\right)^{\frac{1}{4}}\left(\frac{m}{n}\right)^{\frac{1}{2}}\right] \\
&\qquad\times\left[\frac{b_n}{(\ln emn)^{\frac{1}{2}}}\left(\frac{\ln\sqrt{e}n}{\ln\sqrt{e}m}\right)^{\frac{1}{4}}\left(\frac{n}{m}\right)^{\frac{1}{2}}\right] \\
&\leq \left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m^2}{\ln emn}\left(\frac{\ln\sqrt{e}m}{\ln\sqrt{e}n}\right)^{\frac{1}{2}}\left(\frac{m}{n}\right)\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{b_n^2}{\ln emn}\left(\frac{\ln\sqrt{e}n}{\ln\sqrt{e}m}\right)^{\frac{1}{2}}\left(\frac{n}{m}\right)\right]^{\frac{1}{2}} \\
&= \left(\sum_{m=1}^{\infty}\omega(m)ma_m^2\sum_{n=1}^{\infty}\omega(n)nb_n^2\right)^{\frac{1}{2}}.
\end{aligned}$$

By Lemma 2.2, we have (3.1).

For $0 < \epsilon < 1$, setting a'_n as:

$$a'_n = \frac{1}{n(\ln \sqrt{e}n)^{\frac{1+\epsilon}{2}}}, \quad n \in N,$$

then we have

$$\begin{aligned}
\sum_{n=1}^{\infty} n{a'_n}^2 &= \frac{1}{(\ln \sqrt{e})^{1+\epsilon}} + \frac{1}{2(\ln 2\sqrt{e})^{1+\epsilon}} + \sum_{n=3}^{\infty} \frac{1}{n(\ln \sqrt{e}n)^{1+\epsilon}} \\
&< \frac{1}{(\ln \sqrt{e})^{1+\epsilon}} + \frac{1}{2(\ln 2\sqrt{e})^{1+\epsilon}} + \int_{\sqrt{e}}^{\infty} \frac{1}{x(\ln \sqrt{e}x)^{1+\epsilon}} dx \\
&= \frac{1}{(\ln \sqrt{e})^{1+\epsilon}} + \frac{1}{2(\ln 2\sqrt{e})^{1+\epsilon}} + \frac{1}{\epsilon} = \frac{1}{\epsilon}(1 + o(1)) \quad (\epsilon \to 0^+). \qquad (3.2)
\end{aligned}$$

If the constant factor π in (3.1) is not the best possible, then there exists a positive number $K < \pi$, such that (3.1) is valid if we change π to K. In particular, we have

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a'_m a'_n}{\ln emn} < K \sum_{n=1}^{\infty} n{a'_n}^2.$$

By (2.3) and (3.2), we have

$$(\pi + o(1)) < \epsilon \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a'_m a'_n}{\ln emn} < K(1 + o(1)) \quad (\epsilon \to 0^+),$$

and $\pi \leq K$. This contradicts that $K < \pi$. Hence the constant factor π in (3.1) is the best possible. The theorem is proved. □

Remark 3.1. *Inequality (3.1) is more similar to the following Mulholland's inequality for $p = q = 2$ (see [7]):*

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{mn\ln emn} < \frac{\pi}{\sin(\frac{\pi}{p})}\left(\sum_{n=2}^{\infty} n^{-1}a_n^p\right)^{\frac{1}{p}}\left(\sum_{n=2}^{\infty} n^{-1}b_n^q\right)^{\frac{1}{q}}. \tag{3.3}$$

Theorem 3.2. *If $0 < \sum_{n=1}^{\infty} na_n^2 < \infty$, then we have*

$$\sum_{n=1}^{\infty}\frac{1}{n}\left(\sum_{m=1}^{\infty}\frac{a_m}{\ln emn}\right)^2 < \pi^2\sum_{n=1}^{\infty} na_n^2, \tag{3.4}$$

where the constant factor π^2 is the best possible. Inequalities (3.1) and (3.4) are equivalent.

Proof. Since $\sum_{n=1}^{\infty} na_n^2 > 0$, there exists $k_0 \geq 1$, such that for any $k > k_0$, we have $\sum_{n=1}^{k} na_n^2 > 0$, and $b_n(k) = \frac{1}{n}\sum_{m=1}^{k}\frac{|a_m|}{lnemn} > 0 \quad (n \in N)$. By (3.1), we have

$$\begin{aligned}
0 &< \left[\sum_{n=1}^{k} nb_n^2(k)\right]^2 \\
&= \left[\sum_{n=1}^{k}\frac{1}{n}\left(\sum_{m=1}^{k}\frac{|a_m|}{\ln emn}\right)^2\right]^2 \\
&= \left[\sum_{n=1}^{k}\sum_{m=1}^{k}\frac{|a_m|b_n(k)}{\ln emn}\right]^2 < \pi^2\sum_{n=1}^{k} na_n^2\sum_{n=1}^{k} nb_n^2(k).
\end{aligned} \tag{3.5}$$

Thus we find

$$0 < \sum_{n=1}^{k} \frac{1}{n} \left(\sum_{m=1}^{k} \frac{|a_m|}{\ln emn} \right)^2 = \sum_{n=1}^{k} n b_n^2(k) < \pi^2 \sum_{n=1}^{k} n a_n^2. \tag{3.6}$$

It follows that $0 < \sum_{n=1}^{\infty} n b_n^2(\infty) \le \pi^2 \sum_{n=1}^{\infty} n a_n^2 < \infty$. Hence by (3.1), for $k \to \infty$, neither (3.5) nor (3.6) takes equality, and we have

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{m=1}^{\infty} \frac{a_m}{\ln emn} \right)^2 \le \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{m=1}^{\infty} \frac{|a_m|}{\ln emn} \right)^2 < \pi^2 \sum_{n=1}^{\infty} n a_n^2.$$

Inequality (3.4) is valid.

On the other hand, if (3.4) holds, by Cauchy's inequality, we have

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\ln emn} &= \sum_{n=1}^{\infty} \left(\frac{1}{n^{\frac{1}{2}}} \sum_{m=1}^{\infty} \frac{a_m}{\ln emn} \right) \left(n^{\frac{1}{2}} b_n \right) \\ &\le \left[\sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{m=1}^{\infty} \frac{a_m}{\ln emn} \right)^2 \sum_{n=1}^{\infty} n b_n^2 \right]^{\frac{1}{2}}. \end{aligned} \tag{3.7}$$

By (3.4), we have (3.1).

Hence inequalities (3.1) and (3.4) are equivalent. If the constant factor π^2 in (3.4) is not the best possible, we may show that the constant factor π in (3.1) is not the best possible, by using (3.7). This is a contradiction. The theorem is proved. □

Remark 3.2. *Inequality (3.4) is similar to the following equivalent form of (1.1) (see [2]):*

$$\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}\frac{a_m}{m+n+1}\right)^2<\pi^2\sum_{n=0}^{\infty}a_n^2. \tag{3.8}$$

Since inequalities (3.1) and (3.4) are similar to (1.1) and its equivalent form with the best constant factors, we have provided some new results.

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