

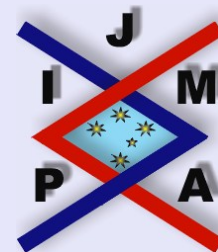
Journal of Inequalities in Pure and Applied Mathematics

A NEW INEQUALITY SIMILAR TO HILBERT'S INEQUALITY

BICHENG YANG

Department of Mathematics
Guangdong Education College
Guangzhou
Guangdong 510303
PEOPLE'S REPUBLIC OF CHINA
EMail: bcyang@pub.guangzhou.gd.cn

©2000 Victoria University
ISSN (electronic): 1443-5756
044-01



volume 3, issue 5, article 75,
2002.

*Received 17 May 2001;
accepted 17 June, 2002.*

Communicated by: J.E. Pečarić

[Abstract](#)

[Contents](#)



[Home Page](#)

[Go Back](#)

[Close](#)

[Quit](#)

Abstract

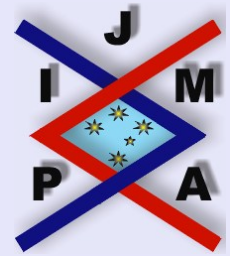
In this paper, we build a new inequality similar to Hilbert's inequality with a best constant factor. As an application, we consider its equivalent form.

2000 Mathematics Subject Classification: 26D15

Key words: Hilbert's inequality, Weight coefficient, Cauchy's inequality.

Contents

1	Introduction	3
2	Some Lemmas	4
3	Main Result and an Application	9
References		



A New Inequality Similar to Hilbert's Inequality

Bicheng Yang

Title Page

Contents



Go Back

Close

Quit

Page 2 of 14

1. Introduction

If $0 < \sum_{n=0}^{\infty} a_n^2 < \infty$ and $0 < \sum_{n=0}^{\infty} b_n^2 < \infty$, then the famous Hilbert's inequality (see Hardy et al. [1]) is given by

$$(1.1) \quad \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{m+n+1} < \pi \left(\sum_{n=0}^{\infty} a_n^2 \sum_{n=0}^{\infty} b_n^2 \right)^{\frac{1}{2}},$$

where the constant factor π is the best possible. Recently, Yang and Debnath [2, 3] and Yang [4, 5] gave (1.1) some extensions and improvements, and Kuang and Debnath [6] considered its strengthened versions and generalizations.

The major objective of this paper is to build a new inequality similar to (1.1), which relates to the double series form as

$$(1.2) \quad \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\ln m + \ln n + 1} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\ln emn}.$$

For this, we must estimate the following weight coefficient

$$(1.3) \quad \omega(n) = \sum_{m=1}^{\infty} \frac{1}{m \ln emn} \left(\frac{\ln \sqrt{en}}{\ln \sqrt{em}} \right)^{\frac{1}{2}} \quad (n \in N),$$

and do some preparatory works.



A New Inequality Similar to Hilbert's Inequality

Bicheng Yang

Title Page

Contents



Go Back

Close

Quit

Page 3 of 14

2. Some Lemmas

Let f have its first four derivatives on $[1, \infty)$ and $(-1)^n f^{(n)}(x) > 0$ ($n = 0, \dots, 4$), and $f(x), f'(x) \rightarrow 0$ ($x \rightarrow \infty$), then (see [6, (2.1)])

$$(2.1) \quad \sum_{k=1}^{\infty} f(k) < \int_1^{\infty} f(x) dx + \frac{1}{2}f(1) - \frac{1}{12}f'(1).$$

Lemma 2.1. For $n \in N$, define $R(n)$ as

$$(2.2) \quad R(n) = \frac{1}{(2 \ln \sqrt{en})^{\frac{1}{2}}} \int_0^{\frac{1}{2 \ln \sqrt{en}}} \frac{1}{(1+u)u^{\frac{1}{2}}} du - \frac{2}{3 \ln en} - \frac{1}{12(\ln en)^2}.$$

Then we have $R(n) > 0$ ($n \in N$).

Proof. Integrating by parts, we have

$$\begin{aligned} \int_0^{\frac{1}{2 \ln \sqrt{en}}} \frac{1}{(1+u)u^{\frac{1}{2}}} du &= 2 \int_0^{\frac{1}{2 \ln \sqrt{en}}} \frac{1}{(1+u)} du^{\frac{1}{2}} \\ &= (2 \ln \sqrt{en})^{\frac{1}{2}} \frac{1}{\ln en} + 2 \int_0^{\frac{1}{2 \ln \sqrt{en}}} u^{\frac{1}{2}} \frac{1}{(1+u)^2} du \\ &= (2 \ln \sqrt{en})^{\frac{1}{2}} \frac{1}{\ln en} + \frac{4}{3} \int_0^{\frac{1}{2 \ln \sqrt{en}}} \frac{1}{(1+u)^2} du^{3/2} \\ &= (2 \ln \sqrt{en})^{\frac{1}{2}} \frac{1}{\ln en} + \frac{1}{3} (2 \ln \sqrt{en})^{\frac{1}{2}} \frac{1}{(\ln en)^2} \\ &\quad + \frac{8}{3} \int_0^{\frac{1}{2 \ln \sqrt{en}}} u^{3/2} \frac{1}{(1+u)^3} du \end{aligned}$$



A New Inequality Similar to
Hilbert's Inequality

Bicheng Yang

Title Page

Contents



Go Back

Close

Quit

Page 4 of 14

$$> (2 \ln \sqrt{en})^{\frac{1}{2}} \frac{1}{\ln en} + \frac{1}{3} (2 \ln \sqrt{en})^{\frac{1}{2}} \frac{1}{(\ln en)^2}.$$

Hence by (2.2), we have

$$R(n) > \frac{1}{\ln en} + \frac{1}{3(\ln en)^2} - \frac{2}{3 \ln en} - \frac{1}{12(\ln en)^2} = \frac{1}{3 \ln en} + \frac{1}{4(\ln en)^2} > 0.$$

The lemma is thus proved. \square

Lemma 2.2. *If $\omega(n)$ is defined by (1.3), then $\omega(n) < \pi$, for $n \in N$.*

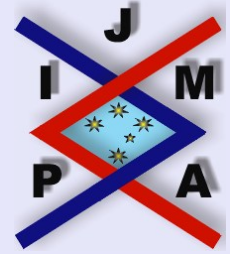
Proof. For fixed $n \in N$, setting

$$f_n(x) = \frac{1}{x \ln enx} \left(\frac{\ln \sqrt{en}}{\ln \sqrt{ex}} \right)^{\frac{1}{2}}, \quad x \in [1, \infty),$$

we find $f_n(1) = \frac{1}{\ln en} (2 \ln \sqrt{en})^{\frac{1}{2}}$, and

$$f'_n(x) = -\frac{1}{x^2 \ln enx} \left(\frac{\ln \sqrt{en}}{\ln \sqrt{ex}} \right)^{\frac{1}{2}} - \frac{1}{x^2 \ln^2 enx} \left(\frac{\ln \sqrt{en}}{\ln \sqrt{ex}} \right)^{\frac{1}{2}} - \frac{1}{2x^2 \ln enx} \cdot \frac{(\ln \sqrt{en})^{\frac{1}{2}}}{(\ln \sqrt{ex})^{\frac{3}{2}}},$$

$$f'_n(1) = -\left(\frac{2}{\ln en} + \frac{1}{\ln^2 en} \right) (2 \ln \sqrt{en})^{\frac{1}{2}}.$$



A New Inequality Similar to Hilbert's Inequality

Bicheng Yang

Title Page

Contents



Go Back

Close

Quit

Page 5 of 14

Setting $u = \frac{\ln \sqrt{ex}}{\ln \sqrt{en}}$ in the following integral, we obtain

$$\int_1^\infty f_n(x) dx = \int_{\frac{1}{2 \ln \sqrt{en}}}^\infty \frac{1}{1+u} \left(\frac{1}{u}\right)^{\frac{1}{2}} du = \pi - \int_0^{\frac{1}{2 \ln \sqrt{en}}} \frac{1}{1+u} \left(\frac{1}{u}\right)^{\frac{1}{2}} du.$$

Hence by (2.1), (2.2) and Lemma 2.1, we have

$$\begin{aligned} \omega(n) &= \sum_{m=1}^\infty f_n(m) \\ &< \int_1^\infty f_n(x) dx + \frac{1}{2} f_n(1) - \frac{1}{12} f_n'(1) \\ &= \pi - \int_0^{1/(2 \ln \sqrt{en})} \frac{1}{1+u} \left(\frac{1}{u}\right)^{\frac{1}{2}} du + \left(\frac{2}{3 \ln en} + \frac{1}{12 \ln^2 en}\right) (2 \ln \sqrt{en})^{\frac{1}{2}} \\ &= \pi - (2 \ln \sqrt{en})^{\frac{1}{2}} R(n) < \pi. \end{aligned}$$

The lemma is proved. □

Lemma 2.3. For $0 < \epsilon < 1$, we have

$$(2.3) \quad \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{1}{mn \ln emn} \left(\frac{1}{\ln \sqrt{em} \ln \sqrt{en}}\right)^{\frac{1+\epsilon}{2}} > \frac{1}{\epsilon} (\pi + o(1)) \quad (\epsilon \rightarrow 0^+).$$

Proof. Setting $u = \frac{\ln \sqrt{ex}}{\ln \sqrt{ey}}$ in the following integral, we find

$$\int_{\sqrt{e}}^\infty \frac{1}{x \ln ex} \left(\frac{1}{\ln \sqrt{ex}}\right)^{\frac{1+\epsilon}{2}} dx$$



A New Inequality Similar to Hilbert's Inequality

Bicheng Yang

Title Page

Contents



Go Back

Close

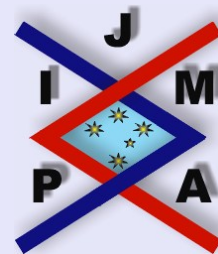
Quit

Page 6 of 14

$$\begin{aligned}
&= \left(\frac{1}{\ln \sqrt{ey}} \right)^{\frac{1+\epsilon}{2}} \int_{\frac{1}{\ln \sqrt{ey}}}^{\infty} \frac{1}{1+u} \left(\frac{1}{u} \right)^{\frac{1+\epsilon}{2}} du \\
&= \left(\frac{1}{\ln \sqrt{ey}} \right)^{\frac{1+\epsilon}{2}} \int_0^{\infty} \frac{1}{1+u} \left(\frac{1}{u} \right)^{\frac{1+\epsilon}{2}} du \\
&\quad - \left(\frac{1}{\ln \sqrt{ey}} \right)^{\frac{1+\epsilon}{2}} \int_0^{\frac{1}{\ln \sqrt{ey}}} \frac{1}{1+u} \left(\frac{1}{u} \right)^{\frac{1+\epsilon}{2}} du \\
&> \left(\frac{1}{\ln \sqrt{ey}} \right)^{\frac{1+\epsilon}{2}} \int_0^{\infty} \frac{1}{1+u} \left(\frac{1}{u} \right)^{\frac{1+\epsilon}{2}} du \\
&\quad - \left(\frac{1}{\ln \sqrt{ey}} \right)^{\frac{1+\epsilon}{2}} \int_0^{\frac{1}{\ln \sqrt{ey}}} \left(\frac{1}{u} \right)^{\frac{1+\epsilon}{2}} du \\
&= \left(\frac{1}{\ln \sqrt{ey}} \right)^{\frac{1+\epsilon}{2}} (\pi + o(1)) - \frac{2}{1-\epsilon} \left(\frac{1}{\ln \sqrt{ey}} \right) (\epsilon \rightarrow 0^+).
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{mn \ln emn} \left(\frac{1}{\ln \sqrt{em} \ln \sqrt{en}} \right)^{\frac{1+\epsilon}{2}} \\
&> \int_{\sqrt{e}}^{\infty} \int_{\sqrt{e}}^{\infty} \frac{1}{xy \ln exy} \left(\frac{1}{\ln \sqrt{ex} \ln \sqrt{ey}} \right)^{\frac{1+\epsilon}{2}} dx dy \\
&= \int_{\sqrt{e}}^{\infty} \frac{1}{y} \left(\frac{1}{\ln \sqrt{ey}} \right)^{\frac{1+\epsilon}{2}} \left[\int_{\sqrt{e}}^{\infty} \frac{1}{x \ln exy} \left(\frac{1}{\ln \sqrt{ex}} \right)^{\frac{1+\epsilon}{2}} dx \right] dy
\end{aligned}$$



A New Inequality Similar to Hilbert's Inequality

Bicheng Yang

Title Page

Contents



Go Back

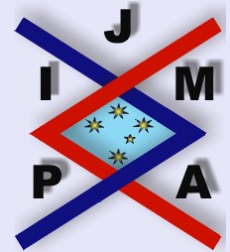
Close

Quit

Page 7 of 14

$$\begin{aligned}
&> (\pi + o(1)) \int_{\sqrt{\epsilon}}^{\infty} \frac{1}{y} \left(\frac{1}{\ln \sqrt{\epsilon} y} \right)^{1+\epsilon} dy - \frac{2}{1-\epsilon} \int_{\sqrt{\epsilon}}^{\infty} \frac{1}{y} \left(\frac{1}{\ln \sqrt{\epsilon} y} \right)^{\frac{1+\epsilon}{2}+1} dy \\
&= (\pi + o(1)) \frac{1}{\epsilon} - \frac{4}{1-\epsilon^2} \\
&= \frac{1}{\epsilon} (\pi + o(1)) \quad (\epsilon \rightarrow 0^+).
\end{aligned}$$

The lemma is proved. □



**A New Inequality Similar to
Hilbert's Inequality**

Bicheng Yang

Title Page

Contents



Go Back

Close

Quit

Page 8 of 14

3. Main Result and an Application

Theorem 3.1. If $0 < \sum_{n=1}^{\infty} na_n^2 < \infty$ and $0 < \sum_{n=1}^{\infty} nb_n^2 < \infty$, then

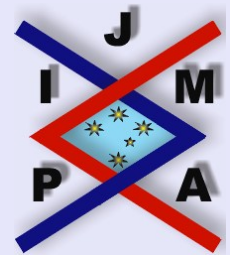
$$(3.1) \quad \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\ln emn} < \pi \left(\sum_{n=1}^{\infty} na_n^2 \sum_{n=1}^{\infty} nb_n^2 \right)^{\frac{1}{2}},$$

where the constant factor π is the best possible.

Proof. By Cauchy's inequality and (1.3), we have

$$\begin{aligned} & \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\ln emn} \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\frac{a_m}{(\ln emn)^{\frac{1}{2}}} \left(\frac{\ln \sqrt{em}}{\ln \sqrt{en}} \right)^{\frac{1}{4}} \left(\frac{m}{n} \right)^{\frac{1}{2}} \right] \\ & \quad \times \left[\frac{b_n}{(\ln emn)^{\frac{1}{2}}} \left(\frac{\ln \sqrt{en}}{\ln \sqrt{em}} \right)^{\frac{1}{4}} \left(\frac{n}{m} \right)^{\frac{1}{2}} \right] \\ &\leq \left[\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m^2}{\ln emn} \left(\frac{\ln \sqrt{em}}{\ln \sqrt{en}} \right)^{\frac{1}{2}} \left(\frac{m}{n} \right) \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{b_n^2}{\ln emn} \left(\frac{\ln \sqrt{en}}{\ln \sqrt{em}} \right)^{\frac{1}{2}} \left(\frac{n}{m} \right) \right]^{\frac{1}{2}} \\ &= \left(\sum_{m=1}^{\infty} \omega(m) ma_m^2 \sum_{n=1}^{\infty} \omega(n) nb_n^2 \right)^{\frac{1}{2}}. \end{aligned}$$

By Lemma 2.2, we have (3.1).



A New Inequality Similar to Hilbert's Inequality

Bicheng Yang

Title Page

Contents



Go Back

Close

Quit

Page 9 of 14

For $0 < \epsilon < 1$, setting a'_n as:

$$a'_n = \frac{1}{n(\ln \sqrt{en})^{\frac{1+\epsilon}{2}}}, \quad n \in N,$$

then we have

$$\begin{aligned} \sum_{n=1}^{\infty} na_n'^2 &= \frac{1}{(\ln \sqrt{e})^{1+\epsilon}} + \frac{1}{2(\ln 2\sqrt{e})^{1+\epsilon}} + \sum_{n=3}^{\infty} \frac{1}{n(\ln \sqrt{en})^{1+\epsilon}} \\ &< \frac{1}{(\ln \sqrt{e})^{1+\epsilon}} + \frac{1}{2(\ln 2\sqrt{e})^{1+\epsilon}} + \int_{\sqrt{e}}^{\infty} \frac{1}{x(\ln \sqrt{ex})^{1+\epsilon}} dx \\ (3.2) \quad &= \frac{1}{(\ln \sqrt{e})^{1+\epsilon}} + \frac{1}{2(\ln 2\sqrt{e})^{1+\epsilon}} + \frac{1}{\epsilon} = \frac{1}{\epsilon}(1 + o(1)) \quad (\epsilon \rightarrow 0^+). \end{aligned}$$

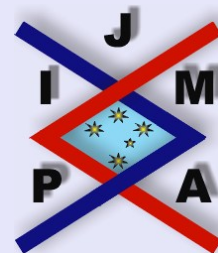
If the constant factor π in (3.1) is not the best possible, then there exists a positive number $K < \pi$, such that (3.1) is valid if we change π to K . In particular, we have

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a'_m a'_n}{\ln emn} < K \sum_{n=1}^{\infty} na_n'^2.$$

By (2.3) and (3.2), we have

$$(\pi + o(1)) < \epsilon \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a'_m a'_n}{\ln emn} < K(1 + o(1)) \quad (\epsilon \rightarrow 0^+),$$

and $\pi \leq K$. This contradicts that $K < \pi$. Hence the constant factor π in (3.1) is the best possible. The theorem is proved. \square



A New Inequality Similar to Hilbert's Inequality

Bicheng Yang

Title Page

Contents



Go Back

Close

Quit

Page 10 of 14

Remark 3.1. Inequality (3.1) is more similar to the following Mulholland's inequality for $p = q = 2$ (see [7]):

$$(3.3) \quad \sum_{n=2}^{\infty} \sum_{m=2}^{\infty} \frac{a_m b_n}{mn \ln emn} < \frac{\pi}{\sin(\frac{\pi}{p})} \left(\sum_{n=2}^{\infty} n^{-1} a_n^p \right)^{\frac{1}{p}} \left(\sum_{n=2}^{\infty} n^{-1} b_n^q \right)^{\frac{1}{q}}.$$

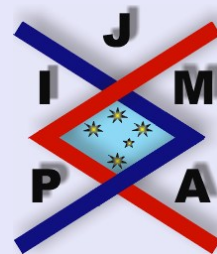
Theorem 3.2. If $0 < \sum_{n=1}^{\infty} n a_n^2 < \infty$, then we have

$$(3.4) \quad \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{m=1}^{\infty} \frac{a_m}{\ln emn} \right)^2 < \pi^2 \sum_{n=1}^{\infty} n a_n^2,$$

where the constant factor π^2 is the best possible. Inequalities (3.1) and (3.4) are equivalent.

Proof. Since $\sum_{n=1}^{\infty} n a_n^2 > 0$, there exists $k_0 \geq 1$, such that for any $k > k_0$, we have $\sum_{n=1}^k n a_n^2 > 0$, and $b_n(k) = \frac{1}{n} \sum_{m=1}^k \frac{|a_m|}{\ln emn} > 0$ ($n \in N$). By (3.1), we have

$$(3.5) \quad \begin{aligned} 0 &< \left[\sum_{n=1}^k n b_n^2(k) \right]^2 \\ &= \left[\sum_{n=1}^k \frac{1}{n} \left(\sum_{m=1}^k \frac{|a_m|}{\ln emn} \right) \right]^2 \\ &= \left[\sum_{n=1}^k \sum_{m=1}^k \frac{|a_m| b_n(k)}{\ln emn} \right]^2 < \pi^2 \sum_{n=1}^k n a_n^2 \sum_{n=1}^k n b_n^2(k). \end{aligned}$$



A New Inequality Similar to Hilbert's Inequality

Bicheng Yang

Title Page

Contents



Go Back

Close

Quit

Page 11 of 14

Thus we find

$$(3.6) \quad 0 < \sum_{n=1}^k \frac{1}{n} \left(\sum_{m=1}^k \frac{|a_m|}{\ln emn} \right)^2 = \sum_{n=1}^k nb_n^2(k) < \pi^2 \sum_{n=1}^k na_n^2.$$

It follows that $0 < \sum_{n=1}^{\infty} nb_n^2(\infty) \leq \pi^2 \sum_{n=1}^{\infty} na_n^2 < \infty$. Hence by (3.1), for $k \rightarrow \infty$, neither (3.5) nor (3.6) takes equality, and we have

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{m=1}^{\infty} \frac{a_m}{\ln emn} \right)^2 \leq \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{m=1}^{\infty} \frac{|a_m|}{\ln emn} \right)^2 < \pi^2 \sum_{n=1}^{\infty} na_n^2.$$

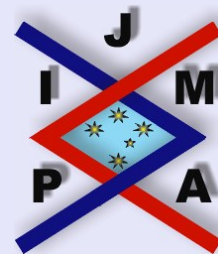
Inequality (3.4) is valid.

On the other hand, if (3.4) holds, by Cauchy's inequality, we have

$$(3.7) \quad \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\ln emn} = \sum_{n=1}^{\infty} \left(\frac{1}{n^{\frac{1}{2}}} \sum_{m=1}^{\infty} \frac{a_m}{\ln emn} \right) \left(n^{\frac{1}{2}} b_n \right) \\ \leq \left[\sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{m=1}^{\infty} \frac{a_m}{\ln emn} \right)^2 \sum_{n=1}^{\infty} nb_n^2 \right]^{\frac{1}{2}}.$$

By (3.4), we have (3.1).

Hence inequalities (3.1) and (3.4) are equivalent. If the constant factor π^2 in (3.4) is not the best possible, we may show that the constant factor π in (3.1) is not the best possible, by using (3.7). This is a contradiction. The theorem is proved. \square



A New Inequality Similar to Hilbert's Inequality

Bicheng Yang

Title Page

Contents



Go Back

Close

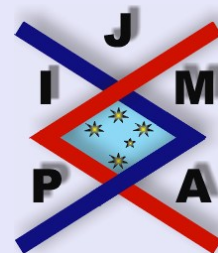
Quit

Page 12 of 14

Remark 3.2. *Inequality (3.4) is similar to the following equivalent form of (1.1) (see [2]):*

$$(3.8) \quad \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} \frac{a_m}{m+n+1} \right)^2 < \pi^2 \sum_{n=0}^{\infty} a_n^2.$$

Since inequalities (3.1) and (3.4) are similar to (1.1) and its equivalent form with the best constant factors, we have provided some new results.



A New Inequality Similar to Hilbert's Inequality

Bicheng Yang

Title Page

Contents



Go Back

Close

Quit

Page 13 of 14

References

- [1] G.H. HARDY, J.E. LITTLEWOOD AND G. POLYA, *Inequalities*, Cambridge Univ. Press, London, 1952.
- [2] B. YANG AND L. DEBNATH, on a new generalization of Hardy-Hilbert's inequality and its application, *J. Math. Anal. Appl.*, **233** (1999), 484–497.
- [3] B. YANG AND L. DEBNATH, Some inequalities involving π and an application to Hilbert's inequality, *Applied Math. Letters*, **129** (1999), 101–105.
- [4] B. YANG, On a strengthened version of the more accurate Hardy-Hilbert's inequality, *Acta Math. Sinica*, **42**(6) (1999), 1103–1110.
- [5] B. YANG, On a strengthened Hardy-Hilbert's inequality, *J. Ineq. Pure and Appl. Math.*, **1**(2), Art. 22 (2000). [ONLINE: http://jipam.vu.edu.au/v1n2/012_00.html]
- [6] J. KUANG AND L. DEBNATH, On new generalizations of Hilbert's inequality and their applications, *J. Math. Anal. Appl.*, **245** (2000), 248–265.
- [7] H.P. MULHOLLAND, Some theorem on Dirichlet series with coefficients and related integrals, *Proc. London Math. Soc.*, **29**(2) (1999), 281–292.



A New Inequality Similar to Hilbert's Inequality

Bicheng Yang

Title Page

Contents



Go Back

Close

Quit

Page 14 of 14