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INTEGRAL MEANS INEQUALITIES FOR FRACTIONAL DERIVATIVES OF SOME GENERAL SUBCLASSES OF ANALYTIC FUNCTIONS

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ABSTRACT. Integral means inequalities are obtained for the fractional derivatives of order $p + \lambda$ $(0 \leqq p \leqq n;\ 0 \leqq \lambda < 1)$ of functions belonging to certain general subclasses of analytic functions. Relevant connections with various known integral means inequalities are also pointed out.

Key words and phrases: Integral means inequalities, Fractional derivatives, Analytic functions, Univalent functions, Extreme points, Subordination.

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1. INTRODUCTION, DEFINITIONS, AND PRELIMINARIES

Let $\mathcal{A}$ denote the class of functions $f(z)$ *normalized* by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

that are *analytic* in the *open* unit disk

$$\mathbb{U} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Also let $\mathcal{A}(n)$ denote the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ of the form:

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \quad (a_k \geqq 0;\ n \in \mathbb{N} := \{1, 2, 3, \ldots\}).$$

We denote by $\mathcal{T}(n)$ the subclass of $\mathcal{A}(n)$ of functions which are *univalent* in $\mathbb{U}$, and by $\mathcal{T}_\alpha(n)$ and $\mathcal{C}_\alpha(n)$ the subclasses of $\mathcal{T}(n)$ consisting of functions which are, respectively, *starlike of order* α $(0 \leqq \alpha < 1)$ and *convex of order* α $(0 \leqq \alpha < 1)$ in $\mathbb{U}$. The classes $\mathcal{A}(n),\ \mathcal{T}(n),\ \mathcal{T}_\alpha(n)$, and $\mathcal{C}_\alpha(n)$ were investigated by Chatterjea [1] (and Srivastava *et al.* [9]). In particular, the following subclasses:

$$\mathcal{T} := \mathcal{T}(1), \quad \mathcal{T}^*(\alpha) := \mathcal{T}_\alpha(1), \quad \text{and} \quad \mathcal{C}(\alpha) := \mathcal{C}_\alpha(1)$$

were considered earlier by Silverman [7].

Next, following the work of Sekine and Owa [4], we denote by $\mathcal{A}(n, \vartheta)$ the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ of the form:

$$f(z) = z - \sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta} a_k z^k \quad (\vartheta \in \mathbb{R};\ a_k \geqq 0;\ n \in \mathbb{N}). \tag{1.1}$$

Finally, the subclasses $\mathcal{T}(n, \vartheta),\ \mathcal{T}_\alpha^*(n, \vartheta)$, and $\mathcal{C}_\alpha(n, \vartheta)$ of the class $\mathcal{A}(n, \vartheta)$ are defined in the same way as the subclasses $\mathcal{T}(n),\ \mathcal{T}_\alpha(n)$, and $\mathcal{C}_\alpha(n)$ of the class $\mathcal{A}(n)$.

We begin by recalling the following useful characterizations of the function classes $\mathcal{T}_\alpha^*(n, \vartheta)$ and $\mathcal{C}_\alpha(n, \vartheta)$ (see Sekine and Owa [4]).

Lemma 1.1. *A function $f(z) \in \mathcal{A}(n, \vartheta)$ of the form (1.1) is in the class $\mathcal{T}_\alpha^*(n, \vartheta)$ if and only if*

$$\sum_{k=n+1}^{\infty} (k - \alpha)\ a_k \leqq 1 - \alpha \quad (n \in \mathbb{N};\ 0 \leqq \alpha < 1). \tag{1.2}$$

Lemma 1.2. *A function $f(z) \in \mathcal{A}(n, \vartheta)$ of the form (1.1) is in the class $\mathcal{C}_\alpha(n, \vartheta)$ if and only if*

$$\sum_{k=n+1}^{\infty} k(k - \alpha)\ a_k \leqq 1 - \alpha \quad (n \in \mathbb{N};\ 0 \leqq \alpha < 1). \tag{1.3}$$

Motivated by the equalities in (1.2) and (1.3) above, Sekine *et al.* [6] defined a general subclass $\mathcal{A}(n; B_k, \vartheta)$ of the class $\mathcal{A}(n, \vartheta)$ consisting of functions $f(z)$ of the form (1.1), which satisfy the following inequality:

$$\sum_{k=n+1}^{\infty} B_k a_k \leqq 1 \quad (B_k > 0;\ n \in \mathbb{N}).$$

Thus it is easy to verify each of the following classifications and relationships:

$$\mathcal{A}(n; k, \vartheta) = \mathcal{T}_0^*(n, \vartheta) =: \mathcal{T}^*(n, \vartheta) = \mathcal{T}(n, \vartheta),$$

$$\mathcal{A}\left(n; \frac{k-\alpha}{1-\alpha}, \vartheta\right) = \mathcal{T}_{\alpha}^{*}(n, \vartheta) \quad (0 \leqq \alpha < 1),$$

and

$$\mathcal{A}\left(n; \frac{k(k-\alpha)}{1-\alpha}, \vartheta\right) = \mathcal{C}_{\alpha}(n, \vartheta) \quad (0 \leqq \alpha < 1).$$

As a matter of fact, Sekine *et al*. [6] also obtained each of the following basic properties of the general classes $\mathcal{A}(n; B_k, \vartheta)$.

Theorem 1.3. *$\mathcal{A}(n; B_k, \vartheta)$ is the convex subfamily of the class $\mathcal{A}(n, \vartheta)$.*

Theorem 1.4. *Let*

$$f_1(z) = z \quad \text{and} \quad f_k(z) = z - \frac{e^{i(k-1)\vartheta}}{B_k} z^k \qquad (1.4)$$
$$(k = n+1, n+2, n+3, \ldots;\ n \in \mathbb{N}).$$

Then $f \in \mathcal{A}(n; B_k, \vartheta)$ if and only if $f(z)$ can be expressed as follows:

$$f(z) = \lambda_1 f_1(z) + \sum_{k=n+1}^{\infty} \lambda_k f_k(z),$$

where

$$\lambda_1 + \sum_{k=n+1}^{\infty} \lambda_k = 1 \quad (\lambda_1 \geqq 0;\ \lambda_k \geqq 0;\ n \in \mathbb{N}).$$

Corollary 1.5. *The extreme points of the class $\mathcal{A}(n; B_k, \vartheta)$ are the functions $f_1(z)$ and $f_k(z)$ $(k \geqq n+1;\ n \in \mathbb{N})$ given by* (1.4).

Applying the concepts of extreme points, fractional calculus, and subordination, Sekine *et al.* [6] obtained several integral means inequalities for higher-order fractional derivatives and fractional integrals of functions belonging to the general classes $\mathcal{A}(n; B_k, \vartheta)$. Subsequently, Sekine and Owa [5] discussed the weakening of the hypotheses for B_k in those results by Sekine *et al.* [6]. In this paper, we investigate the integral means inequalities for the fractional derivatives of $f(z)$ of a general order $p + \lambda$ $(0 \leqq p \leqq n;\ 0 \leqq \lambda < 1)$ of functions $f(z)$ belonging to the general classes $\mathcal{A}(n; B_k, \vartheta)$.

We shall make use of the following definitions of fractional derivatives (*cf.* Owa [3]; see also Srivastava and Owa [8]).

Definition 1.1. The *fractional derivative of order* λ is defined, for a function $f(z)$, by

$$D_z^{\lambda} f(z) := \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta \quad (0 \leqq \lambda < 1), \qquad (1.5)$$

where the function $f(z)$ is analytic in a simply-connected region of the complex z-plane containing the origin and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed by requiring $\log(z-\zeta)$ to be real when $z - \zeta > 0$.

Definition 1.2. Under the hypotheses of Definition 1.1, the *fractional derivative of order* $n + \lambda$ is defined, for a function $f(z)$, by

$$D_z^{n+\lambda} f(z) := \frac{d^n}{dz^n} D_z^{\lambda} f(z) \quad (0 \leqq \lambda < 1;\ n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}).$$

It readily follows from (1.5) in Definition 1.1 that

$$D_z^{\lambda} z^k = \frac{\Gamma(k+1)}{\Gamma(k-\lambda+1)} z^{k-\lambda} \quad (0 \leqq \lambda < 1). \qquad (1.6)$$

We shall also need the concept of subordination between analytic functions and a subordination theorem of Littlewood [2] in our investigation.

Given two functions $f(z)$ and $g(z)$, which are analytic in $\mathbb{U}$, the function $f(z)$ is said to be *subordinate* to $g(z)$ in $\mathbb{U}$ if there exists a function $w(z)$, analytic in $\mathbb{U}$ with

$$w(0)=0 \quad \text{and} \quad |w(z)|<1 \quad (z\in\mathbb{U}),$$

such that

$$f(z)=g(w(z)) \quad (z\in\mathbb{U}).$$

We denote this subordination by

$$f(z)\prec g(z).$$

Theorem 1.6 (Littlewood [2])**.** *If the functions $f(z)$ and $g(z)$ are analytic in $\mathbb{U}$ with*

$$g(z)\prec f(z),$$

then

$$\int_0^{2\pi}\left|g\left(re^{i\theta}\right)\right|^{\mu}d\theta \leqq \int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^{\mu}d\theta \quad (\mu>0;\ 0<r<1).$$

2. THE MAIN INTEGRAL MEANS INEQUALITIES

Theorem 2.1. *Suppose that $f(z)\in\mathcal{A}(n;k^{p+1}B_k,\vartheta)$ and that*

$$\frac{(h+1)^q B_{h+1}\Gamma(h+2-\lambda-p)}{\Gamma(h+1)}\cdot\frac{\Gamma(n+1-p)}{\Gamma(n+2-\lambda-p)}\leqq B_k \quad (k\geqq n+1)$$

for some $h\geqq n$, $0\leqq\lambda<1$, and $0\leqq q\leqq p\leqq n$. Also let the function $f_{h+1}(z)$ be defined by

$$f_{h+1}(z)=z-\frac{e^{ih\vartheta}}{(h+1)^{q+1}B_{h+1}}z^{h+1} \quad \left(f_{h+1}\in A\left(n;k^{q+1}B_k,\vartheta\right)\right). \tag{2.1}$$

Then, for $z=re^{i\theta}$ and $0<r<1$,

$$\int_0^{2\pi}\left|D_z^{p+\lambda}f(z)\right|^{\mu}d\theta\leqq\int_0^{2\pi}\left|D_z^{p+\lambda}f_{h+1}(z)\right|^{\mu}d\theta \quad (0\leqq\lambda<1;\ \mu>0). \tag{2.2}$$

Proof. By virtue of the fractional derivative formula (1.6) and Definition 1.2, we find from (1.1) that

$$\begin{aligned} D_z^{p+\lambda}f(z) &= \frac{z^{1-\lambda-p}}{\Gamma(2-\lambda-p)}\left(1-\sum_{k=n+1}^{\infty}e^{i(k-1)\vartheta}\,\frac{\Gamma(2-\lambda-p)\Gamma(k+1)}{\Gamma(k+1-\lambda-p)}\,a_k\,z^{k-1}\right)\\ &= \frac{z^{1-\lambda-p}}{\Gamma(2-\lambda-p)}\left(1-\sum_{k=n+1}^{\infty}e^{i(k-1)\vartheta}\Gamma(2-\lambda-p)\frac{k!}{(k-p-1)!}\,\Phi(k)a_kz^{k-1}\right), \end{aligned}$$

where

$$\Phi(k):=\frac{\Gamma(k-p)}{\Gamma(k+1-\lambda-p)} \quad (0\leqq\lambda<1;\ k\geqq n+1;\ n\in\mathbb{N}). \tag{2.3}$$

Since $\Phi(k)$ is a *decreasing* function of k, we have

$$0<\Phi(k)\leqq\Phi(n+1)=\frac{\Gamma(n+1-p)}{\Gamma(n+2-\lambda-p)}$$
$$(0\leqq\lambda<1;\ k\geqq n+1;\ n\in\mathbb{N}).$$

Similarly, from (2.1), (1.6), and Definition 1.2, we obtain, for $0 \leqq \lambda < 1$,

$$D_z^{p+\lambda} f_{h+1}(z) = \frac{z^{1-\lambda-p}}{\Gamma(2-\lambda-p)} \left(1 - \frac{e^{ih\vartheta}}{(h+1)^{q+1} B_{h+1}} \cdot \frac{\Gamma(2-\lambda-p)\Gamma(h+2)}{\Gamma(h+2-\lambda-p)} \ z^h \right).$$

For $z = re^{i\theta}$ and $0 < r < 1$, we must show that

$$\int_0^{2\pi} \left| 1 - \sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta} \Gamma(2-\lambda-p) \frac{k!}{(k-p-1)!} \Phi(k) a_k z^{k-1} \right|^{\mu} d\theta$$
$$\leqq \int_0^{2\pi} \left| 1 - \frac{e^{ih\vartheta}}{(h+1)^{q+1} B_{h+1}} \cdot \frac{\Gamma(2-\lambda-p)\Gamma(h+2)}{\Gamma(h+2-\lambda-p)} z^h \right|^{\mu} d\theta, \quad (0 \leqq \lambda < 1; \ \mu > 0).$$

Thus, by applying Theorem 1.6, it would suffice to show that

$$1 - \sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta} \Gamma(2-\lambda-p) \frac{k!}{(k-p-1)!} \Phi(k) a_k z^{k-1} \prec 1 - \frac{e^{ih\vartheta}}{(h+1)^{q+1} B_{h+1}} \cdot \frac{\Gamma(2-\lambda-p)\Gamma(h+2)}{\Gamma(h+2-\lambda-p)} z^h. \tag{2.4}$$

Indeed, by setting

$$1 - \sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta} \Gamma(2-\lambda-p) \frac{k!}{(k-p-1)!} \Phi(k) a_k z^{k-1} = 1 - \frac{e^{ih\vartheta}}{(h+1)^{q+1} B_{h+1}} \cdot \frac{\Gamma(2-\lambda-p)\Gamma(h+2)}{\Gamma(h+2-\lambda-p)} \{w(z)\}^h,$$

we find that

$$\{w(z)\}^h = \frac{(h+1)^{q+1} B_{h+1} \Gamma(h+2-\lambda-p)}{e^{ih\vartheta}\Gamma(h+2)} \cdot \sum_{k=n+1}^{\infty} e^{i(k-1)\vartheta} \frac{k!}{(k-p-1)!} \Phi(k) a_k z^{k-1},$$

which readily yields $w(0) = 0$.

Therefore, we have

$$\begin{aligned}
&|w(z)|^h \\
&\leqq \frac{(h+1)^{q+1} B_{h+1} \Gamma(h+2-\lambda-p)}{\Gamma(h+2)} \sum_{k=n+1}^{\infty} \frac{k!}{(k-p-1)!} \Phi(k) a_k |z|^{k-1} \\
&\leqq |z|^n \frac{(h+1)^{q+1} B_{h+1} \Gamma(h+2-\lambda-p)}{\Gamma(h+2)} \cdot \Phi(n+1) \sum_{k=n+1}^{\infty} \frac{k!}{(k-p-1)!} a_k \\
&= |z|^n \frac{(h+1)^{q+1} B_{h+1} \Gamma(h+2-\lambda-p)}{\Gamma(h+2)} \cdot \frac{\Gamma(n+1-p)}{\Gamma(n+2-\lambda-p)} \sum_{k=n+1}^{\infty} \frac{k!}{(k-p-1)!} a_k \\
&= |z|^n \frac{(h+1)^{q} B_{h+1} \Gamma(h+2-\lambda-p)}{\Gamma(h+1)} \cdot \frac{\Gamma(n+1-p)}{\Gamma(n+2-\lambda-p)} \sum_{k=n+1}^{\infty} \frac{k!}{(k-p-1)!} a_k
\end{aligned}$$

$$\leqq |z|^n \sum_{k=n+1}^{\infty} \frac{k!}{(k-p-1)!} B_k a_k$$

$$\leqq |z|^n \sum_{k=n+1}^{\infty} k^{p+1} B_k a_k \leqq |z|^n < 1 \quad (n \in \mathbb{N}), \tag{2.5}$$

by means of the hypothesis of Theorem 2.1.

In light of the last inequality in (2.5) above, we have the subordination (2.4), which evidently proves Theorem 2.1. □

3. REMARKS AND OBSERVATIONS

First of all, in its special case when $p = q = 0$, Theorem 2.1 readily yields

Corollary 3.1 (cf. Sekine and Owa [5], Theorem 6)**.** *Suppose that* $f(z) \in \mathcal{A}(n; kB_k, \vartheta)$ *and that*

$$\frac{B_{h+1}\Gamma(h+2-\lambda)}{\Gamma(h+1)} \cdot \frac{\Gamma(n+1)}{\Gamma(n+2-\lambda)} \leqq B_k \quad (k \geqq n+1;\ n \in \mathbb{N})$$

for some $h \geqq n$ *and* $0 \leqq \lambda < 1$*. Also let the function* $f_{h+1}(z)$ *be defined by*

$$f_{h+1}(z) = z - \frac{e^{ih\vartheta}}{(h+1)B_{h+1}}\ z^{h+1} \quad (f_{h+1} \in \mathcal{A}(n; kB_k, \vartheta)). \tag{3.1}$$

Then, for $z = re^{i\theta}$ *and* $0 < r < 1$*,*

$$\int_0^{2\pi} \left|D_z^\lambda f(z)\right|^\mu d\theta \leqq \int_0^{2\pi} \left|D_z^\lambda f_{h+1}(z)\right|^\mu d\theta \quad (0 \leqq \lambda < 1;\ \mu > 0). \tag{3.2}$$

A *further* consequence of Corollary 3.1 when $h = n$ would lead us immediately to Corollary 3.2 below.

Corollary 3.2. *Suppose that* $f(z) \in \mathcal{A}(n; kB_k, \vartheta)$ *and that*

$$B_{n+1} \leqq B_k \quad (k \geqq n+1;\ n \in \mathbb{N}). \tag{3.3}$$

Also let the function $f_{n+1}(z)$ *be defined by*

$$f_{n+1}(z) = z - \frac{e^{in\vartheta}}{(n+1)B_{n+1}}\ z^{n+1} \quad (f_{h+1} \in \mathcal{A}(n; kB_k, \vartheta)).$$

Then, for $z = re^{i\theta}$ *and* $0 < r < 1$*,*

$$\int_0^{2\pi} \left|D_z^\lambda f(z)\right|^\mu d\theta \leqq \int_0^{2\pi} \left|D_z^\lambda f_{n+1}(z)\right|^\mu d\theta \quad (0 \leqq \lambda < 1;\ \mu > 0). \tag{3.4}$$

The hypothesis (3.3) in Corollary 3.2 is weaker than the corresponding hypothesis in an earlier result of Sekine *et al.* [6, p. 953, Theorem 6].

Next, for $p = 1$ and $q = 0$, Theorem 2.1 reduces to an integral means inequality of Sekine and Owa [5, Theorem 7] which, for $h = n$, yields another result of Sekine *et al.* [6, p. 953, Theorem 7] under weaker hypothesis as mentioned above.

Finally, by setting $p = q = 1$ in Theorem 2.1, we obtain a slightly improved version of another integral means inequalities of Sekine and Owa [5, Theorem 8] with respect to the parameter λ (see also Sekine *et al.* [6, p. 955, Theorem 8] for the case when $h = n$, just as we remarked above).

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