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ITERATIVE SCHEMES TO SOLVE NONCONVEX VARIATIONAL PROBLEMS

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ABSTRACT. In this paper, we present several algorithms of the projection type to solve a class of nonconvex variational problems.

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1. INTRODUCTION

The theory of variational inequalities is a branch of the mathematical sciences dealing with general equilibrium problems. It has a wide range of applications in economics, operations research, industry, physical, and engineering sciences. Many research papers have been written lately, both on the theory and applications of this field. Important connections with main areas of pure and applied sciences have been made, see for example [1, 12, 13] and the references cited therein.

One of the typical formulations of the variational inequality problem found in the literature is the following

(VI) Find a point $x^* \in C$ and $y^* \in F(x^*)$ satisfying $\langle y^*, x - x^*\rangle \geq 0,$ for all $x \in C,$

where C is a subset of a Hilbert space H and $F : H \rightrightarrows H$ is a set-valued mapping. A tremendous amount of research has been done in the case where C is convex, both on the

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060-02

existence of solutions of (VI) and the construction of solutions, see for example [7, 13, 15, 19]. Only the existence of solutions of (VI) has been considered in the case where C is nonconvex, see for instance [5]. To the best of our knowledge, nothing has been done concerning the construction of solutions in this case.

In this paper we first generalize problem (VI) to take into account the nonconvexity of the set C and then construct a suitable algorithm to solve the generalized (VI). Note that (VI) is usually a reformulation of some minimization problem of some functional over convex sets. For this reason, it does not make sense to generalize (VI) by just replacing the convex sets by nonconvex ones. Also, a straightforward generalization to the nonconvex case of the techniques used when set C is convex cannot be done. This is because these techniques are strongly based on properties of the projection operator over convex sets and these properties do not hold in general when C is nonconvex. Based on the above two arguments, and to take advantage of the techniques used in the convex case, we propose to reformulate problem (VI) when C is convex as the following equivalent problem

$$\text{(VP)} \qquad \text{Find a point } x^* \in C : \ F(x^*) \cap -N(C; x^*) \neq \emptyset,$$

where $N(C;x)$ denotes the normal cone of C at x in the sense of convex analysis. Equivalence of problems (VI) and (VP) will be proved in Proposition 2.3 below. The corresponding problem when C is not convex will be denoted (NVP). This reformulation allows us to consider the resolution of problem (NVP) as the desired suitable generalization of the problem (VI). We point out that the resolution of (VI) with C nonconvex is not, at least from our point of view, a good way for such generalization. Our idea of the generalization is inspired from [5] (see also [18]) where the authors studied the existence of generalized equilibrium.

In the present paper we make use of some recent techniques and ideas from nonsmooth analysis [5, 6] to overcome the difficulties that arise from the nonconvexity of the set C. Specifically, we will be considering the class of uniformly prox-regular sets (see Definition 2.1) which is sufficiently large to include the class of convex sets, p-convex sets (see [8]), $C^{1,1}$ submanifolds (possibly with boundary) of H, the images under a $C^{1,1}$ diffeomorphism of convex sets, and many other nonconvex sets (for more details see [8, 10]).

The paper is organized as follows: In Section 2 we recall some definitions and notation, and prove some useful results that will be needed in the paper. In Section 3 we propose an algorithm to solve problem (NVP) and prove its well-definedness and its convergence under the uniform prox-regularity assumption on C and the strong monotonicity assumption on F. The results proved in Section 3 are extended in Section 4 in two ways: In the first one, we assume that $F = F_1 + F_2$, where F_1 is a strongly monotone set-valued mapping and F_2 is a Hausdorff Lipschitz set-valued mapping not necessarily monotone. In this case F is not necessarily strongly monotone. In the second one, the set C is assumed to be a set-valued mapping of x. In this case, problem (NVP) becomes

$$\text{(SNVP)} \qquad \text{Find a point } x^* \in C(x^*) : \ F(x^*) \cap -N(C(x^*); x^*) \neq \emptyset.$$

2. PRELIMINARIES

Throughout the paper, H will be a Hilbert space. Let C be a nonempty closed subset of H. We denote by $d_C(\cdot)$ or $d(\cdot, C)$ the usual distance function to the subset C, i.e., $d_C(x) := \inf_{u \in C} \|x - u\|$. We recall (see [11]) that *the proximal normal cone* of C at x is given by

$$N^P(C;x) := \{\xi \in H : \exists \alpha > 0 \text{ s.t. } x \in Proj_C(x + \alpha\xi)\},$$

where

$$Proj_C(x) := \{x' \in S : d_C(x) = \|x - x'\|\}.$$

Equivalently (see for example [11]), $N^P(C;x)$ can be defined as the set of all $\xi \in H$ for which there exist $\sigma, \delta > 0$ such that

$$\langle \xi, x' - x \rangle \leq \sigma \|x' - x\|^2 \text{ for all } x' \in (x + \delta I\!B) \cap C.$$

Note that the above inequality is satisfied locally. In Proposition 1.1.5 of [11], the authors give a characterization of $N^P(C;x)$ where the inequality is satisfied globally. For completeness, we reproduce that proposition as the following:-

Lemma 2.1. *Let C be a nonempty closed subset in H, then $\xi \in N^P(C;x)$ if and only if there exists $\sigma > 0$ such that*

$$\langle \xi, x' - x \rangle \leq \sigma \|x' - x\|^2 \text{ for all } x' \in C.$$

We recall also (see [9]) that the *Clarke normal cone* is given by

$$N^C(C;x) = \overline{co}\, [N^P(C;x)],$$

where $\overline{co}[S]$ means the closure of the convex hull of S. It is clear that one always has $N^P(C;x) \subset N^C(C;x)$. The converse is not true in general. Note that $N^C(C;x)$ is always a closed and convex cone and that $N^P(C;x)$ is always a convex cone but may be nonclosed (see [9, 11]). Furthermore, if C is convex all the existing normal cones coincide with the normal cone in the sense of convex analysis $N^{Con}(C;x)$ given by

$$N^{Con}(C;x) := \{y \in H : \langle y, x' - x \rangle \leq 0, \text{ for all } x' \in C\}.$$

We will present an algorithm to solve problem (NVP). The algorithm is an adaptation of the standard projection algorithm that we reproduce below for completeness (for more details concerning this type of projection and convergence analysis in the convex case we refer the reader to [13] and the references therein).

Algorithm 2.1.

(1) Select $x^0 \in H$, $y^0 \in F(x^0)$, and $\rho > 0$.
(2) For $n \geq 0$, compute: $z^{n+1} = x^n - \rho y^n$ and select: $x^{n+1} \in Proj_C(z^{n+1})$, $y^{n+1} \in F(x^{n+1})$.

It is well known that the projection algorithm above has been introduced in the convex case ([13]) and its convergence proved. Observe that Algorithm 2.1 is well defined provided the projection on C is not empty. The convexity assumption on C, made by researchers considering Algorithm 2.1, is not required for its well definedness because it may be well defined, even in the nonconvex case (for example when C is a closed subset of a finite dimensional space, or when C is a compact subset of a Hilbert space, etc.). Rather, convexity is required for its convergence analysis. Our adaptation of the projection algorithm is based on the following two observations:

(1) The sequence of points $\{z^n\}_n$ that it generates must be sufficiently close to C.
(2) The projection operator $Proj_C(\cdot)$ must be Lipschitz on an open set containing the sequence of points $\{z^n\}_n$.

Recently, a new class of nonconvex sets, called *uniformly prox-regular sets* (see [17, 6]) (called proximally smooth sets in the original paper [10]), has been introduced and studied in [10]. It has been successfully used in many nonconvex applications such as optimization, economic models, dynamical systems, differential inclusions, etc. For such applications see [2, 3, 4, 5, 6]. This class seems particularly well suited to overcome the difficulties which arise due to the nonconvexity assumption on C. We take the following characterization proved in [10] as a definition of this class. We point out that the original definition was given in terms of the differentiability of the distance function (see [10]).

Definition 2.1. For a given $r \in]0, +\infty]$, a subset C is uniformly prox-regular with respect to r (we will say uniformly r-prox-regular)(see [10]) if and only if every nonzero proximal normal to C can be realized by an r-ball. This means that for all $\bar{x} \in C$ and all $0 \neq \xi \in N^P(C;\bar{x})$ one has

$$\left\langle \frac{\xi}{\|\xi\|}, x - \bar{x} \right\rangle \leq \frac{1}{2r}\|x - \bar{x}\|^2,$$

for all $x \in C$.

We make the convention $\frac{1}{r} = 0$ for $r = +\infty$. Recall that for $r = +\infty$ the uniform r-prox-regularity of C is equivalent to the convexity of C, which makes this class of great importance.

The following proposition summarizes some important consequences of the uniform prox-regularity needed in the sequel. For the proof of these results we refer the reader to [10, 17].

Proposition 2.2. *Let C be a nonempty closed subset in H and let $r \in]0, +\infty]$. If the subset C is uniformly r-prox-regular then the following hold:*

i) *For all $x \in H$ with $d_C(x) < r$, one has $Proj_C(x) \neq \emptyset$;*
ii) *Let $r' \in (0, r)$. The operator $Proj_C$ is Lipschitz with rank $\frac{r}{r-r'}$ on $C_{r'}$;*
iii) *The proximal normal cone is closed as a set-valued mapping.*
iv) *For all $x \in C$ and all $0 \neq \xi \in N^P(C;x)$ one has*

$$\left\langle \frac{\xi}{\|\xi\|}, x' - x \right\rangle \leq \frac{2}{r}\|x' - x\|^2 + d_C(x'),$$

for all $x' \in H$ with $d_C(x') < r$.

As a direct consequence of Part (iii) of Proposition 2.2, we have $N^C(C;x) = N^P(C;x)$. So, we will denote $N(C;x) := N^C(C;x) = N^P(C;x)$ for such a class of sets.

In order to make clear the concept of r-prox-regular sets, we state the following concrete example: The union of two disjoint intervals $[a, b]$ and $[c, d]$ is r-prox-regular with $r = \frac{c-b}{2}$. The finite union of disjoint intervals is also r-prox-regular and the r depends on the distances between the intervals (for more concrete examples and for a general study of the class of r-prox-regular sets we refer to a forthcoming paper by the first author).

The following proposition establishes the relationship between (VI) and (VP) in the convex case.

Proposition 2.3. *If C is convex, then (VI) $\Longleftrightarrow$ (VP).*

Proof. It follows directly from the above definition of $N^{Con}(C;x)$. □

The next proposition shows that the nonconvex variational problem (NVP) can be rewritten as the following nonconvex variational inequality:

(NVI) Find $x^* \in C$ $y^* \in F(x^*)$ s.t. $\langle y^*, x - x^* \rangle + \frac{\|y^*\|}{2r}\|x - x^*\|^2 \geq 0, \ x \in C.$

Proposition 2.4. *If C is r-prox-regular, then (NVI) $\Longleftrightarrow$ (NVP).*

Proof. ($\Longrightarrow$) Let $x^* \in C$ be a solution of (NVI), i.e., there exists $y^* \in F(x^*)$ such that

$$\langle y^*, x - x^* \rangle + \frac{\|y^*\|}{2r}\|x - x^*\|^2 \geq 0, \ \text{for all}\, x \in C.$$

If $y^* = 0$, then we are done because the vector zero always belongs to any normal cone. If $y^* \neq 0$, then, for all $x \in C$, one has

$$\left\langle \frac{-y^*}{\|y^*\|}, x - x^* \right\rangle \leq \frac{1}{2r}\|x - x^*\|^2.$$

Therefore, by Lemma 2.1 one gets $\frac{-y^*}{\|y^*\|} \in N(C;x^*)$ and so $-y^* \in N(C;x^*)$, which completes the proof of the necessity part.

($\Longleftarrow$) It follows directly from the definition of prox-regular sets in Definition 2.1. $\square$

In what follows we will let C be a uniformly r'-prox-regular subset of H with $r' > 0$ and we will let $r \in (0, r')$. Now, we are ready to present our adaptation of Algorithm 2.1 to the uniform prox-regular case.

3. MAIN RESULTS

3.1. F **Strongly Monotone.** Our first algorithm 3.1 below is proposed to solve problem (NVP).

Algorithm 3.1.

(1) Select $x^0 \in C, y^0 \in F(x^0),$ and $\rho > 0$.
(2) For $n \geq 0$, compute: $z^{n+1} = x^n - \rho y^n$ and select: $x^{n+1} \in Proj_C(z^{n+1})$, $y^{n+1} \in F(x^{n+1})$.

In our analysis we need the following assumptions on F:

Assumptions $\mathcal{A}_1$.

(1) $F : H \rightrightarrows H$ is strongly monotone on C with constant $\alpha > 0$, i.e., there exists $\alpha > 0$ such that $\forall x, x' \in C$
$$\langle y - y', x - x' \rangle \geq \alpha \|x - x'\|^2, \ \ \forall y \in F(x), \ y' \in F(x').$$
(2) F has nonempty compact values in H and is Hausdorff Lipschitz continuous on C with constant $\beta > 0$, i.e., there exists $\beta > 0$ such that $\forall x, x' \in C$
$$\mathcal{H}(F(x), F(x')) \leq \beta \|x - x'\|.$$
Here $\mathcal{H}$ stands for the Hausdorff distance relative to the norm associated with the Hilbert space H defined by
$$\mathcal{H}(A, B) := \max\{\sup_{a \in A} d_B(a), \sup_{b \in B} d_A(b)\}.$$
(3) The constants α and β satisfy the following inequality:
$$\alpha\zeta > \beta\sqrt{\zeta^2 - 1},$$
where $\zeta = \frac{r'}{r' - r}$.

Theorem 3.1. *Assume that $\mathcal{A}_1$ holds and that for each iteration the parameter ρ satisfies the inequalities*
$$\frac{\alpha}{\beta^2} - \epsilon < \rho < \min\left\{\frac{\alpha}{\beta^2} + \epsilon, \frac{r}{\|y^n\| + 1}\right\},$$
where $\epsilon = \frac{\sqrt{(\alpha\zeta)^2 - \beta^2(\zeta^2 - 1)}}{\zeta\beta^2}$, then the sequences $\{z^n\}_n, \ \{x^n\}_n$, and $\{y^n\}_n$ generated by Algorithm 3.1 converge strongly to some $z^*, \ x^*$, and y^* respectively, and x^* is a solution of (NVP).

Proof. From Algorithm 3.1, we have
$$\begin{aligned} \|z^{n+1} - z^n\| &= \left\|(x^n - \rho y^n) - \left(x^{n-1} - \rho y^{n-1}\right)\right\| \\ &= \left\|x^n - x^{n-1} - \rho(y^n - y^{n-1})\right\|. \end{aligned}$$

As the elements $\{x^n\}_n$ belong to C by construction and by using the fact that F is strongly monotone and Hausdorff Lipschitz continuous on C, we have:
$$\left\langle y^n - y^{n-1}, x^n - x^{n-1} \right\rangle \geq \alpha \left\|x^n - x^{n-1}\right\|^2,$$

and

$$\left\|y^n - y^{n-1}\right\| \le \mathcal{H}(F(x^n), F(x^{n-1})) \le \beta \left\|x^n - x^{n-1}\right\|$$

respectively. Note that

$$\begin{aligned}&\left\|x^n - x^{n-1} - \rho(y^n - y^{n-1})\right\|^2 \\ &\qquad = \left\|x^n - x^{n-1}\right\|^2 - 2\rho \left\langle y^n - y^{n-1}, x^n - x^{n-1}\right\rangle + \rho^2 \left\|y^n - y^{n-1}\right\|^2.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}&\left\|x^n - x^{n-1} - \rho(y^n - y^{n-1})\right\|^2 \\ &\qquad \le \left\|x^n - x^{n-1}\right\|^2 - 2\rho\alpha \left\|x^n - x^{n-1}\right\|^2 + \rho^2\beta^2 \left\|x^n - x^{n-1}\right\|^2,\end{aligned}$$

i.e.,

$$\left\|x^n - x^{n-1} - \rho(y^n - y^{n-1})\right\|^2 \le (1 - 2\rho\alpha + \rho^2\beta^2) \left\|x^n - x^{n-1}\right\|^2.$$

So,

$$\left\|x^n - x^{n-1} - \rho(y^n - y^{n-1})\right\| \le \sqrt{1 - 2\rho\alpha + \rho^2\beta^2} \left\|x^n - x^{n-1}\right\|.$$

Finally, we deduce directly that:

$$\left\|z^{n+1} - z^n\right\| \le \sqrt{1 - 2\rho\alpha + \rho^2\beta^2} \left\|x^n - x^{n-1}\right\|.$$

Now, by the choice of ρ in the statement of the theorem, $\rho < \frac{r}{\|y^n\|+1}$, we can easily check that the sequence of points $\{z^n\}_n$ belongs to $C_r := \{x \in H : d_C(x) < r\}$. Consequently, the Lipschitz property of the projection operator on C_r mentioned in Proposition 2.2, yields

$$\begin{aligned}\left\|x^{n+1} - x^n\right\| &= \left\|Proj_C(z^{n+1}) - Proj_C(z^n)\right\| \\ &\le \zeta \left\|z^{n+1} - z^n\right\| \\ &\le \zeta\sqrt{1 - 2\rho\alpha + \rho^2\beta^2} \left\|x^n - x^{n-1}\right\|.\end{aligned}$$

Let $\xi = \zeta\sqrt{1 - 2\rho\alpha + \rho^2\beta^2}$. Our assumption (3) in $\mathcal{A}_1$ and the choice of ρ in the statement of the theorem yield $\xi < 1$. Therefore, the sequence $\{x^n\}_n$ is a Cauchy sequence and hence it converges strongly to some point $x^* \in H$. By using the continuity of the operator F, the strong convergence of the sequences $\{y^n\}_n$ and $\{z^n\}_n$ follows directly from the strong convergence of $\{x^n\}_n$.

Let y^* and z^* be the limits of the sequences $\{y^n\}_n$ and $\{z^n\}_n$ respectively. It is obvious that $z^* = x^* - \rho y^*$ with $x^* \in C$, $y^* \in F(x^*)$. We wish to show that x^* is the solution of our problem (NVP).

By construction we have, for all $n \ge 0$,

$$x^{n+1} \in Proj_C(z^{n+1}) = Proj_C(x^n - \rho y^n),$$

which gives, by the definition of the proximal normal cone,

$$(x^n - x^{n+1}) - \rho y^n \in N(C; x^{n+1}).$$

Using the closedness property of the proximal normal cone in (iii) of Proposition 2.2 and by letting $n \to \infty$ we get

$$\rho y^* \in -N(C; x^*).$$

Finally, as $y^* \in F(x^*)$ we conclude that $-N(C; x^*) \cap F(x^*) \ne \emptyset$ with $x^* \in C$. This completes the proof. □

Remark 3.2. If C is given in an explicit form, then we select, for the starting point, x^0 in C. However, if we do not know the explicit form of C, then the choice of $x^0 \in C$ may not be possible. Assume we know, instead, an explicit form of a δ-neighborhood of C, with $\delta < r/2$. So, we start with a point x^0 in the δ-neighborhood and instead of Algorithm 3.1, we use Algorithm 3.2 below. The convergence analysis of Algorithm 3.2 can be conducted along the same lines under the following choice of ρ:

$$\frac{\alpha}{\beta^2} - \epsilon < \rho < \min\left\{\frac{\alpha}{\beta^2} + \epsilon, \frac{\delta}{\|y^n\| + 1}\right\}.$$

Indeed, if $x^0 \in \delta$-neighborhood of C, then $z^1 := x^0 - \rho y^0$ and so

$$d(z^1, C) \leq d(x^0, C) + \rho\|y^0\| < \delta + \frac{\delta}{\|y^0\| + 1}\|y^0\| < \delta + \delta = 2\delta < r.$$

Therefore, we can project z^1 on C to get $x^1 \in C$, and then all subsequent points of the sequence x^n will be in C.

Algorithm 3.2.

(1) Select $x^0 \in C + \delta\mathbb{B},$ with $0 < 2\delta < r, y^0 \in F(x^0),$ and $\rho > 0$.
(2) For $n \geq 0$, compute: $z^{n+1} = x^n - \rho y^n$ and select: $x^{n+1} \in Proj_C(z^{n+1})$, $y^{n+1} \in F(x^{n+1})$.

Remark 3.3. An inspection of the proof of Theorem 3.1 shows that the sequence $\{y^n\}_n$ is bounded. We state two sufficient conditions ensuring the boundedness of the sequence $\{y^n\}_n$:

(1) The set-valued mapping F is bounded on C.
(2) The set C is bounded and the set-valued mapping F has the linear growth property on C, that is,

$$F(x) \subset \alpha_1(1 + \|x\|)\mathbb{B},$$

for some α_1 and for all $x \in C$.

3.2. F **Not Necessarily Strongly Monotone.** We end this section by noting that our result in Theorem 3.1 can be extended (see Theorem 3.4 below) to the case $F = F_1 + F_2$ where F_1 is a Hausdorff Lipschitz set-valued mapping, strongly monotone on C and F_2 is only a Hausdorff Lipschitz set-valued mapping on C, but not necessarily monotone. It is interesting to point out that, in this case, F is not necessarily strongly monotone on C and so the following result cannot be covered by our previous result. In this case Algorithm 3.1 becomes:

Algorithm 3.3.

(1) Select $x^0 \in C,\ y^0 \in F_1(x^0),\ w^0 \in F_2(x^0)$ and $\rho > 0$.
(2) For $n \geq 0$, compute: $z^{n+1} = x^n - \rho(y^n + w^n)$ and select: $x^{n+1} \in Proj_C(z^{n+1})$, $y^{n+1} \in F_1(x^{n+1}),\ w^{n+1} \in F_2(x^{n+1})$.

The following assumptions on F_1 and F_2 are needed for the proof of the convergence of Algorithm 3.3.

Assumptions $\mathcal{A}_2$.

(1) F_1 is strongly monotone on C with constant $\alpha > 0$.
(2) F_1 and F_2 have nonempty compact values in H and are Hausdorff Lipschitz continuous on C with constant $\beta > 0$ and $\eta > 0$, respectively.
(3) The constants α, ζ, η, and β satisfy the following inequality:

$$\alpha\zeta > \eta + \sqrt{(\beta^2 - \eta^2)(\zeta^2 - 1)}.$$

Theorem 3.4. *Assume that $\mathcal{A}_2$ holds and that for each iteration the parameter ρ satisfies the inequalities*

$$\frac{\alpha\zeta-\eta}{\zeta(\beta^2-\eta^2)}-\varepsilon<\rho<\min\left\{\frac{\alpha\zeta-\eta}{\zeta(\beta^2-\eta^2)}+\varepsilon,\frac{1}{\eta\zeta},\frac{r}{\|y^n+w^n\|+1}\right\},$$

where $\varepsilon=\frac{\sqrt{(\alpha\zeta-\eta)^2-(\beta^2-\eta^2)(\zeta^2-1)}}{\zeta(\beta^2-\eta^2)}$*, then the sequences* $\{z^n\}_n,\ \{x^n\}_n$*, and* $\{y^n\}_n$ *generated by Algorithm 3.3 converge strongly to some* $z^*,\ x^*$*, and* y^* *respectively, and* x^* *is a solution of (NVP) associated to the set-valued mapping* $F=F_1+F_2$*.*

Proof. The proof follows the same lines as the proof of Theorem 3.1 with slight modifications. From Algorithm 3.3, we have

$$\begin{aligned}\left\|z^{n+1}-z^n\right\| &= \left\|[x^n-\rho(y^n+w^n)]-\left[x^{n-1}-\rho(y^{n-1}+w^{n-1}\right]\right\| \\ &\le \left\|x^n-x^{n-1}-\rho(y^n-y^{n-1})\right\|+\rho\left\|w^n-w^{n-1}\right\|.\end{aligned}$$

As the elements $\{x^n\}_n$ belong to C by construction and by using the fact that F_1 is strongly monotone and Hausdorff Lipschitz continuous on C, we have:

$$\left\langle y^n-y^{n-1},x^n-x^{n-1}\right\rangle\ge\alpha\left\|x^n-x^{n-1}\right\|^2,$$

and

$$\left\|y^n-y^{n-1}\right\|\le\mathcal{H}(F_1(x^n),F_1(x^{n-1}))\le\beta\left\|x^n-x^{n-1}\right\|.$$

Note that

$$\begin{aligned}&\left\|x^n-x^{n-1}-\rho(y^n-y^{n-1})\right\|^2 \\ &\qquad=\left\|x^n-x^{n-1}\right\|^2-2\rho\left\langle y^n-y^{n-1},x^n-x^{n-1}\right\rangle+\rho^2\left\|y^n-y^{n-1}\right\|^2.\end{aligned}$$

Thus, a simple computation yields

$$\left\|x^n-x^{n-1}-\rho(y^n-y^{n-1})\right\|^2\le(1-2\rho\alpha+\rho^2\beta^2)\left\|x^n-x^{n-1}\right\|^2.$$

On the other hand, since F_2 is Hausdorff Lipschitz continuous on C, we have

$$\left\|w^n-w^{n-1}\right\|\le\mathcal{H}(F_2(x^n),F_2(x^{n-1}))\le\eta\left\|x^n-x^{n-1}\right\|.$$

Finally,

$$\left\|z^{n+1}-z^n\right\|\le\sqrt{1-2\rho\alpha+\rho^2\beta^2}\left\|x^n-x^{n-1}\right\|+\rho\eta\left\|x^n-x^{n-1}\right\|.$$

Now, by the choice of ρ in the statement of the theorem and the Lipschitz property of the projection operator on C_r mentioned in Proposition 2.2, we have

$$\begin{aligned}\left\|x^{n+1}-x^n\right\| &= \left\|Proj_C(z^{n+1})-Proj_C(z^n)\right\| \\ &\le\zeta\left\|z^{n+1}-z^n\right\| \\ &\le\zeta\left(\sqrt{1-2\rho\alpha+\rho^2\beta^2}+\rho\eta\right)\left\|x^n-x^{n-1}\right\|.\end{aligned}$$

Let $\xi=\zeta\left(\sqrt{1-2\rho\alpha+\rho^2\beta^2}+\rho\eta\right)$. Our assumption (3) in $\mathcal{A}_2$ and the choice of ρ in the statement of the theorem yield $\xi<1$. Therefore, the proof is completed. □

Remark 3.5.

(1) Theorem 3.4 generalizes the main result in [15] to the case where C is nonconvex.
(2) As we have observed in Remark 3.2, Algorithm 3.3 may also be adapted to the case where the starting point x^0 is selected in a δ-neighborhood of the set C with $0<2\delta<r$.

4. EXTENSION

In this section we are interested in extending the results obtained so far to the case where the set C, instead of being fixed, is a set-valued mapping. Besides being a more general case, it also has many applications, see for example [1]. The problem that will be considered is the following:

(SNVP) Find a point $x^* \in C(x^*) : F(x^*) \cap -N(C(x^*); x^*) \neq \emptyset.$

This problem will be called the Set-valued Nonconvex Variational Problem (SNVP). We need the following proposition which is an adaptation of Theorem 4.1 in [6] (see also Theorem 2.1 in [4]) to our problem. We recall the following concept of Lipschitz continuity for set-valued mappings: A set-valued mapping C is said to be Lipschitz if there exists $\kappa > 0$ such that

$$|d(y, C(x)) - d(y', C(x'))| \leq \|y - y'\| + \kappa\|x - x'\|,$$

for all $x, x', y, y' \in H$. In such a case we also say that C is Lipschitz continuous with constant κ. It is easy to see that for set-valued mappings the above concept of Lipschitz continuity is weaker than the Hausdorff Lipschitz continuity.

Proposition 4.1. *Let $r \in]0, +\infty]$ and let $C : H \rightrightarrows H$ be a Lipschitz set-valued mapping with uniformly r-prox-regular values, then, the following closedness property holds: "For any $x^n \to x^*, y^n \to y^*$, and $u^n \to u^*$ with $y^n \in C(x^n)$ and $u^n \in N(C(x^n); y^n)$, one has $u^* \in N(C(x^*); y^*)$".*

Proof. Let $x^n \to x^*, y^n \to y^*$, and $u^n \to u^*$ with $y^n \in C(x^n)$ and $u^n \in N(C(x^n); y^n)$. If $u^* = 0$, then we are done. Assume that $u^* \neq 0$ (hence $u^n \neq 0$ for n large enough). Observe first that $y^* \in C(x^*)$ because C is Lipschitz continuous. As $y^n \to y^*$, for n sufficiently large, $y^n \in y^* + \frac{r}{2}\mathbb{B}$. Therefore, the uniform r-prox-regularity of the images of C and Proposition 2.2 (iv) give

$$\left\langle \frac{u^n}{\|u^n\|}, z - y^n \right\rangle \leq \frac{2}{r}\|z - y^n\|^2 + d_{C(x^n)}(z),$$

for all $z \in H$ with $d_{C(x^n)}(z) < r$. This inequality still holds for n sufficiently large and for all $z \in y^* + \delta\mathbb{B}$ with $0 < \delta < \frac{r}{2}$, because for such z,

$$d_{C(x^n)}(z) \leq \|z - y^*\| + \|y^* - y^n\| \leq \delta + \frac{r}{2} < r.$$

Consequently, the continuity of the distance function with respect to both variables (because C is Lipschitz continuous) and the above inequality give, by letting $n \to +\infty$,

$$\left\langle \frac{u^*}{\|u^*\|}, z - y^* \right\rangle \leq \frac{2}{r}\|z - y^*\|^2 + d_{C(x^*)}(z) \quad \text{for all } z \in y^* + \delta\mathbb{B}.$$

Hence,

$$\left\langle \frac{u^*}{\|u^*\|}, z - y^* \right\rangle \leq \frac{2}{r}\|z - y^*\|^2 \quad \text{for all } z \in (y^* + \delta\mathbb{B}) \cap C(x^*).$$

This ensures, by the equivalent definition (given on page 2) of the proximal normal cone, that $\frac{u^*}{\|u^*\|} \in N(C(x^*); y^*)$ and so $u^* \in N(C(x^*); y^*)$. This completes the proof of the proposition. □

In all that follows, C will be a set-valued mapping with nonempty closed r'-prox-regular values for some $r' > 0$. We will also let $r \in (0, r')$ and $\zeta = \frac{r'}{r'-r}$.

4.1. F **Strongly Monotone.** The next algorithm, Algorithm 4.1, solves problem (SNVP).

Algorithm 4.1.

(1) Select $x^0 \in C(x^0)$, $y^0 \in F(x^0)$, and $\rho > 0$.
(2) For $n \geq 0$, compute: $z^{n+1} = x^n - \rho y^n$ and select: $x^{n+1} \in Proj_{C(x^n)}(z^{n+1})$, $y^{n+1} \in F(x^{n+1})$.

We make the following assumptions on the set-valued mappings F and C:

Assumptions $\mathcal{A}_3$.

(1) F has nonempty compact values and is strongly monotone with constant $\alpha > 0$.
(2) F is Hausdorff Lipschitz continuous and C is Lipschitz continuous with constants $\beta > 0$ and $0 < \kappa < 1$ respectively.
(3) For some constant $0 < k < 1$, the operator $Proj_{C(\cdot)}(\cdot)$ satisfies the condition

$$\left\|Proj_{C(x)}(z) - Proj_{C(y)}(z)\right\| \leq k \left\|x - y\right\|, \quad \text{for all } x, y, z \in H.$$

(4) Let λ be a sufficiently small positive constant such that $0 < \lambda < \frac{r(1-\kappa)}{1+3\kappa}$.
(5) The constants α, β, ζ and k satisfy:

$$\alpha\zeta > \beta\sqrt{\zeta^2 - (1-k)^2}.$$

Theorem 4.2. *Assume that $\mathcal{A}_3$ holds and that for each iteration the parameter ρ satisfies the inequalities*

$$\frac{\alpha}{\beta^2} - \epsilon < \rho < \min\left\{\frac{\alpha}{\beta^2} + \epsilon, \frac{\lambda}{\|y_n\| + 1}\right\},$$

where $\epsilon = \frac{\sqrt{(\alpha\zeta)^2 - \beta^2[\zeta^2 - (1-k)^2])}}{\zeta\beta^2}$, then the sequences $\{z^n\}_n$, $\{x^n\}_n$, and $\{y^n\}_n$ generated by Algorithm 4.1 converge strongly to some z^*, x^*, and y^* respectively, and x^* is a solution of (SNVP).

We prove the following lemma needed in the proof of Theorem 4.2. It is of interest in its own right.

Lemma 4.3. *Under the hypothesis of Theorem 4.2, the sequences of points $\{x^n\}_n$ and $\{z^n\}_n$ generated by Algorithm 4.1 are such that:*

$$z^n \text{ and } z^{n+1} \in [C(x^n)]_r := \{y \in H : d_{C(x^n)}(y) < r\}, \quad \text{for all } n \geq 1.$$

Proof. Observe that by the definition of the algorithm,

$$d(z^1, C(x^0)) = d(x^0 - \rho y^0, C(x^0)) \leq d(x^0, C(x^0)) + \rho\|y^0\| \leq \lambda.$$

For $n = 1$, we have by (2),(3), and (4) of $\mathcal{A}_3$,

$$\begin{aligned} d(z^2, C(x^1)) &= d(x^1 - \rho y^1, C(x^1)) \\ &\leq d(x^1, C(x^1)) - d(x^1, C(x^0)) + \rho\|y^1\| \\ &\leq \kappa\|x^1 - x^0\| + \lambda, \end{aligned}$$

and by the Lipschitz continuity of C, once again, and the first inequality of this proof we get

$$\begin{aligned} d(z^1, C(x^1)) &\leq d(z^1, C(x^0)) + \kappa\|x^1 - x^0\| \\ &= d(x^0 - \rho y^0, C(x^0)) + \kappa\|x^1 - x^0\| \\ &\leq \lambda + \kappa\|x^1 - x^0\|. \end{aligned}$$

On the other hand, we have

$$\begin{aligned}\|x^1 - x^0\| &\le \|x^1 - z^1\| + \|z^1 - x^0\| \\ &= d(z^1, C(x^0)) + \|z^1 - x^0\| \\ &= d(x^0 - \rho y^0, C(x^0)) + \rho\|y^0\| < 2\lambda.\end{aligned}$$

Thus, we see that both $d(z^2, C(x^1))$ and $d(z^1, C(x^1))$ are less than $2\kappa\lambda + \lambda$ which is itself strictly less than r. Similarly, we have for general n,

$$d(z^{n+1}, C(x^n)) \le d(x^n, C(x^n)) + \rho\|y^n\| \le \kappa\|x^n - x^{n-1}\| + \lambda$$

and

$$\begin{aligned}d(z^n, C(x^n)) &\le d(z^n, C(x^{n-1})) + \kappa\|x^n - x^{n-1}\| \\ &\le \kappa\|x^{n-1} - x^{n-2}\| + \lambda + \kappa\|x^n - x^{n-1}\|.\end{aligned}$$

On the other hand,

$$\begin{aligned}\|x^n - x^{n-1}\| &\le \|x^n - z^n\| + \|z^n - x^{n-1}\| \\ &\le d(z^n, C(x^{n-1})) + \lambda \\ &\le d(x^{n-1}, C(x^{n-1})) - d(x^{n-1}, C(x^{n-2})) + 2\lambda \\ &\le \kappa\|x^{n-1} - x^{n-2}\| + 2\lambda.\end{aligned}$$

Hence, using that $\|x^1 - x^0\| < 2\lambda$, we get

$$\|x^n - x^{n-1}\| \le \frac{2\lambda(1-\kappa^n)}{1-\kappa}.$$

Therefore,

$$\begin{aligned}d(z^{n+1}, C(x^n)) &\le \frac{2\kappa\lambda(1-\kappa^n)}{1-\kappa} + \lambda \\ &\le \lambda\frac{1+\kappa-2\kappa^{n+1}}{1-\kappa} \\ &< \frac{\lambda(1+3\kappa)}{1-\kappa} < r,\end{aligned}$$

and

$$\begin{aligned}d(z^n, C(x^n)) &\le \kappa\left\|x^{n-1} - x^{n-2}\right\| + \lambda + \kappa\left\|x^n - x^{n-1}\right\| \\ &\le (\kappa^2+\kappa)\left\|x^{n-1} - x^{n-2}\right\| + 2\lambda\kappa + \lambda \\ &\le (\kappa^2+\kappa)\frac{2\lambda(1-\kappa^{n-1})}{1-\kappa} + 2\lambda\kappa + \lambda \\ &\le \frac{\lambda(1+3\kappa)}{1-\kappa} < r.\end{aligned}$$

This completes the proof. □

Proof of Theorem 4.2. Following the proof of Theorem 3.1 and using the fact that F is strongly monotone and Hausdorff Lipschitz continuous, we get, from Algorithm 4.1,

$$\|z^{n+1} - z^n\| \le \sqrt{1 - 2\rho\alpha + \rho^2\beta^2}\|x^n - x^{n-1}\|.$$

On the other hand, by Lemma 4.3, we have z^n and $z^{n+1} \in [C(x^n)]_r$ and so Proposition 2.2 yields that $Proj_{C(x^n)}(z^n)$ and $Proj_{C(x^n)}(z^{n+1})$ are not empty, and the operator $Proj_{C(x^n)}(\cdot)$ is ζ-Lipschitz on $[C(x^n)]_r$. Then, by the assumption (3) in $\mathcal{A}_3$,

$$\begin{aligned}
\|x^{n+1} - x^n\| &= \|Proj_{C(x^n)}(z^{n+1}) - Proj_{C(x^{n-1})}(z^n)\| \\
&\leq \|Proj_{C(x^n)}(z^{n+1}) - Proj_{C(x^n)}(z^n)\| + \|Proj_{C(x^n)}(z^n) - Proj_{C(x^{n-1})}(z^n)\| \\
&\leq \zeta\|z^{n+1} - z^n\| + k\|x^n - x^{n-1}\| \\
&\leq \left[\zeta\sqrt{1 - 2\rho\alpha + \rho^2\beta^2} + k\right] \|x^n - x^{n-1}\|.
\end{aligned}$$

Let $\xi = \zeta\sqrt{1 - 2\rho\alpha + \rho^2\beta^2} + k$. Our assumptions (4) and (5) in $\mathcal{A}_3$ and the choice of ρ in the statement of the theorem yield $\xi < 1$. As in the proof of Theorem 3.1, we can prove that the sequences $\{x^n\}_n, \{y^n\}_n$, and $\{z^n\}_n$ strongly converge to some $x^*, y^*, z^* \in H$, respectively. It is obvious to see that $z^* = x^* - \rho y^*$ with $x^* \in C(x^*)$, $y^* \in F(x^*)$. We wish to show that x^* is the solution of our problem (SNVP).

By construction we have, for all $n \geq 0$,

$$x^{n+1} \in Proj_{C(x^n)}(z^{n+1}) = Proj_{C(x^n)}(x^n - \rho y^n),$$

which gives, by the definition of the proximal normal cone,

$$(x^n - x^{n+1}) - \rho y^n \in N(C(x^n); x^{n+1}).$$

Using the closedness property of the proximal normal cone in Proposition 4.1 and by letting $n \to \infty$ we get

$$\rho y^* \in -N(C(x^*); x^*).$$

Finally, as $y^* \in F(x^*)$ we conclude that $-N(C(x^*); x^*) \cap F(x^*) \neq \emptyset$ with $x^* \in C(x^*)$. This completes the proof. $\square$

4.2. F **Not Necessarily Strongly Monotone.** We extend Theorem 4.2 to the case $F = F_1 + F_2$, where F_1 is a Hausdorff Lipschitz set-valued mapping strongly monotone and F_2 is only a Hausdorff Lipschitz set-valued mapping. In this case Algorithm 4.1 becomes:

Algorithm 4.2.

(1) Select $x^0 \in C(x^0),\ y^0 \in F_1(x^0),\ w^0 \in F_2(x^0)$ and $\rho > 0$.
(2) For $n \geq 0$, compute: $z^{n+1} = x^n - \rho(y^n + w^n)$ and select: $x^{n+1} \in Proj_{C(x^n)}(z^{n+1})$, $y^{n+1} \in F_1(x^{n+1}),\ w^{n+1} \in F_2(x^{n+1})$.

The following assumptions on F_1 and F_2 are needed for the proof of the convergence of Algorithm 4.2.

Assumptions $\mathcal{A}_4$.

(1) The assumptions on the set-valued mapping C are as in $\mathcal{A}_3$.
(2) F_1 is strongly monotone with constant $\alpha > 0$.
(3) F_1 and F_2 have nonempty compact values and are Hausdorff Lipschitz continuous with constant $\beta > 0$ and $\eta > 0$, respectively.
(4) The constants $\alpha, \beta, \eta, \zeta$, and k satisfy the following inequality:

$$\alpha\zeta > (1-k)\eta + \sqrt{(\beta^2 - \eta^2)[\zeta^2 - (1-k)^2]}.$$

Theorem 4.4. *Assume that $\mathcal{A}_4$ holds and that for each iteration the parameter ρ satisfies the inequalities*

$$\frac{\alpha\zeta - (1-k)\eta}{\zeta(\beta^2 - \eta^2)} - \varepsilon < \rho < \min\left\{\frac{\alpha\zeta - (1-k)\eta}{\zeta(\beta^2 - \eta^2)} + \varepsilon, \frac{1-k}{\zeta\eta}, \frac{r}{\|y^n + w^n\| + 1}\right\},$$

where $\varepsilon = \frac{\sqrt{[\alpha\zeta-(1-k)\eta]^2-(\beta^2-\eta^2)[\zeta^2-(1-k)^2]}}{\zeta(\beta^2-\eta^2)}$*, then the sequences* $\{z^n\}_n,\ \{x^n\}_n$*, and* $\{y^n\}_n$ *generated by Algorithm 4.2 converge strongly to some* $z^*,\ x^*$*, and* y^* *respectively, and* x^* *is a solution of (SNVP) associated to the set-valued mapping* $F = F_1 + F_2$*.*

Proof. As we adapted the proof of Theorem 3.1 to prove Theorem 3.4, we can adapt, in a similar way, the proof of Theorem 4.2 to prove Theorem 4.4. □

Remark 4.5.

(1) Theorem 4.4 generalizes Theorem 3.4 in [14] to the case where C is nonconvex.
(2) As we have observed in Remark 3.2, Algorithms 4.1 and 4.2 may be also adapted to the case where the starting point x^0 is selected in a δ-neighborhood of the set $C(x^0)$ with $0 < 2\delta < r$.

Example 4.1. In many applications (see for example [1]) the set-valued mapping C has the form $C(x) = S + f(x)$, where S is a fixed closed subset in H and f is a point-to-point mapping from H to H. In this case, assumption (3) on C in $\mathcal{A}_3$ and the Lipschitz continuity of C are satisfied provided the mapping f is Lipschitz continuous. Indeed, it is not hard (using the relation below) to show that, if f is γ-Lipschitz then the set-valued mapping C is γ-Lipschitz and satisfies the assumption (3) in $\mathcal{A}_3$ with $k = 2\gamma$. Using the well known relation

$$\bar{x} \in Proj_{S+v}(\bar{u}) \Longleftrightarrow \bar{x} - v \in Proj_S(\bar{u} - v),$$

Algorithms 4.1 and 4.2 can be rewritten in simpler forms. For example, Algorithm 4.2 becomes

Algorithm 4.3.

(1) Select $x^0 \in (I - f)^{-1}(S),\ y^0 \in F_1(x^0),\ w^0 \in F_2(x^0)$ and $\rho > 0$.
(2) For $n \geq 0$, compute: $z^{n+1} = x^n - f(x^n) - \rho(y^n + w^n)$ and select: $x^{n+1} \in Proj_S(z^{n+1}) + f(x^n)$, $y^{n+1} \in F_1(x^{n+1}),\ w^{n+1} \in F_2(x^{n+1})$.

Here I is the Identity operator from H to H.

5. CONCLUSION

The algorithms proposed here can be extended to solve the following general variational problem:

($g-$SNVP) Find a point $x^* \in H$ with $g(x^*) \in C(x^*) : F(x^*) \cap -N(C(x^*); g(x^*)) \neq \emptyset$,

where $g : H \to H$ is a point-to-point mapping. It is obvious that ($g-$SNVP) coincides with (SNVP) when $g = I$. An important reason for considering this general variational problem ($g-$SNVP) is to extend all (or almost all) the types of variational inequalities existing in the literature in the convex case to the nonconvex case by the same way presented in this paper. For instance, when the set-valued mapping C is assumed to have convex values the general variational problem ($g-$SNVP) coincides with the so-called *generalized multivalued quasi-variational inequality* introduced by Noor [16] and studied by himself and many other authors.

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