Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 4, Issue 1, Article 22, 2003

# COMPARISON OF GREEN FUNCTIONS FOR GENERALIZED SCHRÖDINGER OPERATORS ON $C^{1,1}$-DOMAINS

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*Received 16 September, 2002; accepted 10 February, 2003*
*Communicated by A.M. Fink*

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ABSTRACT. We establish some inequalities on the $\frac{1}{2}\Delta$-Green function $G$ on bounded $C^{1,1}$-domain. We use these inequalities to prove the existence of the $\left(\frac{1}{2}\Delta - \mu\right)$-Green function $G_\mu$ and its comparability to $G$, where $\mu$ is in some general class of signed Radon measures. Finally we prove that the choice of this class is essentially optimal.

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*Key words and phrases:* Green function, Schrödinger operator, $3G$-Theorem.

2000 *Mathematics Subject Classification.* 34B27, 35J10.

## 1. INTRODUCTION

The first aim of this paper is to prove some inequalities on the Green function $G$ of $\frac{1}{2}\Delta$ on bounded $C^{1,1}$-domain $\Omega$ in $\mathbb{R}^n$, $n \geq 3$, where $\Delta$ is the Laplacian operator. In particular we give an alternative and shorter proof of the $3G$-Theorem established in [9] using long and sharp discussions. The $3G$-Theorem includes the usual one proved in [11], [4] and [3], which was very useful to obtain some potential theoretic results. The second is to prove a comparison theorem between the Green function $G$ and the Green function $G_\mu$ of the Schrödinger operator $\frac{1}{2}\Delta - \mu$ on $\Omega$, where $\mu$ is allowed to be in some class of signed Radon measures. In contrast to [9], there is no restriction on the sign of $\mu$ in this work. This comparison theorem is very important in the sense that it enables us to deduce some potential theoretic results for $\frac{1}{2}\Delta - \mu$ which are known to hold for $\frac{1}{2}\Delta$. This is stated at the end of the paper. Moreover our result covers the case of signed Radon measures with bounded Newtonian potentials i.e,

$$\sup_{x\in\Omega} \int_\Omega \frac{1}{|x-y|^{n-2}} |\mu|(dy) < +\infty.$$

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ISSN (electronic): 1443-5756

100-02

The Schrödinger operator $\frac{1}{2}\Delta - f$, with $f$ belonging to the Kato class $K_n^{loc}$ which is studied by several authors (see [1], [3], [4], [11]) is just the special case where $\mu$ has the density $f$ with respect to the Lebesgue measure. In particular our results cover the ones proved by Zhao [11]. Finally we show that the choice of this class is essentially optimal.

Our paper is organized as follows.

In Section 2, we give some notations and recall some known results. In Section 3, we prove some inequalities on the Green function of $\frac{1}{2}\Delta$ on bounded $C^{1,1}$-domain. A new and a shorter proof of the $3G$-Theorem established in [9] is given. In Section 4, we introduce a general class of signed Radon measures on $\Omega$ denoted by $\mathcal{K}(\Omega)$ that will be considered in this work. We give some examples and we study some properties of this class. In Section 5, we prove a comparison theorem between the Green functions of $\frac{1}{2}\Delta$ and the Schrödinger operator $\frac{1}{2}\Delta - \mu$, where $\mu$ is in the class $\mathcal{K}(\Omega)$. We also show that when $\mu$ is nonnegative the condition $\mu \in \mathcal{K}(\Omega)$ is necessary for the comparison theorem to hold.

Throughout the paper the letter $C$ will denote a generic positive constant which may vary in value from line to line.

## 2. Preliminaries and Notations

Throughout the paper $\Omega$ denotes a bounded $C^{1,1}$-domain in $\mathbb{R}^n$, $n \geq 3$. This means that for each $z \in \partial\Omega$ there exists a ball $B(z, R_0)$, $R_0 > 0$ and a coordinate system of $\mathbb{R}^n$ such that in these coordinates,

$$B(z, R_0) \cap \Omega = B(z, R_0) \cap \{(x', x_n)/x' \in \mathbb{R}^{n-1},\, x_n > f(x')\},$$

and

$$B(z, R_0) \cap \partial\Omega = B(z, R_0) \cap \{(x', f(x'))/x' \in \mathbb{R}^{n-1}\},$$

where $f$ is a $C^{1,1}$-function.

$\Delta$ denotes the Laplacian operator on $\mathbb{R}^n$ and $G$ its Green function on $\Omega$. For a signed Radon measure $\mu$ on $\Omega$, we denote by $G_\mu$ the $\left(\frac{1}{2}\Delta - \mu\right)$-Green function on $\Omega$, when it exists.

For $x \in \Omega$ let $d(x) = d(x, \partial\Omega)$, the distance from $x$ to the boundary of $\Omega$. We denote by $d(\Omega)$ the diameter of $\Omega$.

Since $\Omega$ is a bounded $C^{1,1}$-domain, then it has the following geometrical property:

There exists $r_0 > 0$ depending only on $\Omega$ such that for any $z \in \partial\Omega$ and $0 < r \leq r_0$ there exist two balls $B_1^z(r)$ and $B_2^z(r)$ of radius $r$ such that $B_1^z(r) \subset \Omega$, $B_2^z(r) \subset \mathbb{R}^n \setminus \overline{\Omega}$ and $\{z\} = \partial B_1^z(r) \cap \partial B_2^z(r)$.

We recall the following interesting estimates on the Green function $G$ which are due to Grüter and Widman [5], Zhao [11] and Hueber [6].

**Theorem 2.1.** *There exists a constant $C > 0$ depending on the diameter of $\Omega$, on the curvature of $\partial\Omega$ and on the dimension $n$ such that*

$$C^{-1} \min\left(1, \frac{d(x)d(y)}{|x-y|^2}\right) \frac{1}{|x-y|^{n-2}} \leq G(x,y) \leq C \min\left(1, \frac{d(x)d(y)}{|x-y|^2}\right) \frac{1}{|x-y|^{n-2}}.$$

*for all $x$, $y \in \Omega$.*

## 3. Inequalities on the Green Function $G$

In this section we first give a new and a simple proof of the $3G$-Theorem established in [9]. We also derive other inequalities on the Green function $G$ that will be used in the next sections.

**Theorem 3.1** ($3G$-Theorem)**.** *There exists a constant* $C = C(\Omega, n) > 0$ *such that for* $x,\ y,\ z \in \Omega$*, we have*

$$\frac{G(x,z)G(z,y)}{G(x,y)} \le C(\frac{d(z)}{d(x)}G(x,z) + \frac{d(z)}{d(y)}G(z,y)).$$

*Proof.* The inequality of the theorem is equivalent to

$$\frac{1}{d(z)G(x,y)} \le C\left(\frac{1}{d(x)G(z,y)} + \frac{1}{d(y)G(x,z)}\right). \tag{3.1}$$

On the other hand, since for $a > 0, b > 0$,

$$\frac{ab}{a+b} \le \min(a,b) \le 2\frac{ab}{a+b},$$

then

$$\frac{d(x)d(y)}{|x-y|^2 + d(x)d(y)} \le \min\left(1, \frac{d(x)d(y)}{|x-y|^2}\right) \le 2\frac{d(x)d(y)}{|x-y|^2 + d(x)d(y)},$$

and hence, from Theorem 2.1, we obtain

$$C^{-1}N(x,y) \le G(x,y) \le CN(x,y),$$

where

$$N(x,y) = \frac{d(x)d(y)}{|x-y|^{n-2}(|x-y|^2 + d(x)d(y))}.$$

Therefore (3.1) is equivalent to

$$\begin{aligned} |x-y|^{n-2}(|x-y|^2 + d(x)d(y)) \le C(&|z-y|^{n-2}(|z-y|^2 + d(z)d(y)) \\ &+ |x-z|^{n-2}(|x-z|^2 + d(x)d(z))). \end{aligned} \tag{3.2}$$

Then, we shall prove (3.2). By symmetry we may assume that $|x-z| \le |y-z|$. We have

$$\begin{aligned} |x-y|^{n-2} &\le (|x-z| + |z-y|)^{n-2} \\ &\le 2^{n-2}|z-y|^{n-2}, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} |x-y|^2 + d(x)d(y) &\le (|x-z| + |z-y|)^2 + (|x-z| + d(z))d(y) \\ &\le 4|z-y|^2 + |z-y|d(y) + d(z)d(y). \end{aligned} \tag{3.4}$$

If $|z-y| \le d(z)$, then

$$|z-y|d(y) \le d(z)d(y). \tag{3.5}$$

If $|z-y| \ge d(z)$, then

$$\begin{aligned} |z-y|d(y) &\le |z-y|(d(z) + |z-y|) \\ &\le 2|z-y|^2. \end{aligned} \tag{3.6}$$

From (3.4), (3.5) and (3.6), we obtain

$$|x-y|^2 + d(x)d(y) \le 6(|z-y|^2 + d(y)d(z)). \tag{3.7}$$

From (3.3) and (3.7), we obtain

$$|x-y|^{n-2}(|x-y|^2 + d(x)d(y)) \le 2^{n+1}|z-y|^{n-2}(|z-y|^2 + d(z)d(y)).$$

This proves (3.2) with $C = 2^{n+1}$. □

**Lemma 3.2.** *There exists a constant $C = C(\Omega, n) > 0$ such that for all $x,\ y \in \Omega$, we have*

$$\frac{d(y)}{d(x)}G(x,y) \leq \frac{C}{|x-y|^{n-2}}.$$

*Proof.* By Theorem 2.1, we have

$$\frac{d(y)}{d(x)}G(x,y) \leq \frac{C}{|x-y|^{n-2}} \min\left(\frac{d(y)}{d(x)}, \frac{d(y)^2}{|x-y|^2}\right).$$

Put $t = \frac{d(y)}{d(x)} > 0$. From the inequality $|x-y| \geq |d(y) - d(x)|$, it follows

$$\begin{aligned}\min\left(\frac{d(y)}{d(x)}, \frac{d(y)^2}{|x-y|^2}\right) &\leq \min\left(\frac{d(y)}{d(x)}, \frac{d(y)^2}{|d(y)-d(x)|^2}\right)\\ &= \min\left(t, \frac{t^2}{(t-1)^2}\right).\end{aligned}$$

Since $\min\left(t, \frac{t^2}{(t-1)^2}\right) \leq 4$, for all $t > 0$, then we obtain

$$\frac{d(y)}{d(x)}G(x,y) \leq \frac{4C}{|x-y|^{n-2}}.$$

By symmetry we also have

$$\frac{d(x)}{d(y)}G(x,y) \leq \frac{4C}{|x-y|^{n-2}}.$$

This ends the proof. □

The usual $3G$-Theorem proved in [3, 4, 11] is well known under the following form which is a simple consequence of Theorem 3.1 and Lemma 3.2.

**Corollary 3.3.** *There exists a constant $C = C(\Omega, n) > 0$ such that for $x,\ y,\ z \in \Omega$, we have*

$$\frac{G(x,z)G(z,y)}{G(x,y)} \leq C\left(\frac{1}{|x-z|^{n-2}} + \frac{1}{|z-y|^{n-2}}\right).$$

## 4. THE CLASS $\mathcal{K}(\Omega)$

**Definition 4.1.** Let $\mu$ be a signed Radon measure on $\Omega$. We say that $\mu$ is in the class $\mathcal{K}(\Omega)$ if it satisfies

$$||\mu|| \equiv \sup_{x\in\Omega} \int_\Omega \frac{d(y)}{d(x)} G(x,y)|\mu|(dy) < +\infty,$$

where $|\mu|$ is the total variation of $\mu$.

In the following we study some properties of the class $\mathcal{K}(\Omega)$ and to this end we first need to prove the following lemma.

**Lemma 4.1.** *For $x,\ y \in \Omega$, we have*
*If $d(x)d(y) \geq |x-y|^2$, then*

$$\frac{1}{3}d(y) \leq d(x) \leq 3d(y).$$

*If $d(x)d(y) \leq |x-y|^2$, then*

$$\max(d(x), d(y)) \leq 2|x-y|.$$

*Proof.* If $d(x)d(y) \geq |x-y|^2$, then in view of the inequality $|x-y| \geq |d(x)-d(y)|$, we obtain

$$d(x)d(y) \geq |d(x)-d(y)|^2,$$

which implies

$$3\,d(x)d(y) \geq d(x)^2 + d(y)^2,$$

and then

$$\frac{1}{3}d(y) \leq d(x) \leq 3\,d(y).$$

If $d(x)d(y) \leq |x-y|^2$, then in view of the inequality $d(x) \geq d(y) - |x-y|$, we obtain

$$d(y)(d(y)-|x-y|) \leq |x-y|^2,$$

which gives

$$\begin{aligned} d(y)^2 &\leq |x-y|^2 + d(y)|x-y| \\ &\leq \left(|x-y| + \frac{1}{2}d(y)\right)^2. \end{aligned}$$

The last inequality yields

$$\frac{1}{2}d(y) \leq |x-y|.$$

Similarly, we have

$$\frac{1}{2}d(x) \leq |x-y|.$$

□

The following proposition provides some interesting examples of measures in the class $\mathcal{K}(\Omega)$.

**Proposition 4.2.** *For $\alpha \in \mathbb{R}$, the measure $\frac{1}{d(y)^\alpha}dy$ is in the class $K(\Omega)$ if and only if $\alpha < 2$.*

*Proof.* We first assume $\alpha < 2$ and we will prove that

$$\sup_{x\in\Omega}\int_\Omega \frac{d(y)}{d(x)}G(x,y)\frac{1}{d(y)^\alpha}dy < +\infty.$$

By Theorem 2.1, we have

$$\sup_{x\in\Omega}\int_\Omega \frac{d(y)}{d(x)}G(x,y)\frac{1}{d(y)^\alpha}dy \leq C\sup_{x\in\Omega}\int_\Omega \min\left(\frac{1}{d(x)d(y)}, \frac{1}{|x-y|^2}\right)\frac{d(y)^{2-\alpha}}{|x-y|^{n-2}}dy. \tag{4.1}$$

On the other hand

$$\begin{aligned} \int_\Omega \min\left(\frac{1}{d(x)d(y)}, \frac{1}{|x-y|^2}\right)&\frac{d(y)^{2-\alpha}}{|x-y|^{n-2}}dy \\ &= \int_{\Omega\cap(d(x)d(y)\geq|x-y|^2)} \cdots dy + \int_{\Omega\cap(d(x)d(y)\leq|x-y|^2)} \cdots dy \\ &\equiv I_1 + I_2. \end{aligned} \tag{4.2}$$

We estimate $I_1$. From Lemma 4.1, we have

$$\begin{aligned}
I_1 &= \int_{\Omega\cap(d(x)d(y)\geq|x-y|^2)} \frac{d(y)^{1-\alpha}}{d(x)|x-y|^{n-2}}dy \\
&\leq Cd(x)^{-\alpha}\int_{|x-y|\leq\sqrt{3}\,d(x)} \frac{1}{|x-y|^{n-2}}dy \\
&\leq Cd(x)^{-\alpha}\int_0^{\sqrt{3}\,d(x)} r dr \\
&\leq Cd(x)^{2-\alpha} \\
&\leq Cd(\Omega)^{2-\alpha}. \qquad (4.3)
\end{aligned}$$

Now we estimate $I_2$. From Lemma 4.1, we have

$$\begin{aligned}
I_2 &= \int_{\Omega\cap(d(x)d(y)\leq|x-y|^2)} \frac{d(y)^{2-\alpha}}{|x-y|^{n}}dy \\
&\leq 2^{2-\alpha}\int_{\Omega} \frac{1}{|x-y|^{n-2+\alpha}}dy \\
&\leq 2^{2-\alpha}w_{n-1}\int_0^{d(\Omega)} r^{1-\alpha}dr \\
&= \frac{2^{2-\alpha}w_{n-1}}{2-\alpha}d(\Omega)^{2-\alpha}, \qquad (4.4)
\end{aligned}$$

where $w_{n-1}$ is the area of the unit sphere $S_{n-1}$ in $\mathbb{R}^n$.

Combining (4.1), (4.2), (4.3) and (4.4), we obtain

$$\sup_{x\in\Omega}\int_\Omega \frac{d(y)}{d(x)}G(x,y)\frac{1}{d(y)^\alpha}dy \leq Cd(\Omega)^{2-\alpha} < +\infty.$$

Now we assume $\alpha \geq 2$ and we will prove that

$$\sup_{x\in\Omega}\int_\Omega \frac{d(y)}{d(x)}G(x,y)\frac{1}{d(y)^\alpha}dy = +\infty.$$

We first remark that when $d(x) \leq \frac{\sqrt{5}-1}{2}|x-y|$, we have

$$d(y) \leq d(x) + |x-y| \leq \frac{\sqrt{5}+1}{2}|x-y|$$

and then $d(x)d(y) \leq |x-y|^2$. By Theorem 2.1, we have

$$\begin{aligned}
\sup_{x\in\Omega}\int_\Omega \frac{d(y)}{d(x)}G(x,y)\frac{1}{d(y)^\alpha}dy &\geq C^{-1}\sup_{x\in\Omega}\int_\Omega \min\left(\frac{1}{d(x)d(y)}, \frac{1}{|x-y|^2}\right)\frac{d(y)^{2-\alpha}}{|x-y|^{n-2}}dy \\
&\geq C^{-1}\sup_{x\in\Omega}\int_{\Omega\cap(d(x)\leq\frac{\sqrt{5}-1}{2}|x-y|)} \frac{d(y)^{2-\alpha}}{|x-y|^{n}}dy. \qquad (4.5)
\end{aligned}$$

Let $z_0 \in \partial\Omega$ and put $x_0$ the center of $B_1^{z_0}(r_0)$. This means $B_1^{z_0}(r_0) = B(x_0, r_0) \subset \Omega$. For $x \in ]z_0, x_0]$, we have

$$|y-x| \leq |y-z_0| + d(x),$$

and

$$\left\{y \in D : d(x) \leq \frac{3-\sqrt{5}}{2}|y-z_0|\right\} \subset \left\{y \in D : d(x) \leq \frac{\sqrt{5}-1}{2}|y-x|\right\}.$$

Hence for $x \in ]z_0, x_0]$, we have

$$\int_{\Omega\cap(d(x)\leq\frac{\sqrt{5}-1}{2}|x-y|)} \frac{d(y)^{2-\alpha}}{|x-y|^n}dy \geq \int_{\Omega\cap(d(x)\leq\frac{3-\sqrt{5}}{2}|y-z_0|)} \frac{|y-z_0|^{2-\alpha}}{(|y-z_0|+d(x))^n}dy. \tag{4.6}$$

From (4.5) and (4.6), we obtain

$$\begin{aligned}\sup_{x\in\Omega}\int_\Omega \frac{d(y)}{d(x)}G(x,y)\frac{1}{d(y)^\alpha}dy &\geq C^{-1}\int_\Omega \frac{1}{|y-z_0|^{n+\alpha-2}}dy \\ &\geq C^{-1}\int_{B_1^{z_0}(r_0)} \frac{1}{|y-z_0|^{n+\alpha-2}}dy \\ &= C^{-1}\int_{|y-x_0|<r_0} \frac{1}{|y-z_0|^{n+\alpha-2}}dy \\ &= C^{-1}\int_{|y|<r_0} \frac{1}{|y-\xi|^{n+\alpha-2}}dy, \qquad (4.7)\end{aligned}$$

where $\xi = z_0 - x_0$ with $|\xi| = r_0$.

We take a spherical coordinate system $(r, \theta_1, \ldots, \theta_{n-1})$ such that $\xi = (|\xi|, 0, \ldots, 0)$. Then, we have

$$\int_{|y|<r_0} \frac{1}{|y-\xi|^{n+\alpha-2}}dy = w_{n-2}\int_0^{r_0} r^{n-1}\int_0^\pi \frac{(\sin\theta_1)^{n-2}}{(r^2+r_0^2-2rr_0\cos\theta_1)^{\frac{n+\alpha}{2}-1}}d\theta_1 dr. \tag{4.8}$$

By making the change of variables $t = \tan\frac{\theta_1}{2}$, we obtain

$$\begin{aligned}&\int_0^\pi \frac{(\sin\theta_1)^{n-2}}{(r^2+r_0^2-2rr_0\cos\theta_1)^{\frac{n+\alpha}{2}-1}}d\theta_1 \\ &\qquad = 2^{n-1}\int_0^{+\infty} \frac{t^{n-2}(1+t^2)^{\frac{\alpha-n}{2}}}{((r+r_0)^2t^2+(r_0-r)^2)^{\frac{n+\alpha}{2}-1}}dt \\ &\qquad = \frac{2^{n-1}(r_0+r)^{1-n}}{(r_0-r)^{\alpha-1}}\int_0^{+\infty} \frac{s^{n-2}\left(1+(\frac{r_0-r}{r_0+r})^2s^2\right)^{\frac{\alpha-n}{2}}}{(s^2+1)^{\frac{n+\alpha}{2}-1}}ds \\ &\qquad \geq \frac{k}{(r_0-r)^{\alpha-1}},\end{aligned}$$

where $k = k(r_0, \alpha, n) > 0$.

This implies

$$\int_0^{r_0} r^{n-1}\int_0^\pi \frac{(\sin\theta_1)^{n-2}}{(r^2+r_0^2-2rr_0\cos\theta_1)^{\frac{n+\alpha}{2}-1}}d\theta_1 dr \geq k\int_0^{r_0} \frac{r^{n-1}}{(r_0-r)^{\alpha-1}}dr = +\infty. \tag{4.9}$$

From (4.7), (4.8) and (4.9), we obtain

$$\sup_{x\in\Omega}\int_\Omega \frac{d(y)}{d(x)}G(x,y)\frac{1}{d(y)^\alpha}dy = +\infty.$$

This ends the proof. □

Now we compare the class $\mathcal{K}(\Omega)$ with the class of signed Radon measures with bounded Newtonian potentials. A signed Radon measure $\mu$ is said to be of bounded Newtonian potential if $\sup_{x\in\Omega}\int_\Omega \frac{1}{|x-y|^{n-2}}|\mu|(dy) < +\infty$.

**Proposition 4.3.** *The class $\mathcal{K}(\Omega)$ properly contains the class of signed Radon measures with bounded Newtonian potentials.*

*Proof.* From definitions and Lemma 3.2, it is clear that the class of signed Radon measures with bounded Newtonian potentials is contained in $\mathcal{K}(\Omega)$. In the sequel we will prove that for $1 \leq \alpha$,

$$\int_\Omega \frac{1}{d(y)^\alpha} dy = +\infty$$

and then

$$\sup_{x\in\Omega} \int_\Omega \frac{1}{|x-y|^{n-2}} \frac{1}{d(y)^\alpha} dy = +\infty.$$

In particular for $1 \leq \alpha < 2$, $\frac{1}{d(y)^\alpha} dy$ does not define a bounded Newtonian potential and by Proposition 4.2, we know that $\frac{1}{d(y)^\alpha} dy \in \mathcal{K}(\Omega)$.

Without loss of generality we assume that $0 \in \partial\Omega$. We know that there exists $R_0 > 0$ such that

$$B(0,R_0) \cap \Omega = B(0,R_0) \cap \{(x', x_n)/x' \in \mathbb{R}^{n-1},\, x_n > f(x')\},$$

and

$$B(0,R_0) \cap \partial\Omega = B(0,R_0) \cap \{(x', f(x'))/x' \in \mathbb{R}^{n-1}\},$$

where $f$ is a $C^{1,1}$-function.
By the continuity of $f$, there exists $\rho_0 \in \left]0, \frac{R_0}{4}\right[$ such that for $|y'| < \rho_0$, we have $|f(y')| < \frac{R_0}{2}$.

Hence for all $y = (y', y_n)$ such that $|y'| < \rho_0$ and $0 < y_n - f(y') < \frac{R_0}{4}$, we have $(y', f(y')) \in \partial\Omega$ and $y \in B(0,R_0) \cap \Omega$ which give $d(y) \leq y_n - f(y')$.

Using these observations we have

$$\begin{aligned}
\int_\Omega \frac{1}{d(y)^\alpha} dy &\geq \int_{\Omega\cap B(0,R_0)} \frac{1}{d(y)^\alpha} dy \\
&\geq \int_{|y'|<\rho_0} \int_{0<y_n - f(y')<\frac{R_0}{4}} \frac{1}{(y_n - f(y'))^\alpha} dy_n dy' \\
&= \int_{|y'|<\rho_0} dy' \int_0^{\frac{R_0}{4}} \frac{1}{r^\alpha} dr = +\infty.
\end{aligned}$$

□

We next prove that the Kato class $K_n^{loc}$ is properly contained in $\mathcal{K}(\Omega)$. For the reader's convenience we recall the definition of the Kato class $K_n^{loc}$.

**Definition 4.2.** A Borel measurable function $f$ on $\Omega$ is in the Kato class $K_n^{loc}$ if it satisfies

$$\lim_{r\to 0} \sup_{x\in\Omega} \int_{(|x-y|<r)\cap\Omega} \frac{|f(y)|}{|x-y|^{n-2}} dy = 0.$$

**Proposition 4.4.** *The class $\mathcal{K}(\Omega)$ properly contains the Kato class $K_n^{loc}$.*

*Proof.* Let $f$ be in $K_n^{loc}$. We have

$$\lim_{r\to 0} \sup_{x\in\Omega} \int_{(|x-y|<r)\cap\Omega} \frac{|f(y)|}{|x-y|^{n-2}} dy = 0.$$

Then, there exists $r > 0$ such that

$$\sup_{x\in\Omega} \int_{(|x-y|<r)\cap\Omega} \frac{|f(y)|}{|x-y|^{n-2}} dy \leq 1. \tag{4.10}$$

This yields

$$\sup_{x\in\Omega} \int_{(|x-y|<r)\cap\Omega} |f(y)| dy \leq r^{n-2}.$$

On the other hand, since $\overline{\Omega}$ is a compact subset then there are $x_1, \dots, x_p \in \Omega,\ p \in \mathbb{N}^*$ such that $\overline{\Omega} = \cup_{i=1}^{p} B(x_i, r) \cap \overline{\Omega}$. Hence the last inequality gives

$$\int_\Omega |f(y)|dy \le pr^{n-2}.$$

It follows that

$$\sup_{x\in\Omega} \int_{(|x-y|\ge r)\cap\Omega} \frac{|f(y)|}{|x-y|^{n-2}} dy \le p. \tag{4.11}$$

From (4.10) and (4.11) we obtain

$$\sup_{x\in\Omega} \int_\Omega \frac{|f(y)|}{|x-y|^{n-2}} dy \le p + 1 < +\infty.$$

This means that $f(y)dy$ defines a bounded Newtonian potential and the result holds from Proposition 4.3. □

## 5. THE GREEN FUNCTION FOR $\frac{1}{2}\Delta - \mu$

In this section we prove that when $\mu \in \mathcal{K}(\Omega)$ the Green function $G_\mu$ of the Schrödinger operator $\frac{1}{2}\Delta - \mu$ exists and it is comparable to $G$. We first prove the following result.

**Theorem 5.1.** *There exists a constant $C = C(\Omega, n) > 0$ such that for all $\mu \in \mathcal{K}(\Omega)$ and all nonnegative superharmonic function $h$ on $\Omega$, we have*

$$\int_\Omega G(x,y)h(y)|\mu|(dy) \le C||\mu||h(x),$$

*for all $x \in \Omega$.*

*Proof.* By the $3G$-Theorem, we have

$$\int_\Omega G(x,y)G(y,z)|\mu|(dy) \le 2C||\mu||G(x,z), \tag{5.1}$$

for all $x,\ z \in \Omega$.

Now let $h$ be a nonnegative superharmonic function on $\Omega$; there is an increasing sequence $(h_n)_n$ of nonnegative measurable functions on $\Omega$ such that

$$h(x) = \sup_n \int_\Omega G(x,z)h_n(z)dz,$$

for all $x \in \Omega$.

From (5.1), we have

$$\int_\Omega \int_\Omega G(x,y)G(y,z)|\mu|(dy)h_n(z)dz \le 2C||\mu|| \int_\Omega G(x,z)h_n(z)dz,$$

for all $x \in \Omega$.

By the Fubini's theorem, we obtain

$$\int_\Omega G(x,y) \int_\Omega G(y,z)h_n(z)dz|\mu|(dy) \le 2C||\mu|| \int_\Omega G(x,z)h_n(z)dz,$$

for all $x \in \Omega$.

When $n$ tends to $+\infty$, we obtain

$$\int_\Omega G(x,y)h(y)|\mu|(dy) \le 2C||\mu||h(x),$$

for all $x \in \Omega$. □

**Corollary 5.2.** *Let* $\mu \in \mathcal{K}(\Omega)$. *Then*

$$\sup_{x\in\Omega} \int_\Omega G(x,y)|\mu|(dy) < +\infty.$$

Let $\mu$ be a signed Radon measure in the class $\mathcal{K}(\Omega)$, i.e. $||\mu|| < +\infty$. The Jordan decomposition into positive and negative parts says that $\mu = \mu^+ - \mu^-$ and $|\mu| = \mu^+ + \mu^-$. From Corollary 5.2, the functions

$$x \to \int_\Omega G(x,y)\mu^+(dy) \text{ and } x \to \int_\Omega G(x,y)\mu^-(dy)$$

are two continuous potentials on $\Omega$, and the real continuous function

$$x \to \int_\Omega G(x,y)\mu(dy)$$

corresponds to the difference of these two potentials. Hence from the perturbed theory studied in [2], it follows that there exists a Green function $G_\mu$ for the Schrödinger operator $\frac{1}{2}\Delta - \mu$ on $\Omega$ satisfying the resolvent equation:

$$G(x,y) = G_\mu(x,y) + \int_\Omega G(x,z)G_\mu(z,y)d\mu(z),$$

for all $x,\ y \in \Omega$.

Our main result is the following.

**Theorem 5.3.** *Assume that* $\mu \in \mathcal{K}(\Omega)$ *with* $||\mu||$ *sufficiently small. Then the Green functions* $G$ *and* $G_\mu$ *are comparable, i.e. there is a constant* $C = C(\Omega, n, ||\mu||) > 0$ *such that*

$$C^{-1}G \leq G_\mu \leq CG.$$

*Proof.* We have the resolvent equation:

$$\begin{aligned} G(x,y) &= G_\mu(x,y) + \int_\Omega G(x,z)G_\mu(z,y)d\mu(z) \\ &\equiv G_\mu(x,y) + G * G_\mu(x,y). \end{aligned}$$

Then

$$G_\mu = G - G * G_\mu.$$

By iteration we obtain

$$G_\mu = G + \sum_{m\geq 1} (-1)^m G^{*m+1}, \tag{5.2}$$

where

$$G^{*2}(x,y) \equiv G * G(x,y) = \int_\Omega G(x,z)G(z,y)d\mu(z),$$

and

$$G^{*m+1} = G^{*m} * G.$$

From the $3G$-Theorem, we have

$$\begin{aligned} &\frac{1}{G(x,y)} \int_\Omega G(x,z)G(z,y)|\mu|(dz) \\ &\qquad \leq C\left(\int_\Omega \frac{d(z)}{d(x)}G(x,z)|\mu|(dz) + \int_\Omega \frac{d(z)}{d(y)}G(z,y)|\mu|(dz)\right) \\ &\qquad \leq 2C||\mu||. \end{aligned}$$

In particular, we have

$$|G^{*2}| \leq 2C||\mu||G.$$

By recurrence, we obtain

$$|G^{*m+1}| \leq (2C||\mu||)^m G. \tag{5.3}$$

When $||\mu||$ is sufficiently small so that $2C||\mu|| < \frac{1}{2}$, we obtain, from (5.2) and (5.3),

$$\begin{aligned}|G_\mu - G| &\leq \sum_{m\geq 1}(2C||\mu||)^m G\\ &= \frac{2C||\mu||}{1-2C||\mu||}G,\end{aligned}$$

which yields

$$\left(\frac{1-4C||\mu||}{1-2C||\mu||}\right) G \leq G_\mu \leq \frac{1}{1-2C||\mu||}\, G.$$

□

Recall that when $\mu$ is a nonnegative Radon measure, we know by [8] that the Green function $G_\mu$ of $\frac{1}{2}\Delta - \mu$ exists and satisfies the resolvent equation:

$$G(x,y) = G_\mu(x,y) + \int_\Omega G(x,z)G_\mu(z,y)d\mu(z),$$

for all $x,\ y \in \Omega$.

Next we show that in this case, the condition $\mu \in \mathcal{K}(\Omega)$ is necessary and sufficient for the comparability result.

**Lemma 5.4.** *There exists a constant $C = C(\Omega, n) > 0$ such that*

$$C^{-1}d(x) \leq \int_\Omega G(x,y)dy \leq Cd(x),$$

*for all $x \in \Omega$.*

*Proof.* From Theorem 2.1, we have

$$G(x,y) \leq \frac{C}{|x-y|^{n-2}} \min\left(1, \frac{d(x)d(y)}{|x-y|^2}\right),$$

for all $x,\ y \in \Omega$.

If $d(y) \leq 2|x-y|$, then

$$G(x,y) \leq 2C\frac{d(x)}{|x-y|^{n-1}}. \tag{5.4}$$

If $d(y) \geq 2|x-y|$, then $d(x) \geq d(y) - |x-y| \geq |x-y|$, which implies

$$G(x,y) \leq \frac{C}{|x-y|^{n-2}} \leq C\frac{d(x)}{|x-y|^{n-1}}. \tag{5.5}$$

Combining (5.4) and (5.5), we obtain

$$G(x,y) \leq 2C\frac{d(x)}{|x-y|^{n-1}},$$

for all $x,\ y \in \Omega$.

This yields

$$\begin{aligned}\int_\Omega G(x,y)dy &\leq 2Cd(x)\int_\Omega \frac{1}{|x-y|^{n-1}}dy\\ &\leq 2Cd(x)\int_{0\leq|x-y|\leq d(\Omega)}\frac{1}{|x-y|^{n-1}}dy\\ &= 2Cw_{n-1}d(x)\int_0^{d(\Omega)}dr\\ &= C_1 d(x).\end{aligned}$$

From Theorem 2.1, we also have

$$\frac{C^{-1}}{|x-y|^{n-2}}\min\left(1,\frac{d(x)d(y)}{|x-y|^2}\right)\leq G(x,y),$$

for all $x,\ y\in\Omega$.

This implies

$$C^{-1}\frac{d(x)d(y)}{d(\Omega)^n}\leq G(x,y),$$

for all $x,\ y\in\Omega$.

Hence

$$C^{-1}d(\Omega)^{-n}d(x)\int_\Omega d(y)dy\leq\int_\Omega G(x,y)dy,$$

which means

$$C_2 d(x)\leq\int_\Omega G(x,y)dy,$$

for all $x\in\Omega$. □

**Theorem 5.5.** *Let $\mu$ be a nonnegative Radon measure. Then, the Green function $G_\mu$ of $\frac{1}{2}\Delta-\mu$ on $\Omega$ is comparable to $G$ if and only if $\mu\in\mathcal{K}(\Omega)$.*

*Proof.* We have the integral equation:

$$G(x,y)=G_\mu(x,y)+\int_\Omega G(x,z)G_\mu(z,y)d\mu(z),$$

for all $x,\ y\in\Omega$.

We first assume that $G_\mu$ and $G$ are comparable which means that there exists a constant $C\geq 1$ such that

$$C^{-1}G\leq G_\mu\leq G.$$

Hence

$$\int_\Omega G(x,z)G(z,y)d\mu(z)\leq(C-1)G(x,y),$$

for all $x,\ y\in\Omega$.

This implies

$$\int_\Omega\int_\Omega G(x,z)G(z,y)d\mu(z)dy\leq(C-1)\int_\Omega G(x,y)dy.$$

for all $x\in\Omega$.

Using the Fubini's theorem, it follows that

$$\int_\Omega G(x,z)\int_\Omega G(z,y)dyd\mu(z)\leq(C-1)\int_\Omega G(x,y)dy.$$

for all $x\in\Omega$.

From Lemma 5.4, we deduce that

$$\int_\Omega d(z)G(x,z)d\mu(z) \le C'd(x),$$

for all $x \in \Omega$.

This means that

$$\sup_{x\in\Omega}\int_\Omega \frac{d(z)}{d(x)}G(x,z)d\mu(z) \le C',$$

and then $\mu \in \mathcal{K}(\Omega)$.

Now let $\mu \in \mathcal{K}(\Omega)$, which means $||\mu|| < +\infty$. By Theorem 5.3 the Green function $G_{\frac{\mu}{8C||\mu||}}$ of the Schrödinger operator $\Delta - \frac{\mu}{8C||\mu||}$ is comparable to $G$. This means that there exists $C > 1$ such that

$$C^{-1}G \le G_{\frac{\mu}{8C||\mu||}} \le G.$$

By Theorem 1 in [10], it follows that

$$C^{-8C||\mu||}G \le G_\mu \le G,$$

which ends the proof. □

**Remark 5.6.** In view of the paper [7], our results hold also when we replace the Laplace operator by an elliptic operator

$$L = \sum_{i,j=1}^n a_{i,j}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i \frac{\partial}{\partial x_i}$$

which is uniformly elliptic with bounded Hölder continuous coefficients $a_{i,j},\ b_i$.

**Remark 5.7.** The comparison theorem serves as a main tool to obtain some potential-theoretic results. For example it implies the equivalence of $\left(\frac{1}{2}\Delta - \mu\right)$-potential and $\frac{1}{2}\Delta$-potential of any measure with support contained in $\Omega$ and then the equivalence of $\left(\frac{1}{2}\Delta - \mu\right)$-capacity and $\frac{1}{2}\Delta$-capacity of any set in $\Omega$. These equivalences say that the fine topology, polar sets, etc. are the same for $\frac{1}{2}\Delta$ and $\frac{1}{2}\Delta - \mu$. Following the argument in [7], the comparison theorem also implies the equivalence of $\left(\frac{1}{2}\Delta - \mu\right)$-harmonic measure and $\frac{1}{2}\Delta$-harmonic measure on $\partial\Omega$. This gives rise to a boundary Harnack principle and a comparison theorem for nonnegative $\left(\frac{1}{2}\Delta - \mu\right)$-solutions and nonnegative $\frac{1}{2}\Delta$-solutions vanishing continuously on a part of $\partial\Omega$ (see [4]).

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