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# RELATIVELY B-PSEUDOMONOTONE VARIATIONAL INEQUALITIES OVER PRODUCT OF SETS

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ABSTRACT. In this paper, we consider variational inequality problem over the product of sets which is equivalent to the problem of system of variational inequalities. These two problems are studied for single valued maps as well as for multivalued maps. New concept of pseudomonotonicity in the sense of Brézis is introduced to prove the existence of a solution of our problems. As an application of our results, the existence of a coincidence point of two families of operators is also established.

*Key words and phrases:* Variational inequalities, System of variational inequalities, Relatively B-pseudomonotone maps, Coincidence point.

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## 1. INTRODUCTION

It is mentioned by J.P. Aubin in his book [3] that the Nash equilibrium problem [12, 13] for differentiable functions can be formulated in the form of a variational inequality problem over product of sets (for short, VIPPS). Not only the Nash equilibrium problem but also various equilibrium-type problems, like, traffic equilibrium, spatial equilibrium, and general equilibrium programming problems, from operations research, economics, game theory, mathematical physics and other areas, can also be uniformly modelled as a (VIPPS), see for example [8, 11, 14] and the references therein. Pang [14] decomposed the original variational inequality problem defined on the product of sets into a system of variational inequalities (for short, SVI), which is easy to solve, to establish some solution methods for (VIPPS). Later, it was found that these two problems, (VIPPS) and (SVI), are equivalent. In the recent past (VIPPS) or (SVI) has been considered and studied by many authors, see for example [1, 7, 8, 9, 10, 11] and references therein. Konnov [9] extended the concept of (pseudo) monotonicity and established the

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existence results for a solution of (VIPPS). Recently, Ansari and Yao [2] introduced the system of generalized implicit variational inequalities and proved the existence of its solution. They derived the existence results for a solution of system of generalized variational inequalities and used their results as tools to establish the existence of a solution of system of optimization problems, which include the Nash equilibrium problem as a special case, for nondifferentiable functions.

In this paper, we study (VIPPS) or (SVI) without using arguments from generalized monotonicity as considered in [9]. For this, we introduce the concept of relatively B-pseudomonotonicity which extends in a natural way the well-known pseudomonotonicity in the sense of Brézis [4] (see also [5]). By using a fixed point theorem of Chowdhury and Tan [6], we establish the existence results for a solution of (VIPPS) or (SVI) under relatively B-pseudomonotonicity. As an application of our results, we prove the existence of a coincidence point for two families of nonlinear operators.

We also consider the generalized variational inequality problem over product of sets (for short, GVIPPS) and system of generalized variational inequalities (for short, SGVI). We adopt the technique of Yang and Yao [16] to establish the existence of a solution of (GVIPPS) or (SGVI) by using the existence results for a solution of (VIPPS) or (SVIP).

## 2. FORMULATIONS AND PRELIMINARIES

Let $I$ be a finite index set, that is, $I = \{1, 2, \dots, n\}$. For each $i \in I$, let $X_i$ be a topological vector space with its dual $X_i^*$, $K_i$ a nonempty and convex subset of $X_i$, $K = \prod_{i\in I} K_i$, $X = \prod_{i\in I} X_i$, and $X^* = \prod_{i\in I} X_i^*$. We denote by $\langle\cdot,\cdot\rangle$ the pairing between $X_i^*$ and $X_i$. For each $i \in I$, when $X_i$ is a normed space, its norm is denoted by $\|\cdot\|_i$ and the product norm on $X$ will be denoted by $\|\cdot\|$. For each $x \in X$, we write $x = (x_i)_{i\in I}$, where $x_i \in X_i$, that is, for each $x \in X$, $x_i \in X_i$ denotes the $i$th component of $x$. For each $i \in I$, let $f_i : K \to X_i^*$ be a nonlinear map. We consider the following *variational inequality problem over product of sets* (for short, VIPPS): Find $\bar{x} \in K$ such that

$$\sum_{i\in I} \langle f_i(\bar{x}), y_i - \bar{x}_i\rangle \geq 0, \quad \text{for all } y_i \in K_i,\ i \in I. \tag{2.1}$$

Of course, if we define the mapping $f : K \to X^*$ by

$$f(x) = \big(f_i(x)\big)_{i\in I}, \tag{2.2}$$

then (VIPPS) can be equivalently re-written as the usual variational inequality problem of finding $\bar{x} \in K$ such that

$$\langle f(\bar{x}), y - \bar{x}\rangle \geq 0, \quad \text{for all } y \in K.$$

Very recently, Konnov [9] proved some existence results for a solution of (VIPPS) under relatively pseudomonotonicity or strongly relative pseudomonotonicity assumptions in the setting of Banach spaces.

We also consider the following problem of *system of variational inequalities* which has been studied by many authors because of its applications in various equilibrium-type problems from operations research, economics, game theory, mathematical physics and other areas, see for example, [1, 8, 10, 11, 14] and references therein:

(SVIP) Find $\bar{x} \in X$ such that for each $i \in I, \langle f_i(\bar{x}), y_i - \bar{x}_i\rangle \geq 0,$ for all $y_i \in K_i$.

It is easy to see that these two problems, (VIPPS) and (SVIP), are equivalent. Indeed, that (SVIP) implies (VIPPS) is obvious. The reverse implication holds if we let $y_j = \bar{x}_j$ for all $j \neq i$.

For each $i \in I$, when $f_i$ and $f$ are multivalued maps, then (VIPPS) and (SVIP) are called *generalized variational inequality problem over product sets* and *system of generalized variational inequalities*, respectively. More precisely, for each $i \in I$, let $F_i : K \to 2^{X_i^*}$ be a multivalued map with nonempty values so that if we set

$$F = (F_i : i \in I), \tag{2.3}$$

then $F : K \to 2^{X^*}$ is a multivalued map with nonempty values. We shall also consider the following problems:

(GVIPPS) Find $\bar{x} \in K$ and $\bar{u} \in F(\bar{x})$ such that $\sum_{i \in I} \langle \bar{u}_i, y_i - \bar{x}_i \rangle \geq 0,$ for all $y_i \in K_i,\ i \in I,$

where $u_i$ is the $i$th component of $u$.

(SGVIP) Find $\bar{x} \in K$ and $\bar{u} \in F(\bar{x})$ such that $\langle \bar{u}_i, y_i - \bar{x}_i \rangle \geq 0,$ for all $y_i \in K_i,\ i \in I,$

where $u_i$ is the $i$th component of $u$. As in the single valued case, it is easy to see that (GVIPPS) and (SGVIP) are equivalent.

In [2], Ansari and Yao used (SGVIP) as a tool to prove the existence of a solution of Nash equilibrium problem for nondifferentiable functions.

For every nonempty set $A$, we denote by $2^A$ (respectively, $\mathcal{F}(A)$) the family of all subsets (respectively, finite subsets) of $A$. If $A$ is a nonempty subset of a vector space, then $coA$ denotes the convex hull of $A$.

The following result of Chowdhury and Tan [6] will be used to establish the main result of this paper.

**Theorem 2.1.** *Let $K$ be a nonempty and convex subset of a topological vector space (not necessarily Hausdorff) $X$ and $T : K \to 2^K$ a multivalued map. Assume that the following conditions hold:*

(i) *For all $x \in K$, $T(x)$ is convex.*
(ii) *For each $A \in \mathcal{F}(K)$ and for all $y \in coA$, $T^{-1}(y) \cap coA$ is open in $coA$.*
(iii) *For each $A \in \mathcal{F}(K)$ and all $x, y \in coA$ and every net $\{x_\alpha\}_{\alpha \in \Gamma}$ in $K$ converging to $x$ such that $ty + (1-t)x \notin T(x_\alpha)$ for all $\alpha \in \Gamma$ and for all $t \in [0,1]$, we have $y \notin T(x)$.*
(iv) *There exist a nonempty, closed and compact subset $D$ of $K$ and an element $\tilde{y} \in D$ such that $\tilde{y} \in T(x)$ for all $x \in K \setminus D$.*
(v) *For all $x \in D$, $T(x)$ is nonempty.*

*Then there exists $\hat{x} \in K$ such that $\hat{x} \in T(\hat{x})$.*

## 3. EXISTENCE RESULTS FOR (VIPPS) AND (SVIP)

**Definition 3.1.** The map $f : K \to X^*$, defined by (2.2), is said to be *relatively B-pseudomonotone* (respectively, *relatively demimonotone*) if for each $x \in K$ and every net $\{x^\alpha\}_{\alpha \in \Gamma}$ in $K$ converging to $x$ (respectively, weakly to $x$) with

$$\liminf_\alpha \left[ \sum_{i \in I} \langle f_i(x^\alpha), x_i - x_i^\alpha \rangle \right] \geq 0$$

we have

$$\sum_{i \in I} \langle f_i(x), y_i - x_i \rangle \geq \limsup_\alpha \left[ \sum_{i \in I} \langle f_i(x^\alpha), y_i - x_i^\alpha \rangle \right] \quad \text{for all } y \in K.$$

Of course, if $I$ is a singleton set, then the above definition reduces to the definition of a pseudomonotone map, introduced by Brézis [4] (see also [5] and [3, p. 410]).

Now we are ready to establish the main result of this paper on the existence of a solution of (VIPPS) under relatively B-pseudomonotonicity assumption.

**Theorem 3.1.** *For each $i \in I$, let $K_i$ be a nonempty and convex subset of a real topological vector space (not necessarily, Hausdorff) $X_i$. Let $f$, defined by (2.2), be relatively B-pseudomonotone such that for each $A \in \mathcal{F}(K)$, $x \mapsto \sum_{i\in I}\langle f_i(x), y_i - x_i\rangle$ is upper semicontinuous on $coA$. Assume that there exists a nonempty, closed and compact subset $D$ of $K$ and an element $\tilde{y} \in D$ such that for all $x \in K \setminus D$, $\sum_{i\in I}\langle f_i(x), \tilde{y}_i - x_i\rangle < 0$. Then (VIPPS) has a solution.*

*Proof.* For each $x \in K$, define a multivalued map $T : K \to 2^K$ by

$$T(x) = \left\{ y \in K : \sum_{i\in I}\langle f_i(x), y_i - x_i\rangle < 0 \right\}.$$

Then for all $x \in K$, $T(x)$ is convex. Let $A \in \mathcal{F}(K)$, then for all $y \in coA$,

$$[T^{-1}(y)]^c \cap coA = \left\{ x \in coA : \sum_{i\in I}\langle f_i(x), y_i - x_i\rangle \geq 0 \right\}$$

is closed in $coA$ by upper semicontinuity of the map $x \mapsto \sum_{i\in I}\langle f_i(x), y_i - x_i\rangle$ on $coA$. Hence $T^{-1}(y) \cap coA$ is open in $coA$.

Suppose that $x, y \in coA$ and $\{x^\alpha\}_{\alpha\in\Gamma}$ is a net in $K$ converging to $x$ such that

$$\sum_{i\in I}\langle f_i(x^\alpha), (ty_i + (1-t)x_i) - x_i^\alpha\rangle \geq 0, \quad \text{for all } \alpha \in \Gamma \text{ and all } t \in [0,1].$$

For $t = 0$, we have

$$\sum_{i\in I}\langle f_i(x^\alpha), x_i - x_i^\alpha\rangle \geq 0, \quad \text{for all } \alpha \in \Gamma,$$

and therefore

$$\liminf_\alpha \left[ \sum_{i\in I}\langle f_i(x^\alpha), x_i - x_i^\alpha\rangle \right] \geq 0.$$

By the relatively B-pseudomonotonicity of $f$, we have

$$\sum_{i\in I}\langle f_i(x), y_i - x_i\rangle \geq \limsup_\alpha \left[ \sum_{i\in I}\langle f_i(x^\alpha), y_i - x_i^\alpha\rangle \right]. \tag{3.1}$$

For $t = 1$, we have

$$\sum_{i\in I}\langle f_i(x^\alpha), y_i - x_i^\alpha\rangle \geq 0, \quad \text{for all } \alpha \in \Gamma,$$

and therefore,

$$\liminf_{\alpha\in\Gamma} \left[ \sum_{i\in I}\langle f_i(x^\alpha), y_i - x_i^\alpha\rangle \right] \geq 0. \tag{3.2}$$

From (3.1) and (3.2), we obtain

$$\sum_{i\in I}\langle f_i(x), y_i - x_i\rangle \geq 0,$$

and thus $y \notin T(x)$.

Assume that for all $x \in D$, $T(x)$ is nonempty. Then all the conditions of Theorem 2.1 are satisfied. Hence there exists $\hat{x} \in K$ such that $\hat{x} \in T(\hat{x})$, that is,

$$0 = \sum_{i \in I} \langle f_i(\hat{x}), \hat{x}_i - \hat{x}_i \rangle < 0,$$

a contradiction. Thus there exists $\bar{x} \in K$ such that $T(\bar{x}) = \emptyset$, that is,

$$\sum_{i \in I} \langle f_i(\bar{x}), y_i - \bar{x}_i \rangle \geq 0, \quad \text{for all } y_i \in K_i \; i \in I.$$

Hence $\bar{x}$ is a solution of (VIPPS). □

**Corollary 3.2.** *For each $i \in I$, let $K_i$ be a nonempty, closed and convex subset of a real reflexive Banach space $X_i$. Let $f$, defined by (2.2), be relatively demimonotone such that for each $A \in \mathcal{F}(K)$, $x \mapsto \sum_{i \in I} \langle f_i(x), y_i - x_i \rangle$ is upper semicontinuous on $\mathrm{co}A$. Assume that there exists $\tilde{y} \in K$ such that*

$$\lim_{||x|| \to \infty, \; x \in K} \sum_{i \in I} \langle f_i(x), \tilde{y}_i - x_i \rangle < 0. \tag{3.3}$$

*Then (VIPPS) has a solution.*

*Proof.* Let $\alpha = \lim_{||x|| \to \infty, \; x \in K} \sum_{i \in I} \langle f_i(x), \tilde{y}_i - x_i \rangle < 0$. Then by (3.3), $\alpha < 0$. Let $r > 0$ be such that $||\tilde{y}|| \leq r$ and $\sum_{i \in I} \langle f_i(x), \tilde{y}_i - x_i \rangle < \frac{\alpha}{2}$ for all $x \in K$ with $||x|| > r$. For each $i \in I$, let $K_i^r = \{x_i \in K_i : ||x_i||_i \leq r\}$, and we denote by $K^r = \prod_{i \in I} K_i^r$. Then $K^r$ is a nonempty and weakly compact subset of $K$. Note that for any $x \in K \setminus K^r$, $\sum_{i \in I} \langle f_i(x), \tilde{y}_i - x_i \rangle < \frac{\alpha}{2} < 0$, and the conclusion follows from Theorem 3.1. □

As an application of Corollary 3.2, we establish the existence of a coincidence point for two families of nonlinear operators.

**Corollary 3.3.** *For each $i \in I$, let $X_i$ be a real reflexive Banach space. Let $f, g : X \to X_i^*$ be defined as $f(x) = (f_i(x))_{i \in I}$ and $g(x) = (g_i(x))_{i \in I}$, respectively, for all $x \in X$, where for each $i \in I$, $g_i : X \to X_i^*$ is a nonlinear operator. Assume that $(f - g)$ is relatively demimonotone and for each $A \in \mathcal{F}(X)$, $x \mapsto \sum_{i \in I} \langle f_i(x), y_i - x_i \rangle$ is upper semicontinuous on $\mathrm{co}A$. Further, assume that there exists $\tilde{y} \in X$ such that*

$$\lim_{||x|| \to \infty, \; x \in X} \sum_{i \in I} \langle (f_i - g_i)(x), \tilde{y}_i - x_i \rangle < 0.$$

*Then there exists $\bar{x} \in X$ such that $f_i(\bar{x}) = g_i(\bar{x})$ for each $i \in I$.*

*Proof.* From the Corollary 3.2, there exists $\bar{x} \in X$ such that for each $i \in I$,

$$\langle f_i(\bar{x}), y_i - \bar{x}_i \rangle \geq \langle g_i(\bar{x}), y_i - \bar{x}_i \rangle, \quad \text{for all } y_i \in X_i.$$

Therefore we have, $f_i(\bar{x}) = g_i(\bar{x})$ for each $i \in I$. □

Finally, we give another application of Corollary 3.2 in the setting of Hilbert spaces.

**Corollary 3.4.** *For each $i \in I$, let $(X_i, \langle \cdot, \cdot \rangle)$ be a real Hilbert space and $K_i$ a nonempty, closed and convex subset of $X_i$. Let $f$, defined by (2.2), be relatively demimonotone such that for each $A \in \mathcal{F}(K)$, $x \mapsto \sum_{i \in I} \langle f_i(x), y_i - x_i \rangle$ is lower semicontinuous on $\mathrm{co}A$. Assume that there exists $\tilde{y} \in K$ such that*

$$\lim_{||x|| \to \infty, \; x \in K} \sum_{i \in I} \langle x_i - f_i(x), \tilde{y}_i - x_i \rangle < 0.$$

*Then there exists $\bar{x} \in K$ such that for each $i \in I$, $f_i(\bar{x}) = \bar{x}_i$.*

*Proof.* For each $i \in I$, define a nonlinear operator $S_i : K \to X_i$ by $S_i(x) = x_i - f_i(x)$ for all $x \in K$. Then obviously, for each $i \in I$, $S_i$ satisfies all the conditions of Corollary 3.2. Hence there exists $\bar{x} \in K$ such that for each $i \in I$, $\langle S_i(\bar{x}), y_i - \bar{x}_i \rangle \geq 0$ for all $y_i \in K_i$. For each $i \in I$, let $y_i = f_i(\bar{x})$, we have $||\bar{x}_i - f_i(\bar{x})|| \leq 0$. Therefore, for each $i \in I$, $f_i(\bar{x}) = \bar{x}_i$. □

## 4. EXISTENCE RESULTS FOR (GVIPPS) AND (SGVIP)

In this section, we adopt the technique of Yang and Yao [16] to derive the existence results for a solution of (GVIPPS) and (SGVIP) by using the results of Section 3.

**Definition 4.1.** The multivalued map $F : K \to 2^{X^*}$, defined by (2.3), is said to be *relatively B-pseudomonotone* (respectively, *relatively demimonotone*) if for each $x \in K$ and every net $\{x^\alpha\}_{\alpha \in \Gamma}$ in $K$ converging to $x$ (respectively, weakly to $x$) with

$$\liminf_\alpha \left[ \sum_{i \in I} \langle u_i^\alpha, x_i - x_i^\alpha \rangle \right] \geq 0, \quad \text{for all } u_i^\alpha \in F_i(x^\alpha)$$

we have, for all $u_i \in F_i(x)$

$$\sum_{i \in I} \langle u_i, y_i - x_i \rangle \geq \limsup_\alpha \left[ \sum_{i \in I} \langle u_i^\alpha, y_i - x_i^\alpha \rangle \right] \quad \text{for all } u_i^\alpha \in F_i(x^\alpha) \text{ and } y \in K.$$

**Definition 4.2.** Let $Z$ be topological vector space and $U$ a subset of $Z$. Let $G : U \to 2^{Z^*}$ be a multivalued map and $g : U \to Z^*$ a single valued map. $g$ is called a *selection* of $G$ on $U$ if $g(x) \in G(x)$ for all $x \in U$. Furthermore, the function $g$ is called a *continuous selection* of $G$ on $U$ if it is continuous on $U$ and a selection of $G$ on $U$.

For further details on continuous selections of multivalued maps, we refer to [15].

It follows from (2.2) and (2.3) that if $f : K \to X^*$, defined by (2.2), is a selection of $F : K \to 2^{X^*}$, defined by (2.3), then for each $i \in I$, $f_i : K \to X_i^*$ is a selection of $F_i : K \to 2^{X_i^*}$ on $K$.

**Lemma 4.1.** *If $f : K \to X^*$, defined by (2.2), is a selection of $F : K \to 2^{X^*}$, defined by (2.3), on $K$, then every solution of (VIPPS) is a solution of (GVIPPS).*

*Proof.* Assume that $\bar{x} \in K$ is a solution of (VIPPS). Then

$$\sum_{i \in I} \langle f_i(\bar{x}), y_i - \bar{x}_i \rangle \geq 0, \quad \text{for all } y_i \in K_i,\ i \in I.$$

Let $\bar{u}_i = f_i(\bar{x})$, so that $\bar{u} = f(\bar{x})$. Since $f$ is a selection of $F$, we have $\bar{u} \in F(\bar{x})$ such that

$$\sum_{i \in I} \langle \bar{u}_i, y_i - \bar{x}_i \rangle \geq 0, \quad \text{for all } y_i \in K_i,\ i \in I.$$

Hence $(\bar{x}, \bar{u})$ is a solution of (GVIPPS). □

**Lemma 4.2.** *Let $f : K \to X^*$, defined by (2.2), be a selection of a multivalued map $F : K \to 2^{X^*}$, defined by (2.3), on $K$. If $F$ is relatively B-pseudomonotone (respectively, relatively demimonotone), then $f$ is also relatively B-pseudomonotone (respectively, relatively demimonotone).*

*Proof.* Let $u_i = f_i(x)$ and $u_i^\alpha = f_i(x^\alpha)$ for all $x \in K$, so that $u = f(x)$ and $u^\alpha = f(x^\alpha)$. Then the result follows from the definitions. □

In the remaining part of the paper, we shall assume that the pairing $\langle \cdot, \cdot \rangle$ is continuous.

**Theorem 4.3.** *For each $i \in I$, let $K_i$ be a nonempty and convex subset of a real topological vector space (not necessarily, Hausdorff) $X_i$. Assume that*

(i) *$F : K \to 2^{X^*}$, defined by (2.3), is a relatively B-pseudomonotone multivalued map;*

(ii) *there exists a continuous selection* $f : K \to X^*$, *defined by (2.2), of* $F$ *on* $K$;

(iii) *there exist a nonempty, closed and compact subset* $D$ *of* $K$ *and an element* $\tilde{y} \in D$ *such that for all* $x \in K \setminus D$, $\sum_{i\in I}\langle f_i(x), \tilde{y}_i - x_i\rangle < 0$.

*Then (GVIPPS) has a solution.*

*Proof.* From condition (ii), there exists a continuous functions $f : K \to X^*$ such that $f(x) \in F(x)$, for all $x \in K$. Since the pairing $\langle\cdot,\cdot\rangle$ is continuous, we have the map $x \mapsto \sum_{i\in I}\langle f_i(x), y_i - x_i\rangle$ is continuous on $K$. By Lemma 4.2, $f$ is relatively B-pseudomonotone. Therefore by Theorem 3.1, there exists a solution $\bar{x} \in K$ of (VIPPS). For each $i \in I$, let $\bar{u}_i = f_i(\bar{x}) \in F_i(\bar{x})$. Then by Lemma 4.1, $(\bar{x}, \bar{u})$ is a solution of (GVIPPS). □

**Corollary 4.4.** *For each* $i \in I$, *let* $K_i$ *be a nonempty and convex subset of a real reflexive Banach space* $X_i$. *Assume that*

(i) $F : K \to 2^{X^*}$, *defined by (2.3), is a relatively demimonotone multivalued map;*

(ii) *there exists a continuous selection* $f : K \to X^*$, *defined by (2.2), of* $F$ *on* $K$;

(iii) *there exists* $\tilde{y} \in K$ *such that*

$$\lim_{||x||\to\infty,\ x\in K} \sum_{i\in I}\langle f_i(x), \tilde{y}_i - x_i\rangle < 0.$$

*Then (GVIPPS) has a solution.*

*Proof.* It follows by using the argument of Theorem 4.3 and Corollary 3.2. □

## REFERENCES


[1] Q.H. ANSARI AND J.-C. YAO, A fixed point theorem and its applications to the system of variational inequalities, *Bull. Austral. Math. Soc.*, **59** (1999), 433–442.

[2] Q.H. ANSARI AND J.-C. YAO, System of generalized variational inequalities and their applications, *Appl. Anal.*, **76** (3-4) (2000), 203–217.

[3] J.P. AUBIN, *Mathematical Methods of Game Theory and Economic*, North-Holland, Amsterdam, 1982.

[4] H. BRÉZIS, Equations et inequations non linéaires dans les espaces vectoriels en dualité, *Ann. Inst. Fourier (Grenoble)*, **18** (1968), 115–175.

[5] H. BRÉZIS, L. NIRENBERG AND G. STAMPACCHIA, A remark on Ky Fan's minimax principle, *Boll. Un. Math. Ital.*, **6**(4) (1972), 293–300.

[6] M.S.R. CHOWDHURY AND K.-K. TAN, Generalization of Ky Fan minimax inequality with applications to generalized variational inequalities for pseudo-monotone operators and fixed point theorems, *J. Math. Anal. Appl.*, **204** (1996), 910–926.

[7] C. COHEN AND F. CHAPLAIS, Nested monotony for variational inequalities over product of spaces and convergence of iterative algorithms, *J. Optim. Theory Appl.*, **59** (1988), 360–390.

[8] M. FERRIS AND J.-S. PANG, Engineering and economic applications of complementarity problems, *SIAM Rev.*, **39** (1997), 669–713.

[9] I.V. KONNOV, Relatively monotone variational inequalities over product sets, *Oper. Res. Lett.*, **28** (2001), 21–26.

[10] S. MAKLER-SCHEIMBERG, V.H. NGUYEN AND J.J. STRODIOT, Family of perturbation methods for variational inequalities, *J. Optim. Theory Appl.*, **89** (1996), 423–452.

[11] A. NAGURNEY, *Network Economics: A Variational Inequality Approach*, Kluwer Academic Publishers, Dordrecht-Boston-London, 1993.

[12] J.F. NASH, Equilibrium point in $n$-person games, *Proc. Nat. Acad. Sci. USA*, **36** (1950), 48–49.

[13] J.F. NASH, Noncooperative games, *Ann. Math.*, **54** (1951), 286–295.

[14] J.-S. PANG, Asymmetric variational inequalities over product of sets: Applications and iterative methods, *Math. Prog.*, **31** (1985), 206–219.

[15] D. REPOVS AND P.V. SEMENOV, *Continuous Selections of Multivalued Mappings*, Kluwer Academic Publishers, Dordrecht/Boston/London, (1998).

[16] X. Q. YANG AND J.C. YAO, Gap functions and existence of set-valued vector variational inequalities, *J. Optim. Theory Appl.*, (2002) (to appear).