Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 4, Issue 3, Article 58, 2003

AN INTEGRAL APPROXIMATION IN THREE VARIABLES

A. SOFO

SCHOOL OF COMPUTER SCIENCE AND MATHEMATICS
VICTORIA UNIVERSITY OF TECHNOLOGY
PO BOX 14428, MCMC 8001,
VICTORIA, AUSTRALIA.
sofo@csm.vu.edu.au
URL: http://rgmia.vu.edu.au/sofo

Received 15 November, 2002; accepted 25 August, 2003
Communicated by C.E.M. Pearce

ABSTRACT. In this paper we will investigate a method of approximating an integral in three independent variables. The Ostrowski type inequality is established by the use of Peano kernels and provides a generalisation of a result given by Pachpatte.

Key words and phrases: Ostrowski inequality, Three independent variables, Partial derivatives.

2000 *Mathematics Subject Classification.* Primary 26D15; Secondary 41A55.

1. INTRODUCTION

The numerical estimation of the integral, or multiple integral of a function over some specified interval is important in many scientific applications. Generally speaking, the error bound for the midpoint rule is about one half of the trapezoidal rule and Stewart [14] has a nice geometrical explanation of this generality. The speed of convergence of an integral is also important and Weideman [15] has some pertinent examples illustrating perfect, algebraic, geometric, super-geometric and sub-geometric convergence for periodic functions.

In particular, we shall establish an Ostrowski type inequality for a triple integral which provides a generalisation or extension of a result given by Pachpatte [10].

In 1938 Ostrowski [7] obtained a bound for the absolute value of the difference of a function to its average over a finite interval. The following definitions will be used in this exposition

$$\mathcal{M}\left(f\right) := \frac{1}{b-a}\int_a^b f\left(t\right)dt, \tag{1.1}$$

$$I_T\left(f\right) := \frac{f\left(b\right)+f\left(a\right)}{2} \tag{1.2}$$

ISSN (electronic): 1443-5756

125-02

and

$$I_M(f) := f\left(\frac{a+b}{2}\right). \tag{1.3}$$

The Ostrowski result is given by:

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), that is,*

$$\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty.$$

Then we have the inequality

$$|f(x) - \mathcal{M}(f)| \le \left(\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right)(b-a)\|f'\|_\infty \tag{1.4}$$

for all $x \in [a,b]$.

The constant $\frac{1}{4}$ is the best possible.

Improvements of the result (1.4) has also been obtained by Dedić, Matić and Pearce [2], Pearce, Pečarić, Ujević and Varošanec [11], Dragomir [3] and Sofo [12]. For a symmetrical point $x \in \left[a, \frac{a+b}{2}\right]$, very recently Guessab and Schmeisser [4] studied the more general quadrature formula

$$\mathcal{M}(f) - \left[\frac{f(x) + f(a+b-x)}{2}\right] = E(f;x)$$

where $E(f;x)$ is the remainder.

For $x = \frac{a+b}{2}$ and f defined on $[a,b]$ with Lipschitz constant M, then

$$|\mathcal{M}(f) - I_M(f)| \le \frac{M(b-a)}{4}.$$

For $x = a$, then

$$|\mathcal{M}(f) - I_T(f)| \le \frac{M(b-a)}{4}.$$

The following result, which is a generalisation of Theorem 1.1, was given by Milovanović [6, p. 468] in 1975 concerning a function, f, of several variables.

Theorem 1.2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function defined on $D = \{(x_1,\dots,x_m) \,|\, a_i \le x_i \le b_i,\ (i = 1,\dots,m)\}$ and let $\left|\frac{\partial f}{\partial x_i}\right| \le M_i$ $(M_i > 0,\ i = 1,\dots,m)$ in D. Furthermore, let $x \mapsto p(x)$ be integrable and $p(x) > 0$ for every $x \in D$. Then for every $x \in D$, we have the inequality:*

$$\left|f(x) - \frac{\int_D p(y) f(y)\,dy}{\int_D p(y)\,dy}\right| \le \frac{\sum_{i=1}^m M_i \int_D p(y)|x_i - y_i|\,dy}{\int_D p(y)\,dy}. \tag{1.5}$$

In 2001, Barnett and Dragomir [1] obtained the following Ostrowski type inequality for double integrals.

Theorem 1.3. *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be continuous on $[a,b] \times [c,d]$, $f''_{x,y} = \frac{\partial^2 f}{\partial x \partial y}$ exist on $(a,b) \times (c,d)$ and is bounded, that is,*

$$\left\|f''_{s,t}\right\|_\infty := \sup_{(x,y)\in(a,b)\times(c,d)} \left|\frac{\partial^2 f(x,y)}{\partial x \partial y}\right| < \infty,$$

then we have the inequality:

$$\left|\int_a^b \int_c^d f(s,t)\,dsdt - (b-a)\int_c^d f(x,t)\,dt - (d-c)\int_a^b f(s,y)\,ds + (d-c)(b-a)f(x,y)\right| \leq \left[\frac{(b-a)^2}{4} + \left(x-\frac{a+b}{2}\right)^2\right]\left[\frac{(d-c)^2}{4} + \left(y-\frac{c+d}{2}\right)^2\right]\left\|f''_{s,t}\right\|_\infty \tag{1.6}$$

for all $(x,y) \in [a,b]\times[c,d]$.

Pachpatte [8], obtained an inequality in the vein of (1.6) but used elementary analysis in his proof.

Pachpatte [9] also obtains a discrete version of an inequality with two independent variables. Hanna, Dragomir and Cerone [5] obtained a further complementary result to (1.6) and Sofo [13] further improved the result (1.6).

2. TRIPLE INTEGRALS

In three independent variables Pachpatte obtains several results. For discrete variables he obtains a result in [9] and in [10] for continuous variables he obtained the following.

Theorem 2.1. *Let* $\Delta := [a,k]\times[b,m]\times[c,n]$ *for* $a,b,c,k,m,n \in \mathbb{R}^+$ *and* $f(r,s,t)$ *be differentiable on* Δ. *Denote the partial derivatives by* $D_1 f(r,s,t) = \frac{\partial}{\partial r} f(r,s,t)$; $D_2 f(r,s,t) = \frac{\partial}{\partial s}$, $D_3 f(r,s,t) = \frac{\partial}{\partial t}$ *and* $D_3D_2D_1 f = \frac{\partial^3 f}{\partial t \partial s \partial r}$. *Let* $F(\Delta)$ *be the clan of continuous functions* $f : \Delta \to \mathbb{R}$ *for which* $D_1 f$, $D_2 f$, $D_3 f$, $D_3D_2D_1 f$ *exist and are continuous on* Δ. *For* $f \in F(\Delta)$ *we have*

$$\begin{aligned}
&\Bigg|\int_a^k \int_b^m \int_c^n f(r,s,t)\,dtdsdr \\
&- \frac{1}{8}(k-a)(m-b)(n-c)\left[f(a,b,c)+f(k,m,n)\right] \\
&+ \frac{1}{4}(m-b)(n-c)\int_a^k \left[f(r,b,c)+f(r,m,n)+f(r,m,c)+f(r,b,n)\right]dr \\
&+ \frac{1}{4}(k-a)(n-c)\int_b^m \left[f(a,s,c)+f(k,s,n)+f(a,s,n)+f(k,s,c)\right]ds \\
&+ \frac{1}{4}(k-a)(m-b)\int_c^n \left[f(a,b,t)+f(k,m,t)+f(k,b,t)+f(a,m,t)\right]dt \\
&- \frac{1}{2}(k-a)\int_b^m \int_c^n \left[f(a,s,t)+f(k,s,t)\right]dtds \\
&- \frac{1}{2}(m-b)\int_a^k \int_c^n \left[f(r,b,t)+f(r,m,t)\right]dtdr \\
&- \frac{1}{2}(n-c)\int_a^k \int_b^m \left[f(r,s,c)+f(r,s,n)\right]dsdr\Bigg| \\
&\leq \int_a^k \int_b^m \int_c^n \left|D_3D_2D_1 f(r,s,t)\right|dtdsdr.
\end{aligned} \tag{2.1}$$

The following theorem establishes an Ostrowski type identity for an integral in three independent variables.

Theorem 2.2. *Let* $f : [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \to \mathbb{R}$ *be a continuous mapping such that the following partial derivatives* $\frac{\partial^{i+j+k} f(\cdot,\cdot,\cdot)}{\partial x^i \partial y^j \partial z^k}$; $i = 0, \ldots, n-1$, $j = 0, \ldots, m-1$; $k = 0, \ldots, p-1$ *exist and are continuous on* $[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ *. Also, let*

$$P_n(x,r) := \begin{cases} \dfrac{(r-a_1)^n}{n!}; & r \in [a_1, x), \\ \dfrac{(r-b_1)^n}{n!}; & r \in [x, b_1], \end{cases} \tag{2.2}$$

$$Q_m(y,s) := \begin{cases} \dfrac{(s-a_2)^m}{m!}; & s \in [a_2, y), \\ \dfrac{(s-b_2)^m}{m!}; & s \in [y, b_2], \end{cases} \tag{2.3}$$

and

$$S_p(z,t) := \begin{cases} \dfrac{(t-a_3)^p}{p!}; & t \in [a_3, z), \\ \dfrac{(t-b_3)^p}{p!}; & s \in [z, b_3], \end{cases} \tag{2.4}$$

then for all $(x,y,z) \in [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ *we have the identity*

$$\begin{aligned} V &:= \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(r,s,t)\,dtdsdr \\ &\quad - \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\sum_{k=0}^{p-1} X_i(x)\,Y_j(y)\,Z_k(z)\,\frac{\partial^{i+j+k} f(x,y,z)}{\partial x^i \partial y^j \partial z^k} \\ &\quad + (-1)^p \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} X_i(x)\,Y_j(y)\int_{a_3}^{b_3} S_p(z,t)\,\frac{\partial^{i+j+p} f(x,y,t)}{\partial x^i \partial y^j \partial t^p}dt \\ &\quad + (-1)^m \sum_{i=0}^{n-1}\sum_{k=0}^{p-1} X_i(x)\,Z_k(z)\int_{a_2}^{b_2} Q_m(y,s)\,\frac{\partial^{i+m+k} f(x,s,z)}{\partial x^i \partial s^m \partial z^k}ds \\ &\quad + (-1)^n \sum_{j=0}^{m-1}\sum_{k=0}^{p-1} Y_j(y)\,Z_k(z)\int_{a_1}^{b_1} P_n(x,r)\,\frac{\partial^{n+j+k} f(r,y,z)}{\partial r^n \partial y^j \partial z^k}dr \\ &\quad - (-1)^{m+p} \sum_{i=0}^{n-1} X_i(x)\int_{a_2}^{b_2}\int_{a_3}^{b_3} Q_m(y,s)\,S_p(z,t)\,\frac{\partial^{i+m+p} f(x,s,t)}{\partial x^i \partial s^m \partial t^p}dtds \\ &\quad - (-1)^{n+p} \sum_{j=0}^{m-1} Y_j(y)\int_{a_1}^{b_1}\int_{a_3}^{b_3} P_n(x,r)\,S_p(z,t)\,\frac{\partial^{n+j+p} f(r,y,t)}{\partial r^n \partial y^j \partial t^p}dtdr \\ &\quad - (-1)^{n+m} \sum_{k=0}^{p-1} Z_k(z)\int_{a_1}^{b_1}\int_{a_2}^{b_2} P_n(x,r)\,Q_m(y,s)\,\frac{\partial^{n+m+k} f(r,s,z)}{\partial r^n \partial s^m \partial z^k}dsdr \\ &= -(-1)^{n+m+p}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} P_n(x,r)\,Q_m(y,s)\,S_p(z,t)\,\frac{\partial^{n+m+p} f(r,s,t)}{\partial r^n \partial s^m \partial t^p}dtdsdr, \end{aligned} \tag{2.5}$$

where

$$X_i(x) := \frac{(b_1 - x)^{i+1} + (-1)^i (x - a_1)^{i+1}}{(i+1)!}, \tag{2.6}$$

$$Y_j(y) := \frac{(b_2 - y)^{j+1} + (-1)^j (y - a_2)^{j+1}}{(j+1)!}, \tag{2.7}$$

and

$$Z_k(z) := \frac{(b_3 - z)^{k+1} + (-1)^k (z - a_3)^{k+1}}{(k+1)!}. \tag{2.8}$$

Proof. We have an identity, see [5]

$$\int_{a_1}^{b_1} g(r)\,dr = \sum_{i=0}^{n-1} X_i(x)\, g^{(i)}(x) + (-1)^n \int_{a_1}^{b_1} P_n(x,r)\, g^{(n)}(r)\,dr. \tag{2.9}$$

Now for the partial mapping $f(\cdot, s, t)$, $s \in [a_2, b_2]$, we have

$$\int_{a_1}^{b_1} f(r,s,t)\,dr = \sum_{i=0}^{n-1} X_i(x) \frac{\partial^i f}{\partial x^i} + (-1)^n \int_{a_1}^{b_1} P_n(x,r) \frac{\partial^n f}{\partial r^n} dr \tag{2.10}$$

for every $r \in [a_1, b_1]$, $s \in [a_2, b_2]$ and $t \in [a_3, b_3]$.

Now integrate over $s \in [a_2, b_2]$

$$\begin{aligned} \int_{a_1}^{b_1} \int_{a_2}^{b_2} & f(r,s,t)\,dsdr \\ &= \sum_{i=0}^{n-1} X_i(x) \int_{a_2}^{b_2} \frac{\partial^i f}{\partial x^i} ds + (-1)^n \int_{a_1}^{b_1} P_n(x,r) \left(\int_{a_2}^{b_2} \frac{\partial^n f}{\partial r^n} ds \right) dt \end{aligned} \tag{2.11}$$

for all $x \in [a_1, b_1]$.

From (2.9) for the partial mapping $\frac{\partial^i f}{\partial x^i}$ on $[a_2, b_2]$ we have,

$$\begin{aligned} \int_{a_2}^{b_2} & \frac{\partial^i}{\partial x^i} f(x,s,t)\,ds \\ &= \sum_{j=0}^{m-1} Y_j(y) \frac{\partial^j}{\partial y^j} \left(\frac{\partial^i f}{\partial x^i} \right) + (-1)^m \int_{a_2}^{b_2} Q_m(y,s) \frac{\partial^m}{\partial s^m} \left(\frac{\partial^i f}{\partial x^i} \right) ds \\ &= \sum_{j=0}^{m-1} Y_j(y) \frac{\partial^{i+j} f}{\partial x^i \partial y^j} + (-1)^m \int_{a_2}^{b_2} Q_m(y,s) \frac{\partial^{i+m} f}{\partial x^i \partial s^m} ds. \end{aligned} \tag{2.12}$$

Also, from (2.8)

$$\int_{a_2}^{b_2} \frac{\partial^n f}{\partial r^n} ds = \sum_{j=0}^{m-1} Y_j(y) \frac{\partial^{j+n} f}{\partial y^j \partial r^n} + (-1)^m \int_{a_2}^{b_2} Q_m(y,s) \frac{\partial^m}{\partial s^m} \left(\frac{\partial^n f}{\partial r^n} \right) ds. \tag{2.13}$$

From (2.11) substitute (2.12) and (2.13), so that

$$\begin{aligned}
(2.14)\quad & \int_{a_1}^{b_1}\int_{a_2}^{b_2} f(r,s,t)\,dsdr \\
&= \sum_{i=0}^{n-1} X_i(x)\left[\sum_{j=0}^{m-1} Y_j(y)\frac{\partial^{i+j} f}{\partial x^i \partial y^j} + (-1)^m \int_{a_2}^{b_2} Q_m(y,s)\frac{\partial^{i+m} f}{\partial x^i \partial s^m} ds\right] \\
&\quad + (-1)^n \int_{a_1}^{b_1} P_n(x,r)\left[\sum_{j=0}^{m-1} Y_j(y)\frac{\partial^{j+n} f}{\partial y^j \partial r^n}\right. \\
&\qquad \left. + (-1)^m \int_{a_2}^{b_2} Q_m(y,s)\frac{\partial^m}{\partial s^m}\left(\frac{\partial^n f}{\partial r^n}\right) ds\right] dt \\
&= \sum_{i=0}^{n-1} X_i(x)\sum_{j=0}^{m-1} Y_j(y)\frac{\partial^{i+j} f}{\partial x^i \partial y^j} + (-1)^m \sum_{i=0}^{n-1} X_i(x)\int_{a_2}^{b_2} Q_m(y,s)\frac{\partial^{i+m} f}{\partial x^i \partial s^m} ds \\
&\quad + (-1)^n \sum_{j=0}^{m-1} Y_j(y)\int_{a_1}^{b_1} P_n(x,r)\frac{\partial^{j+n} f}{\partial y^j \partial r^n} \\
&\quad + (-1)^{n+m}\int_{a_1}^{b_1}\int_{a_2}^{b_2} P_n(x,r)\,Q_m(y,s)\frac{\partial^{n+m} f}{\partial s^m \partial r^n} dsdr
\end{aligned}$$

Now integrate (2.14) for $t \in [a_3, b_3]$

$$\begin{aligned}
(2.15)\quad & \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(r,s,t)\,dtdsdr = \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} X_i(x)\,Y_j(y)\int_{a_3}^{b_3}\frac{\partial^{i+j} f}{\partial x^i \partial y^j} dt \\
&\quad + (-1)^m \sum_{i=0}^{n-1} X_i(x)\int_{a_2}^{b_2} Q_m(y,s)\left(\int_{a_3}^{b_3}\frac{\partial^{i+m} f}{\partial x^i \partial s^m} dt\right) ds \\
&\quad + (-1)^n \sum_{j=0}^{m-1} Y_j(y)\int_{a_2}^{b_1} P_n(x,r)\left(\int_{a_3}^{b_3}\frac{\partial^{j+n}}{\partial y^j \partial r^n} dt\right) dr \\
&\quad + (-1)^{n+m}\int_{a_1}^{b_1}\int_{a_2}^{b_2} P_n(x,r)\,Q_m(y,s)\left(\int_{a_3}^{b_3}\frac{\partial^{n+m} f}{\partial s^m \partial r^n} dt\right) dsdr.
\end{aligned}$$

From (2.9),

$$\begin{aligned}
(2.16)\quad \int_{a_3}^{b_3}\frac{\partial^{i+j} f}{\partial x^i \partial y^j} dt &= \sum_{k=0}^{p-1} Z_k(z)\frac{\partial^k}{\partial z^k}\left(\frac{\partial^{i+j} f}{\partial x^i \partial y^j}\right) \\
&\quad + (-1)^p \int_{a_3}^{b_3} S_p(z,t)\frac{\partial^p}{\partial t^p}\left(\frac{\partial^{i+j} f}{\partial x^i \partial y^j}\right) dt,
\end{aligned}$$

$$\begin{aligned}
(2.17)\quad \int_{a_3}^{b_3}\frac{\partial^{i+m} f}{\partial x^i \partial s^m} dt &= \sum_{k=0}^{p-1} Z_k(z)\frac{\partial^k}{\partial z^k}\left(\frac{\partial^{i+m} f}{\partial x^i \partial s^m}\right) \\
&\quad + (-1)^p \int_{a_3}^{b_3} S_p(z,t)\frac{\partial^p}{\partial t^p}\left(\frac{\partial^{i+m} f}{\partial x^i \partial s^m}\right) dt,
\end{aligned}$$

$$\int_{a_3}^{b_3} \frac{\partial^{j+n} f}{\partial y^j \partial r^n} dt = \sum_{k=0}^{p-1} Z_k(z) \frac{\partial^k}{\partial z^k}\left(\frac{\partial^{j+n} f}{\partial y^j \partial r^n}\right) + (-1)^p \int_{a_3}^{b_3} S_p(z,t) \frac{\partial^p}{\partial t^p}\left(\frac{\partial^{j+n} f}{\partial y^j \partial r^n}\right) dt,$$

and

$$\int_{a_3}^{b_3} \frac{\partial^{n+m} f}{\partial s^m \partial r^n} dt = \sum_{k=0}^{p-1} Z_k(z) \frac{\partial^k}{\partial z^k}\left(\frac{\partial^{n+m} f}{\partial r^n \partial s^m}\right) + (-1)^p \int_{a_3}^{b_3} S_p(z,t) \frac{\partial^p}{\partial t^p}\left(\frac{\partial^{n+m} f}{\partial r^n \partial s^m}\right) dt. \tag{2.18}$$

Putting (2.16), (2.17) and (2.18) into (2.15) we arrive at the identity (2.5). □

At the midpoint of the interval

$$\bar{x} = \frac{a_1+b_1}{2},\ \bar{y} = \frac{a_2+b_2}{2},\ \bar{z} = \frac{a_3+b_3}{2}$$

we have the following corollary.

Corollary 2.3. *Under the assumptions of Theorem 2.2, we have the identity*

$$\begin{aligned}
\bar{V} :=& \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(r,s,t)\,dtdsdr \\
&- \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\sum_{k=0}^{p-1} X_i(\bar{x}) Y_j(\bar{y}) Z_k(\bar{z}) \frac{\partial^{i+j+k} f(\bar{x},\bar{y},\bar{z})}{\partial x^i \partial y^j \partial z^k} \\
&+ (-1)^p \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} X_i(\bar{x}) Y_j(\bar{y}) \int_{a_3}^{b_3} S_p(\bar{z},t) \frac{\partial^{i+j+p} f(\bar{x},\bar{y},t)}{\partial x^i \partial y^j \partial t^p} dt \\
&+ (-1)^m \sum_{i=0}^{n-1}\sum_{k=0}^{p-1} X_i(\bar{x}) Z_k(\bar{z}) \int_{a_2}^{b_2} Q_m(\bar{y},s) \frac{\partial^{i+m+k} f(\bar{x},s,\bar{z})}{\partial x^i \partial s^m \partial z^k} ds \\
&+ (-1)^n \sum_{j=0}^{m-1}\sum_{k=0}^{p-1} Y_j(\bar{y}) Z_k(\bar{z}) \int_{a_1}^{b_1} P_n(\bar{x},r) \frac{\partial^{n+j+k} f(r,\bar{y},\bar{z})}{\partial r^n \partial y^j \partial z^k} dr \\
&- (-1)^{m+p} \sum_{i=0}^{n-1} X_i(\bar{x}) \int_{a_2}^{b_2}\int_{a_3}^{b_3} Q_m(\bar{y},s) S_p(\bar{z},t) \frac{\partial^{i+m+p} f(\bar{x},s,t)}{\partial x^i \partial s^m \partial t^p} dtds \\
&- (-1)^{n+p} \sum_{j=0}^{m-1} Y_j(\bar{y}) \int_{a_1}^{b_1}\int_{a_3}^{b_3} P_n(\bar{x},r) S_p(\bar{z},t) \frac{\partial^{n+j+p} f(r,\bar{y},t)}{\partial r^n \partial y^j \partial t^p} dtdr \\
&- (-1)^{n+m} \sum_{k=0}^{p-1} Z_k(\bar{z}) \int_{a_1}^{b_1}\int_{a_2}^{b_2} P_n(\bar{x},r) Q_m(\bar{y},s) \frac{\partial^{n+m+k} f(r,s,\bar{z})}{\partial r^n \partial s^m \partial z^k} dsdr \\
=& -(-1)^{n+m+p} \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} P_n(\bar{x},r) Q_m(\bar{y},s) S_p(\bar{z},t) \frac{\partial^{n+m+p} f(r,s,t)}{\partial r^n \partial s^m \partial t^p} dtdsdr.
\end{aligned} \tag{2.19}$$

The identity (2.5) will now be utilised to establish an inequality for a function of three independent variables which will furnish a refinement for the inequality (2.1) given by Pachpatte.

Theorem 2.4. *Let* $f:[a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]\to\mathbb{R}$ *be continuous on* $(a_1,b_1)\times(a_2,b_2)\times(a_3,b_3)$ *and the conditions of Theorem 2.2 apply. Then we have the inequality*

$$|V|\le\begin{cases}\left[\dfrac{(x-a_1)^{n+1}+(b_1-x)^{n+1}}{(n+1)!}\right]\left[\dfrac{(y-a_2)^{m+1}+(b_2-y)^{m+1}}{(m+1)!}\right]\\ \qquad\times\left[\dfrac{(z-a_3)^{p+1}+(b_3-z)^{p+1}}{(p+1)!}\right]\left\|\dfrac{\partial^{n+m+p}f}{\partial r^n\partial s^m\partial t^p}\right\|_\infty\\ \qquad\text{if }\dfrac{\partial^{n+m+p}f}{\partial r^n\partial s^m\partial t^p}\in L_\infty\left([a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]\right);\\[2ex] \dfrac{1}{n!m!p!}\left[\dfrac{(x-a_1)^{n\beta+1}+(b_1-x)^{n\beta+1}}{n\beta+1}\right]^{\frac{1}{\beta}}\left[\dfrac{(y-a_2)^{m\beta+1}+(b_2-y)^{m\beta+1}}{m\beta+1}\right]^{\frac{1}{\beta}}\\ \qquad\times\left[\dfrac{(z-a_3)^{p\beta+1}+(b_3-z)^{p\beta+1}}{p\beta+1}\right]^{\frac{1}{\beta}}\left\|\dfrac{\partial^{n+m+p}f}{\partial r^n\partial s^m\partial t^p}\right\|_\alpha\\ \qquad\text{if }\dfrac{\partial^{n+m+p}f}{\partial r^n\partial s^m\partial t^p}\in L_\alpha\left([a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]\right),\quad\alpha>1,\ \alpha^{-1}+\beta^{-1}=1;\\[2ex] \dfrac{1}{8n!m!p!}\left[(x-a_1)^n+(b_1-x)^n+|(x-a_1)^n-(b_1-x)^n|\right]\\ \qquad\times\left[(y-a_2)^m+(b_2-y)^m+|(y-a_2)^m-(b_2-y)^m|\right]\\ \qquad\times\left[(z-a_3)^p+(b_3-z)^p+|(z-a_3)^p-(b_3-z)^p|\right]\left\|\dfrac{\partial^{n+m+p}f}{\partial r^n\partial s^m\partial t^p}\right\|_1\\ \qquad\text{if }\dfrac{\partial^{n+m+p}f}{\partial r^n\partial s^m\partial t^p}\in L_1\left([a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]\right);\end{cases}$$

for all $(x,y,z)\in[a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]$, *where*

$$\left\|\frac{\partial^{n+m+p}f}{\partial r^n\partial s^m\partial t^p}\right\|_\infty=\sup_{(r,s,t)\in[a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]}\left|\frac{\partial^{n+m+p}f}{\partial r^n\partial s^m\partial t^p}\right|<\infty,$$

and

$$\left\|\frac{\partial^{n+m+p}f}{\partial r^n\partial s^m\partial t^p}\right\|_\alpha=\left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\left|\frac{\partial^{n+m+p}f}{\partial r^n\partial s^m\partial t^p}\right|^\alpha dtdsdr\right)^{\frac{1}{\alpha}}<\infty. \tag{2.20}$$

Proof.

$$\begin{aligned}|V|&=\left|\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}P_n(x,r)\,Q_m(y,s)\,S_p(z,t)\,\frac{\partial^{n+m+p}f(r,s,t)}{\partial r^n\partial s^m\partial t^p}dtdsdr\right|\\&\le\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\left|P_n(x,r)\,Q_m(y,s)\,S_p(z,t)\right|\left|\frac{\partial^{n+m+p}f(r,s,t)}{\partial r^n\partial s^m\partial t^p}\right|dtdsdr.\end{aligned}$$

Using Hölder's inequality and property of the modulus and integral, then we have that

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\left|P_n(x,r)\,Q_m(y,s)\,S_p(z,t)\right|\left|\frac{\partial^{n+m+p}f(r,s,t)}{\partial r^n\partial s^m\partial t^p}\right|dtdsdr \tag{2.21}$$

$$\leq \begin{cases} \left\| \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \right\|_\infty \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left| P_n(x,r) Q_m(y,s) S_p(z,t) \right| dt ds dr, \\ \left\| \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \right\|_\alpha \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left| P_n(x,r) Q_m(y,s) S_p(z,t) \right|^\beta dt ds dr \right)^{\frac{1}{\beta}}, \\ \hfill \alpha > 1,\ \alpha^{-1} + \beta^{-1} = 1; \\ \left\| \frac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \right\|_1 \sup_{(r,s,t) \in [a_1,b_1] \times [a_2,b_2] \times [a_3,b_3]} \left| P_n(x,r) Q_m(y,s) S_p(z,t) \right|. \end{cases}$$

From (2.21) and using (2.2), (2.3) and (2.4)

$$\begin{aligned} &\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left| P_n(x,r) Q_m(y,s) S_p(z,t) \right| dt ds dr \\ &\quad = \int_{a_1}^{b_1} \left| P_n(x,r) \right| dr \int_{a_2}^{b_2} \left| Q_m(y,s) \right| ds \int_{a_3}^{b_3} \left| S_p(z,t) \right| dt \\ &\quad = \left[\int_{a_1}^{x} \frac{(r-a_1)^n}{n!} dr + \int_{x}^{b_1} \frac{(b_1-r)^n}{n!} dr \right] \left[\int_{a_2}^{y} \frac{(s-a_2)^m}{m!} ds + \int_{y}^{b_2} \frac{(b_2-s)^m}{m!} ds \right] \\ &\qquad \times \left[\int_{a_3}^{z} \frac{(t-a_3)^p}{p!} dt + \int_{z}^{b_3} \frac{(b_3-t)^p}{p!} dt \right] \\ &\quad = \frac{\left[(x-a_1)^{n+1} + (b_1-x)^{n+1} \right] \left[(y-a_2)^{m+1} + (b_2-y)^{m+1} \right]}{(n+1)!\,(m+1)!} \\ &\qquad \times \frac{\left[(z-a_3)^{p+1} + (b_3-z)^{p+1} \right]}{(p+1)!} \end{aligned}$$

giving the first inequality in (2.20).

Now, if we again use (2.21) we have

$$\begin{aligned} &\left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left| P_n(x,r) Q_m(y,s) S_p(z,t) \right|^\beta dt ds dr \right)^{\frac{1}{\beta}} \\ &\quad = \left(\int_{a_1}^{b_1} \left| P_n(x,r) \right|^\beta dr \right)^{\frac{1}{\beta}} \left(\int_{a_2}^{b_2} \left| Q_m(y,s) \right|^\beta ds \right)^{\frac{1}{\beta}} \left(\int_{a_3}^{b_3} \left| S_p(z,t) \right|^\beta dt \right)^{\frac{1}{\beta}} \\ &\quad = \frac{1}{n!m!p!} \left[\int_{a_1}^{x} (r-a_1)^{n\beta} dr + \int_{x}^{b_1} (b_1-r)^{n\beta} dr \right]^{\frac{1}{\beta}} \\ &\qquad \times \left[\int_{a_2}^{y} (s-a_2)^{m\beta} ds + \int_{y}^{b_2} (b_2-s)^{m\beta} ds \right]^{\frac{1}{\beta}} \\ &\qquad \times \left[\int_{a_3}^{z} (t-a_3)^{p\beta} dt + \int_{z}^{b_3} (b_3-t)^{p\beta} dt \right]^{\frac{1}{\beta}} \\ &\quad = \frac{1}{n!m!p!} \left[\frac{(x-a_1)^{n\beta+1} + (b_1-x)^{n\beta+1}}{n\beta+1} \right]^{\frac{1}{\beta}} \left[\frac{(y-a_2)^{m\beta+1} + (b_2-y)^{m\beta+1}}{m\beta+1} \right]^{\frac{1}{\beta}} \\ &\qquad \times \left[\frac{(z-a_3)^{p\beta+1} + (b_3-z)^{p\beta+1}}{p\beta+1} \right]^{\frac{1}{\beta}} \end{aligned}$$

producing the second inequality in (2.20).

Finally, we have

$$\begin{aligned}
&\sup_{(r,s,t)\in[a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]} \left|P_n(x,r)\,Q_m(y,s)\,S_p(z,t)\right| \\
&= \sup_{r\in[a_1,b_1]} \left|P_n(x,r)\right| \sup_{s\in[a_2,b_2]} \left|Q_m(y,s)\right| \sup_{t\in[a_3,b_3]} \left|S_p(z,t)\right| \\
&= \max\left\{\frac{(x-a_1)^n}{n!},\frac{(b_1-x)^n}{n!}\right\} \max\left\{\frac{(y-a_2)^m}{m!},\frac{(b_2-y)^m}{m!}\right\} \\
&\qquad \times \max\left\{\frac{(z-a_3)^p}{p!},\frac{(b_3-z)^p}{p!}\right\} \\
&= \frac{1}{n!m!p!}\left[\frac{(x-a_1)^n+(b_1-x)^n}{2}+\left|\frac{(x-a_1)^n-(b_1-x)^n}{2}\right|\right] \\
&\qquad \times \left[\frac{(y-a_2)^m+(b_2-y)^m}{2}+\left|\frac{(y-a_2)^m-(b_2-y)^m}{2}\right|\right] \\
&\qquad \times \left[\frac{(z-a_3)^p+(b_3-z)^p}{2}+\left|\frac{(z-a_3)^p-(b_3-z)^p}{2}\right|\right],
\end{aligned}$$

giving us the third inequality in (2.20) and we have used the fact that for $A>0,\ B>0$ then

$$\max\{A,B\}=\frac{A+B}{2}+\left|\frac{A-B}{2}\right|.$$

Hence the theorem is completely solved. □

The following corollary is a consequence of Theorem 2.4.

Corollary 2.5. *Under the assumptions of Corollary 2.3, we have the inequality*

$$\left|\bar{V}\right| \le \begin{cases}
\left[\dfrac{(b_1-a_1)^{n+1}(b_2-a_2)^{m+1}(b_3-a_3)^{p+1}}{2^{n+m+p}(n+1)!\,(m+1)!\,(p+1)!}\right]\left\|\dfrac{\partial^{n+m+p}f}{\partial r^n\partial s^m\partial t^p}\right\|_\infty, \\[2ex]
\dfrac{1}{2^{n+m+p}n!m!p!}\left[\dfrac{(b_1-a_1)^{n\beta+1}(b_2-a_2)^{m\beta+1}(b_3-a_3)^{p\beta+1}}{(n\beta+1)(m\beta+1)(p\beta+1)}\right]^{\frac{1}{\beta}} \\
\qquad\qquad \times\left\|\dfrac{\partial^{n+m+p}f}{\partial r^n\partial s^m\partial t^p}\right\|_\alpha, \\[2ex]
\dfrac{1}{2^{n+m+p}n!m!p!}(b_1-a_1)^n(b_2-a_2)^m(b_3-a_3)^p\left\|\dfrac{\partial^{n+m+p}f}{\partial r^n\partial s^m\partial t^p}\right\|_1,
\end{cases}$$

where $\|\cdot\|_\alpha\ (\alpha\in[1,\infty))$ *are the Lebesgue norms on* $[a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]$.

The following two corollaries concern the estimation of V at the end points.

Corollary 2.6. *Under the assumptions of Theorem 2.4 we have, for $x = a_1$, $y = a_2$ and $z = a_3$, the inequality*

$$
\begin{aligned}
&\left|V\left(a_1, a_2, a_3\right)\right| \\
&:= \left| \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} f\left(r, s, t\right) dtdsdr - \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \sum_{k=0}^{p-1} X_i\left(a_1\right) Y_j\left(a_2\right) Z_k\left(a_3\right) \frac{\partial^{i+j+k} f}{\partial x^i \partial y^j \partial z^k} \right. \\
&\quad + (-1)^p \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} X_i\left(a_1\right) Y_j\left(a_2\right) \int_{a_3}^{b_3} \bar{S}_p\left(a_3, t\right) \frac{\partial^{i+j+p} f}{\partial x^i \partial y^j \partial t^p} dt \\
&\quad + (-1)^m \sum_{i=0}^{n-1} \sum_{k=0}^{p-1} X_i\left(a_1\right) Z_k\left(a_3\right) \int_{a_2}^{b_2} \bar{Q}_m\left(a_2, s\right) \frac{\partial^{i+m+k} f}{\partial x^i \partial s^m \partial z^k} ds \\
&\quad + (-1)^n \sum_{j=0}^{m-1} \sum_{k=0}^{p-1} Y_j\left(a_2\right) Z_k\left(a_3\right) \int_{a_1}^{b_1} \bar{P}_n\left(a_1, r\right) \frac{\partial^{n+j+k} f}{\partial r^n \partial y^j \partial z^k} dr \\
&\quad - (-1)^{m+p} \sum_{i=0}^{n-1} X_i\left(a_1\right) \int_{a_2}^{b_2} \int_{a_3}^{b_3} \bar{Q}_m\left(a_2, s\right) \bar{S}_p\left(a_3, t\right) \frac{\partial^{i+m+p} f}{\partial x^i \partial s^m \partial t^p} dtds \\
&\quad - (-1)^{n+p} \sum_{j=0}^{m-1} Y_j\left(a_2\right) \int_{a_1}^{b_1} \int_{a_3}^{b_3} \bar{P}_n\left(a_1, r\right) \bar{S}_p\left(a_3, t\right) \frac{\partial^{n+j+p} f}{\partial r^n \partial y^j \partial t^p} dtdr \\
&\quad \left. - (-1)^{n+m} \sum_{k=0}^{p-1} Z_k\left(a_3\right) \int_{a_1}^{b_1} \int_{a_2}^{b_2} \bar{P}_n\left(a_1, r\right) \bar{Q}_m\left(a_2, s\right) \frac{\partial^{n+m+k} f}{\partial r^n \partial s^m \partial z^k} dsdr \right| \\
&\le \begin{cases}
\dfrac{\left(b_1 - a_1\right)^{n+1} \left(b_2 - a_2\right)^{m+1} \left(b_3 - a_3\right)^{p+1}}{(n+1)!\,(m+1)!\,(p+1)!} \left\| \dfrac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \right\|_\infty, \\
\qquad\qquad \text{if } \dfrac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \in L_\infty\left(\left[a_1, b_1\right] \times \left[a_2, b_2\right] \times \left[a_3, b_3\right]\right); \\
\dfrac{\left(b_1 - a_1\right)^{n+\frac{1}{\beta}}}{n!\left(n\beta + 1\right)^{\frac{1}{\beta}}} \cdot \dfrac{\left(b_2 - a_2\right)^{m+\frac{1}{\beta}}}{m!\left(m\beta + 1\right)^{\frac{1}{\beta}}} \cdot \dfrac{\left(b_3 - a_3\right)^{p+\frac{1}{\beta}}}{p!\left(p\beta + 1\right)^{\frac{1}{\beta}}} \left\| \dfrac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \right\|_\alpha, \\
\text{if } \dfrac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \in L_\alpha\left(\left[a_1, b_1\right] \times \left[a_2, b_2\right] \times \left[a_3, b_3\right]\right), \quad \alpha > 1,\ \alpha^{-1} + \beta^{-1} = 1; \\
\dfrac{\left(b_1 - a_1\right)^n \left(b_2 - a_2\right)^m \left(b_3 - a_3\right)^p}{n!m!p!} \left\| \dfrac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \right\|_1, \\
\qquad\qquad \text{if } \dfrac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \in L_1\left(\left[a_1, b_1\right] \times \left[a_2, b_2\right] \times \left[a_3, b_3\right]\right),
\end{cases}
\end{aligned}
$$

where

$$X_i\left(a_1\right) := \frac{\left(b_1 - a_1\right)^{i+1}}{(i+1)!}, \quad Y_j\left(a_2\right) := \frac{\left(b_2 - a_2\right)^{j+1}}{(j+1)!}, \quad Z_k\left(a_3\right) := \frac{\left(b_3 - a_3\right)^{k+1}}{(k+1)!}.$$

$$\bar{P}_n\left(a_1, r\right) = \frac{\left(r - b_1\right)^n}{n!}, \quad r \in \left[a_1, b_1\right]; \quad \bar{Q}_m\left(a_2, s\right) = \frac{\left(s - b_2\right)^m}{m!}, \quad s \in \left[a_2, b_2\right]$$

and

$$\bar{S}_p(a_3,t) = \frac{(t-b_3)^p}{p!}; \quad t \in [a_3,b_3].$$

Corollary 2.7. *Under the assumptions of Theorem 2.4 we have, for $x = b_1$, $y = b_2$ and $z = b_3$, the inequality*

$$\begin{aligned}
&|V(b_1,b_2,b_3)| \\
&:= \Bigg| \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(r,s,t)\,dtdsdr - \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\sum_{k=0}^{p-1} X_i(a_1)\,Y_j(a_2)\,Z_k(a_3)\,\frac{\partial^{i+j+k} f}{\partial x^i \partial y^j \partial z^k} \\
&\quad + (-1)^p \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} X_i(b_1)\,Y_j(b_2) \int_{a_3}^{b_3} \bar{S}_p(b_3,t)\,\frac{\partial^{i+j+p} f}{\partial x^i \partial y^j \partial t^p} dt \\
&\quad + (-1)^m \sum_{i=0}^{n-1}\sum_{k=0}^{p-1} X_i(b_1)\,Z_k(b_3) \int_{a_2}^{b_2} \bar{Q}_m(b_2,s)\,\frac{\partial^{i+m+k} f}{\partial x^i \partial s^m \partial z^k} ds \\
&\quad + (-1)^n \sum_{j=0}^{m-1}\sum_{k=0}^{p-1} Y_j(b_2)\,Z_k(b_3) \int_{a_1}^{b_1} \bar{P}_n(b_1,r)\,\frac{\partial^{n+j+k} f}{\partial r^n \partial y^j \partial z^k} dr \\
&\quad - (-1)^{m+p} \sum_{i=0}^{n-1} X_i(b_1) \int_{a_2}^{b_2}\int_{a_3}^{b_3} \bar{Q}_m(b_2,s)\,\bar{S}_p(b_3,t)\,\frac{\partial^{i+m+p} f}{\partial x^i \partial s^m \partial t^p} dtds \\
&\quad - (-1)^{n+p} \sum_{j=0}^{m-1} Y_j(b_2) \int_{a_1}^{b_1}\int_{a_3}^{b_3} \bar{P}_n(b_1,r)\,\bar{S}_p(b_3,t)\,\frac{\partial^{n+j+p} f}{\partial r^n \partial y^j \partial t^p} dtdr \\
&\quad - (-1)^{n+m} \sum_{k=0}^{p-1} Z_k(b_3) \int_{a_1}^{b_1}\int_{a_2}^{b_2} \bar{P}_n(b_1,r)\,\bar{Q}_m(b_2,s)\,\frac{\partial^{n+m+k} f}{\partial r^n \partial s^m \partial z^k} dsdr \Bigg| \\
&\le \begin{cases}
\dfrac{(b_1-a_1)^{n+1}(b_2-a_2)^{m+1}(b_3-a_3)^{p+1}}{(n+1)!\,(m+1)!\,(p+1)!} \left\| \dfrac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \right\|_\infty, \\
\qquad\qquad \text{if } \dfrac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \in L_\infty([a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]); \\
\dfrac{(b_1-a_1)^{n+\frac{1}{\beta}}}{n!\,(n\beta+1)^{\frac{1}{\beta}}} \cdot \dfrac{(b_2-a_2)^{m+\frac{1}{\beta}}}{m!\,(m\beta+1)^{\frac{1}{\beta}}} \cdot \dfrac{(b_3-a_3)^{p+\frac{1}{\beta}}}{p!\,(p\beta+1)^{\frac{1}{\beta}}} \left\| \dfrac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \right\|_\alpha, \\
\text{if } \dfrac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \in L_\alpha([a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]), \quad \alpha>1,\ \alpha^{-1}+\beta^{-1}=1; \\
\dfrac{(b_1-a_1)^{n}(b_2-a_2)^{m}(b_3-a_3)^{p}}{n!m!p!} \left\| \dfrac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \right\|_1, \\
\qquad\qquad \text{if } \dfrac{\partial^{n+m+p} f}{\partial r^n \partial s^m \partial t^p} \in L_1([a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]),
\end{cases}
\end{aligned}$$

where

$$X_i(b_1) := \frac{(-1)^i (b_1-a_1)^{i+1}}{(i+1)!},\ Y_j(b_2) := \frac{(-1)^j (b_2-a_2)^{j+1}}{(j+1)!},\ Z_k(b_3) := \frac{(-1)^k (b_3-a_3)^{k+1}}{(k+1)!}.$$

$$\bar{P}_n (b_1, r) = \frac{(r - a_1)^n}{n!}; \quad r \in [a_1, b_1],$$
$$\bar{Q}_m (b_2, s) = \frac{(s - a_2)^m}{m!}; \quad s \in [a_2, b_2]$$

and

$$\bar{S}_p (b_3, t) = \frac{(t - a_3)^p}{p!}; \quad t \in [a_3, b_3].$$

REFERENCES

[1] N.S. BARNETT AND S.S. DRAGOMIR, An Ostrowski type inequality for double integrals and applications for cubature formulae, *Soochow J. Math.,* **27**(1) (2001), 1–10.

[2] L. DEDIĆ, M. MATIĆ AND J.E. PEČARIĆ, On some generalisations of Ostrowski inequality for Lipschitz functions and functions of bounded variation, *Math. Ineq. & Appl.,* **3**(1) (2001), 1–14.

[3] S.S. DRAGOMIR, A generalisation of Ostrowski's integral inequality for mappings whose derivatives belong to $L_\infty [a, b]$ and applications in numerical integration, *J. KSIAM,* **5**(2) (2001), 117–136.

[4] A. GUESSAB AND G. SCHMEISSER, Sharp integral inequalities of the Hermite-Hadamard type, *J. of Approximation Theory,* **115** (2002), 260–288.

[5] G. HANNA, S.S. DRAGOMIR AND P. CERONE, A general Ostrowski type inequality for double integrals, *Tamsui Oxford J. Mathematical Sciences,* **18**(1) (2002), 1–16.

[6] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Inequalities Involving Functions and their Integrals and Derivatives,* Kluwer Academic Publishers, Dordrecht, 1994.

[7] A. OSTROWSKI, Uber die Absolutabweichung einer differentienbaren Funktionen von ihren Integralmittelwert, *Comment. Math. Hel,* **10** (1938), 226-227.

[8] B.G. PACHPATTE, On a new Ostrowski type inequality in two independent variables, *Tamkang J. Math.,* **32**(1) (2001), 45–49.

[9] B.G. PACHPATTE, Discrete inequalities in three independent variables, *Demonstratio Math.,* **31** (1999), 849–854.

[10] B.G. PACHPATTE, On an inequality of Ostrowski type in three independent variables, *J. Math. Analysis and Applications,* **249** (2000), 583–591.

[11] C.E.M. PEARCE, J.E. PEČARIĆ, N. UJEVIĆ AND S. VAROSANEC, Generalisation of some inequalities of Ostrowski-Grüss type, *Math. Ineq. & Appl.,* **3**(1) (2000), 25–34.

[12] A. SOFO, Integral inequalities for $n-$times differentiable mappings, 65-139. *Ostrowski Type Inequalities and Applications in Numerical Integration.* Editors: S.S. Dragomir and Th. M. Rassias. Kluwer Academic Publishers, 2002.

[13] A. SOFO, Double integral inequalities based on multi-branch Peano kernels, *Math. Ineq. & Appl.,* **5**(3) (2002), 491–504.

[14] J. STEWART, *Calculus: Early Transcendentals,* 3rd Edition. Brooks/Cole, Pacific Grove, 1995.

[15] J.A.C. WEIDEMAN, Numerical integration of periodic functions: A few examples, *The American Mathematical Monthly,* Vol. **109**, 21-36, 2002.