Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 4, Issue 4, Article 74, 2003

THE SPREAD OF THE SHAPE OPERATOR AS CONFORMAL INVARIANT

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Received 01 May, 2003; accepted 27 September, 2003
Communicated by S.S. Dragomir

ABSTRACT. The notion the spread of a matrix was first introduced fifty years ago in algebra. In this article, we define the spread of the shape operator by applying the same idea to submanifolds of Riemannian manifolds. We prove that the spread of shape operator is a conformal invariant for any submanifold in a Riemannian manifold. Then, we prove that, for a compact submanifold of a Riemannian manifold, the spread of the shape operator is bounded above by a geometric quantity proportional to the Willmore-Chen functional. For a complete non-compact submanifold, we establish a relationship between the spread of the shape operator and the Willmore-Chen functional. In the last section, we obtain a necessary and sufficient condition for a surface of rotation to have finite integral of the spread of the shape operator.

Key words and phrases: Principal curvatures, Shape operator, Extrinsic scalar curvature, Surfaces of rotation.

2000 *Mathematics Subject Classification.* 53B25, 53B20, 53A30.

1. INTRODUCTION

In the classic matrix theory *spread of a matrix* has been defined by Mirsky in [7] and then mentioned in various references, as for example [6]. Let $A \in M_n(\mathbb{C})$, $n \geq 3$, and let $\lambda_1, \dots, \lambda_n$ be the characteristic roots of A. The *spread* of A is defined to be $s(A) = \max_{i,j} |\lambda_i - \lambda_j|$. Let us denote by $||A||$ the Euclidean norm of the matrix A, i.e.: $||A||^2 = \sum_{i,j=1}^{m,n} |a_{ij}|^2$. We use also the classical notation E_2 for the sum of all 2-square principal subdeterminants of A. If $A \in M_n(\mathbb{C})$ then we have the following inequalities (see [6]):

$$s(A) \leq \left(2||A||^2 - \frac{2}{n}|trA|^2\right)^{\frac{1}{2}}, \tag{1.1}$$

$$s(A) \leq \sqrt{2}||A||. \tag{1.2}$$

ISSN (electronic): 1443-5756

060-03

If $A \in M_n(\mathbb{R})$, then:

$$s(A) \leq \left[2\left(1-\frac{1}{n}\right)(trA)^2 - 4E_2(A)\right]^{\frac{1}{2}}, \tag{1.3}$$

with equality if and only if $n-2$ of the characteristic roots of A are equal to the arithmetic mean of the remaining two.

Consider now an isometrically immersed submanifold M^n of dimension $n \geq 2$ in a Riemannian manifold $(\bar{M}^{n+s}, \bar{g})$. Then the Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y),$$

$$\bar{\nabla}_X \xi = -A_\xi X + D_X \xi,$$

for every $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(\nu M)$. Take a vector $\eta \in \nu_p M$ and consider the linear mapping $A_\eta : T_pM \to T_pM$. Let us consider the eigenvalues $\lambda_\eta^1, \dots, \lambda_\eta^n$ of A_η. We put

$$L_\eta(p) = \sup_{i=1,\dots,n}(\lambda_\eta^i) - \inf_{i=1,\dots,n}(\lambda_\eta^i). \tag{1.4}$$

L_η is the spread of the shape operator in the direction η. We define *the spread of the shape operator at the point* p by

$$L(p) = \sup_{\eta \in \nu_p M} L_\eta(p). \tag{1.5}$$

Suppose M is a compact submanifold of $\bar{M}$.

Let us remark that when M^2 is a surface we have

$$L_\nu^2(p) = (\lambda_\nu^1(p) - \lambda_\nu^2(p))^2 = 4(|H(p)|^2 - K(p)),$$

where ν is the normal vector at p, H is the mean curvature, and K is the Gaussian curvature. In [1] it is proved that for a surface M^2 in $\mathbb{E}^{2+s}$ the geometric quantity $(|H|^2 - K)dV$ is a conformal invariant. As a corollary, one obtains for an orientable surface in $\mathbb{E}^{2+s}$ that $L_\nu^2 dV$ is a conformal invariant.

Let $\xi_{n+1}, \dots, \xi_{n+s}$ be an orthonormal frame in the normal fibre bundle νM. Let us recall the definition of *the extrinsic scalar curvature* from [2]:

$$ext = \frac{2}{n(n-1)} \sum_{r=1}^{s} \sum_{i<j} \lambda_{n+r}^i \lambda_{n+r}^j.$$

In [2] it is proved that for a submanifold M^n of a Riemannian manifold $(\bar{M}, \bar{g})$, the geometric quantity $(|H|^2 - ext)g$ is invariant under any conformal change of metric. If M is compact (see also [2]), this result implies that for M, a n-dimensional compact submanifold of a Riemannian manifold $(\bar{M}, \bar{g})$, the geometric quantity $\int (|H|^2 - ext)^{\frac{n}{2}} dV$ is a conformal invariant.

Let us prove the following fact.

Proposition 1.1. *Let M^n be a submanifold of the Riemannian manifold $(\bar{M}, \bar{g})$. Then the spread of the shape operator is a conformal invariant.*

Proof. The context and the idea of the proof are similar to the one given in [3, pp. 204-205]. Let us consider ρ a nowhere vanishing positive function on $\bar{M}$. We have the conformal change of metric in the ambient space $\bar{M}$ given by

$$\bar{g}^* = \rho^2 \bar{g}.$$

Let us denote by h and h^* the second fundamental forms of M in $(\bar{M}, \bar{g})$ and $(\bar{M}, \bar{g}^*)$, respectively. Then we have (see [3]):

$$g(A_\xi^* X, Y) = g(A_\xi X, Y) + g(X,Y)\bar{g}(U, \xi),$$

where U is the vector field defined by $U = (d\rho)^{\#}$. Let $e_1, \dots, e_n$ be the principal normal directions of A_ξ with respect to g. Then $\rho^{-1}e_1, \dots, \rho^{-1}e_n$, form an orthonormal frame of M with respect to g^*, and they are the principal directions of A^*_ξ. Therefore

$$\begin{aligned} L^*(p) &= \sup_{\xi^* \in \nu_p M; ||\xi^*||_* = 1} L^*_{\xi^*} \\ &= \sup_{\xi^* \in \nu_p M; ||\xi^*||_* = 1} \left(\sup_{i=1,\dots,n} (\lambda^i_\xi)^* - \inf_{j=1,\dots,n} (\lambda^j_\xi)^* \right) \\ &= \sup_{\xi \in \nu_p M; ||\xi|| = 1} \left[\sup_{i=1,\dots,n} \left(\lambda^i_\xi + \bar{g}(U,\xi) \right) - \inf_{j=1,\dots,n} \left(\lambda^j_\xi + \bar{g}(U,\xi) \right) \right] \\ &= \sup_{\xi \in \nu_p M; ||\xi|| = 1} \left[\sup_{i=1,\dots,n} \left(\lambda^i_\xi \right) - \inf_{j=1,\dots,n} \left(\lambda^j_\xi \right) \right] = L(p). \end{aligned}$$

This proves the proposition. □

When M is a surface, both L and $L^2 dV$ are conformal invariants.

The *shape discriminant* of the submanifold M in $\bar{M}$ w.r.t. a normal direction η was discussed in [9]. Let A_η be the shape operator associated with an arbitrary normal vector η at p. The shape discriminant of η is defined by

$$D_\eta = 2||A_\eta||^2 - \frac{2}{n}(trace\, A_\eta)^2, \tag{1.6}$$

where $||A_\eta||^2 = (\lambda^1_\eta)^2 + \cdots + (\lambda^n_\eta)^2$, at every point $p \in M \subset \bar{M}$.

The following pointwise double inequality was proved in [9]:

$$D_\eta \bigg/ \binom{n}{2} \le L^2_\eta \le D_\eta, \tag{1.7}$$

We will use this inequality later on. The proof of this fact is algebraically related to the proof of Chen's fundamental inequality with classical curvature invariants (see [4]). The alternate proof of this result is presented in [10].

2. GEOMETRIC INEQUALITIES ON COMPACT SUBMANIFOLDS

In this section, we study the relationship between the spread of the shape operator's spectrum and the conformal invariant from [2]. The main result is Proposition 2.1. For its proof we need a few preliminary steps.

Proposition 2.1. *Let M^n be a compact submanifold of a Riemannian manifold $\bar{M}^{n+s}$. Then the following inequality holds:*

$$\left(\int_M L dV \right)^2 (vol(M))^{\frac{n}{2n-2}} \le 2n(n-1) \left(\int_M (|H|^2 - ext)^{\frac{n}{2}} dV \right)^{\frac{2}{n}}. \tag{2.1}$$

The equality holds if and only if either $n = 2$ or M is a totally umbilical submanifold of dimension $n \ge 3$.

Before presenting the proof, let us see what this inequality means. For any conformal diffeomorphism ϕ of the ambient space $\bar{M}$, the quantity

$$\left(\int_{\phi(M)} L dV_\phi \right)^2 (vol(\phi(M))^{\frac{n}{2n-2}}$$

is bounded above by the conformal invariant geometric quantity expressed in (2.1).

First, let us prove the following.

Lemma 2.2. *Let $M^n \subset \bar{M}^{n+s}$ be a compact submanifold and p an arbitrary point in M. Consider an orthonormal normal frame $\xi_1, \dots, \xi_s$ at p and let D_α be the shape discriminant corresponding to ξ_α, where $\alpha = 1, \dots, s$. Then we have*

$$\frac{1}{2n(n-1)} \sum_{\alpha=1}^{s} D_\alpha = |H|^2 - ext. \tag{2.2}$$

Proof. Since

$$H = \frac{1}{n} \sum_{\alpha=1}^{s} \left(\sum_{i=1}^{n} \lambda_\alpha^i \right) \xi_\alpha,$$

$$ext = \frac{2}{n(n-1)} \sum_{\alpha=1}^{s} \sum_{i<j} \lambda_\alpha^i \lambda_\alpha^j,$$

we have

$$|H|^2 - ext = \frac{1}{n^2} \sum_{\alpha=1}^{s} \sum_{i=1}^{n} (\lambda_\alpha^i)^2 - \frac{2}{n^2(n-1)} \sum_{\alpha=1}^{s} \sum_{i<j} \lambda_\alpha^i \lambda_\alpha^j. \tag{2.3}$$

A direct computation yields

$$D_\alpha = \frac{2(n-1)}{n} \sum_{i=1}^{n} (\lambda_\alpha^i)^2 - \frac{4}{n} \sum_{i<j} \lambda_\alpha^i \lambda_\alpha^j. \tag{2.4}$$

Summing from $\alpha = 1$ to $\alpha = s$ in (2.4) and comparing the result with (2.3) one may get (2.2). □

From the cited result in [2] and the previous lemma, we have:

Corollary 2.3. *If M is a compact submanifold in the ambient space $\bar{M}$, then*

$$\int_M \left(\sum_{\alpha=1}^{s} D_\alpha \right)^{\frac{n}{2}} dV$$

is a conformal invariant.

Let us remark that for $n = 2$ this is a well-known fact.

Lemma 2.4. *Let M be a submanifold in the arbitrary ambient space $\bar{M}$. With the previous notations we have*

$$4(|H|^2 - ext) \le \sum_{\alpha} L_\alpha^2(p) \le 2n(n-1)(|H|^2 - ext)$$

at each point $p \in M$. The equalities holds if and only if p is an umbilical point.

Proof. This is a direct consequence of Lemma 2.2 and (1.7). □

Proof. We may prove now Proposition 2.1. Let p be an arbitrary point of M and let η_0 be a normal direction such that $L(p) = L_{\eta_0}(p)$. Consider the completion of η_0 up to a orthonormal normal base $\eta_0 = \eta_1, \dots, \eta_s$. Then we have

$$L^2(p) = L_{\eta_0}^2(p) \le \sum_{\alpha=1}^{s} L_\alpha^2(p) \le 2n(n-1)(|H|^2 - ext). \tag{2.5}$$

By applying Hölder's inequality, one has:

$$\left(\int_M LdV\right)^2 \leq \left(\int_M L^2 dV\right)(vol(M)).$$

Applying Hölder's inequality one more time yields

$$\int_M \left(|H|^2 - ext\right) dV \leq \left(\int_M \left(|H|^2 - ext\right)^{\frac{n}{2}} dV\right)^{\frac{2}{n}} (vol(M))^{\frac{n-2}{n}}.$$

Therefore, by using the inequality established in Lemma 2.4, we have

$$\begin{aligned}\left(\int_M LdV\right)^2 &\leq \left(\int_M L^2 dV\right)(vol(M)) \\ &\leq 2n(n-1)vol(M)\int_M \left(|H|^2 - ext\right) dV \\ &\leq 2n(n-1)\left(vol(M)\right)^{\frac{2n-2}{n}}\left(\int_M \left(|H|^2 - ext\right)^{\frac{n}{2}} dV\right)^{\frac{2}{n}}.\end{aligned}$$

Let us discuss when the equality case may occur. We have seen that we get an identity if $n = 2$.

Now, let us assume $n \geq 3$. The first inequality in (2.5) is equality at p if there exist $s-1$ umbilical directions (i.e. $L_\alpha(p) = 0$ for $s = 2, \dots, n$). The second inequality in (2.5) is equality if and only if p is an umbilical point (see [9]). Finally, the two Hölder inequalities are indeed equalities if and only if there exist real numbers θ and μ satisfying $L(p) = \theta$ and $|H|^2 - ext = \mu$ at every $p \in M$. The first equality conditions impose pointwise $L(p) = 0$, which yields $\theta = \mu = 0$. This means that M is totally umbilical. □

3. THE NONCOMPACT CASE

Let M be an n-dimensional noncompact submanifold of an $(n+d)$-dimensional Riemannian manifold $(\bar{M}, g)$.

Proposition 3.1. *Let $M^n \subset \bar{M}^{n+d}$ be a complete noncompact submanifold and $\eta_1, \dots, \eta_d$ an orthonormal basis of the normal bundle. Suppose that $\sum \lambda_\alpha^i \lambda_\alpha^j \geq 0$ and $L_\alpha \in L^2(M)$. Then*

$$\int_M (|H|^2 - ext)dV < \infty.$$

Proof. We use the inequality (1.7). It is sufficient to prove locally the inequality:

$$|H|^2 - ext \leq \sum_{i=1}^{d} D_i$$

This is true since, elementary, the following inequality holds:

$$(\lambda_\alpha^1)^2 + \dots + (\lambda_\alpha^d)^2 - \frac{2n}{n-1}\sum_{i<j}\lambda_\alpha^i\lambda_\alpha^j \leq 2[(\lambda_\alpha^1)^2 + \dots + (\lambda_\alpha^d)^2] - \frac{2}{n}\left\{\sum_{i=1}^{d}(\lambda_\alpha^i)\right\}^2.$$

This is equivalent to

$$n(n-1)\sum_{i=1}^{d}(\lambda_\alpha^i)^2 - 2n^2\sum_{i<j}\lambda_\alpha^i\lambda_\alpha^j \leq 2(n-1)^2\sum_{i=1}^{d}(\lambda_\alpha^i)^2 - 4(n-1)\sum_{i<j}\lambda_\alpha^i\lambda_\alpha^j$$

or

$$(n^2 - 3n + 2)\left\{\sum_{i=1}^{d}(\lambda_\alpha^i)^2\right\} + 2(n^2 - 2n + 2)\sum_{i<j}\lambda_\alpha^i\lambda_\alpha^j \geq 0,$$

which holds by using the hypothesis and that $n \geq 2$.

The inequality is the α-component of the invariant inequality we are going to prove. By adding up d such inequalities and by considering the improper integral on M of the appropriate functions, the conclusion follows. This is due to

$$\int_M (|H|^2 - ext)dV \leq \int_M \sum_{i=1}^{d} D_i dV \leq \binom{n}{2}\sum_{i=1}^{d}\int_M L_i^2 dV$$

by the first inequality in 1.7. □

In the next proposition we establish a relation between $\int_M [L(p)]^2 dV$ and the Willmore-Chen integral $\int_M (|H| - ext)dV$, studied in [2].

Proposition 3.2. *Let $M^n \subset \bar{M}^{n+d}$ be a complete noncompact orientable submanifold. If $L(p) \in L^2(M)$, then $\int_M (|H|^2 - ext)dV < \infty$.*

Proof. By direct computation, we have:

$$\begin{aligned}
\int_M (|H|^2 - ext)dV &= \frac{1}{n^2(n-1)}\int_M \sum_{\alpha=1}^{d}\sum_{i<j}(\lambda_\alpha^1 - \lambda_\alpha^j)^2 dV \\
&\leq \frac{1}{n^2(n-1)}\int_M \sum_{\alpha=1}^{d}\sum_{i<j} L^2(p) dV \\
&= \frac{d}{2n}\int_M L^2(p)dV.
\end{aligned} \tag{3.1}$$

□

Let us discuss now two examples. First, let us consider *the catenoid* defined by

$$f_c(u, v) = \left(c\cos u\ \cosh\frac{v}{c}, c\ \sin u\ \cosh\frac{v}{c}, v\right).$$

Using the classical formulas for example from [8] one finds:

$$\lambda_1 = -\lambda_2 = \frac{1}{c}\cosh^{-2}\frac{v}{c}.$$

Therefore, we have

$$\int_{-\infty}^{\infty} L(p)dv = \int_{-\infty}^{\infty}\frac{2}{c}\cosh^{-2}\frac{v}{c}dv = 4\int_{-\infty}^{\infty}\frac{e^t dt}{e^{2t}+1} = 4\pi < \infty.$$

Let us consider *the pseudosphere* whose profile functions are given by (see, for example [5]):

$$c_1(v) = ae^{-v/a}$$
$$c_2(v) = \int_0^v \sqrt{1 - e^{-2t/a}}dt$$

for $0 \leq v < \infty$. For simplicity, let us consider just the "upper" part of the pseudosphere. We have

$$\lambda_1 = \frac{e^{v/a}}{a}\sqrt{1 - e^{-2v/a}},$$

$$\lambda_2 = -\left(ae^{v/a}\sqrt{1-e^{-2v/a}}\right)^{-1}.$$

Remark that:

$$\int_M LdV = \int_0^\infty \frac{e^{t/a}}{a\sqrt{1-e^{-2t/a}}}dt = \frac{1}{2}\int_1^\infty \frac{dy}{\sqrt{y-1}} = \infty.$$

A natural question is to find a characterization for surfaces of rotation that have finite integral of the spread of shape operator.

Consider surfaces of revolution whose profile curves are described as $c(s) = (y(s), s)$ (see, for example, [8]). Then we have the following.

Proposition 3.3. *Let M be a surface of rotation in Euclidean 3-space defined by*

$$f(s,t) = (y(s)\cos t, y(s)\sin t, s).$$

Then the integral of the spread of the shape operator on M is finite if and only if there exists an integrable $C^\infty(\mathbb{R})$ function $f > 0$ which satisfies the following second order differential equation:

$$-yy'' = 1 + (y')^2 \pm f(s)y(1+(y')^2)^{\frac{3}{2}}.$$

Proof. For the proof, we use the classical formulas from [5, p. 228]. We have for $\lambda_1 = k_{meridian}$, and respectively for $\lambda_2 = k_{parallel}$:

$$\lambda_1 = \frac{-y''}{[1+(y')2]^{\frac{3}{2}}},$$
$$\lambda_2 = \frac{1}{y[1+(y')^2]^{\frac{1}{2}}}.$$

Then, the condition that the integral is finite means that there exists an integrable function $f > 0$ such that

$$\int_R |\lambda_1 - \lambda_2| ds = \int_R f(s)ds.$$

If we assume that $f \in C^\infty$, then the equality between the function under the integral holds everywhere and a straightforward computation yields the claimed equality. □

For example, for the catenoid $f(s) = 0$.

REFERENCES

[1] B.Y. CHEN, An invariant of conformal mappings, *Proc. Amer. Math. Soc.*, **40** (1973), 563–564.

[2] B.Y. CHEN, Some conformal invariants of submanifolds and their applications, *Boll. Un. Mat. Ital.*, **(4) 10** (1974) 380–385.

[3] B.Y. CHEN, *Total Mean Curvature and Submanifolds of Finite Type*, World Scientific, 1984.

[4] B.Y. CHEN, Mean curvature and shape operator of isometric immersions in real-space-forms, *Glasgow Math.J.*, **38** (1996) 87–97.

[5] A. GRAY, *Modern Differential Geometry of Curves and Surfaces with Matematica*, CRC Press, Second edition, 1998.

[6] M. MARCUS AND H. MINC, *A survey of matrix theory and matrix inequalities*, Prindle, Weber & Schmidt, 1969.

[7] L. MIRSKY, The spread of a matrix, *Mathematika*, **3**(1956), 127–130.

[8] M. SPIVAK, *A Comprehensive Introduction to Differential Geometry*, Volume three, Publish or Perish, Inc., Second edition, 1979.

[9] B. SUCEAVĂ, Some theorems on austere submanifolds, *Balkan. J. Geom. Appl.,* **2** (1997), 109–115.

[10] B. SUCEAVĂ, Remarks on B.Y. Chen's inequality involving classical invariants, *Anal. Şti. Univ. Al. I. Cuza Iaşi, Matematică,* **XLV** (1999), 405–412.