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WEIGHTED WEAK TYPE INEQUALITIES FOR THE HARDY OPERATOR WHEN $p = 1$

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Abstract

The paper studies the weighted weak type inequalities for the Hardy operator as an operator from weighted L^p to weighted weak L^q in the case $p = 1$. It considers two different versions of the Hardy operator and characterizes their weighted weak type inequalities when $p = 1$. It proves that for the classical Hardy operator, the weak type inequality is generally weaker when $q < p = 1$. The best constant in the inequality is also estimated.

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Contents

1. Introduction

The classical Hardy operator I is the integral operator $If(x) = \int_c^x f(t)dt$, where the lower limit c in the integral is generally taken to be 0 or $-\infty$, depending on the underlying space considered. In [4], Hardy first studied this operator from L^p to the weighted $L^p_{x^{-p}}$ when $p > 1$. The boundedness of this operator from L^p_u to L^q_v for general weights u, v and different pairs of indices p and q was considered in [12], [2], [11] and [16]. The boundedness of I is expressed by the strong type inequality

$$\left(\int If(x)^q v(x)dx\right)^{\frac{1}{q}} \leq C\left(\int f(y)^p u(y)dy\right)^{\frac{1}{p}}, \quad f \geq 0,$$

which is also called the weighted norm inequality when $p, q \geq 1$. When $p < 1$, the integral on the right hand side is no longer a norm, and the inequality is of little interest. Like other integral operators, the weighted strong type inequality for I always implies the weighted weak type inequality

$$\left(\int_{\{x:If(x)>\lambda\}} v(x)dx\right)^{\frac{1}{q}} \leq \frac{C}{\lambda}\left(\int f(y)^p u(y)dy\right)^{\frac{1}{p}}, \quad f \geq 0, \lambda > 0.$$

It is known that when $1 \leq p \leq q < \infty$, both the weighted strong type and weak type inequalities for the classical Hardy operator impose the same condition on the weights u and v. That is, for given u and v, either both inequalities hold or both fail. We say that the weighted strong type and weak type inequalities are equivalent. However, when $q < p$ and $1 < p < \infty$, the equivalence does not hold in general. Characteristics of weighted weak type inequalities for the

Hardy operator and modified Hardy operators were studied in [1], [3], [5], [7], [9], [10], [13], and [14]. This paper looks at the Hardy Operator and considers the weighted weak type inequalities in the special case $p = 1$.

The case $p = 1$ is subtle, because in this case we need to consider two different operators. If $p \neq 1$, considering inequalities for I from L^p_u to L^q_v is readily reduced to considering them for the operator

$$I_w f(x) = \int_c^x f(t)w(t)dt$$

from L^p_w to L^q_v, where $w = u^{1-p'}$ with $\frac{1}{p} + \frac{1}{p'} = 1$.

However, when $p = 1$, the inequalities for I do not reduce to those for the operator I_w, so we need to deal with them separately. In Section 2, a more general operator than I_w is considered. Instead of considering I_w, we consider the operator I_μ,

$$I_\mu f(x) = \int_c^x f(t)d\mu(t), \tag{1.1}$$

where μ is the σ-finite measure of the underlying space.

In Theorem 2.2, we show that the weighted weak type and strong type inequalities for I_μ are still equivalent. In Theorem 2.4, the weak type inequality for I, when $p = 1$ and $0 < q < \infty$, is considered. We will see that when $0 < q < 1 = p$, the weighted weak type inequality is weaker in general.

Throughout the paper, λ is an arbitrary positive number, acting in the weak type inequalities. The conventions of $0 \cdot \infty = 0, 0/0 = 0$, and $\infty/\infty = 0$ are used.

2. The Case $p = 1$ for the Hardy Operator

First let us consider the operator I_μ defined in (1.1), with $c = -\infty$ for convenience. The strong type inequalities for I_μ when $p = 1$ was studied in [15], and we state the result in the following proposition.

Proposition 2.1. *Suppose $0 < q < \infty$, and μ, ν are σ-finite measures on $\mathbb{R}$. The strong type inequality*

$$\left(\int_{-\infty}^{\infty} I_\mu f(x)^q d\nu\right)^{\frac{1}{q}} \le C \int_{-\infty}^{\infty} f(y) d\mu, \qquad f \ge 0, \tag{2.1}$$

holds if and only if

$$\int_E d\nu < \infty, \quad \text{where } E = \left\{x \in \mathbb{R} : \int_{-\infty}^{x} d\mu > 0\right\}. \tag{2.2}$$

In the next theorem, we show that condition (2.2) is also necessary and sufficient for the weak type inequality, in other words, the strong type and weak type inequalities for I_μ are equivalent when $p = 1$.

Theorem 2.2. *Suppose $0 < q < \infty$, and μ, ν are σ-finite measures on $\mathbb{R}$. Then the weak type inequality*

$$(\nu\{x : I_\mu f(x) > \lambda\})^{\frac{1}{q}} \le \frac{C}{\lambda} ||f||_{L^1_{d\mu}}, \quad f \ge 0, \tag{2.3}$$

and the strong type inequality (2.1) are equivalent.

Proof. Because the strong type inequality of an operator always implies the weak type inequality, we only need to prove (2.2) is also necessary for the weak type inequality (2.3).

Since $\int_{-\infty}^{x} d\mu$ is a non-decreasing function, the set E is an interval of the form $E = (z, \infty)$ or $E = [z, \infty)$. Suppose $E \neq \emptyset$, otherwise the proof is trivial.

If $z \neq -\infty$, then we firstly suppose that z is an atom for μ. Set $f(t) = (1/\mu\{z\})\chi_{\{z\}}(t)$. Since $z \in (-\infty, x]$ for every $x \in E$ we have $I_\mu f(x) = 1$. Thus

$$\begin{aligned}\left(\int_E d\nu\right)^{\frac{1}{q}} &\leq \left(\left\{x : I_\mu f(x) > \frac{1}{2}\right\}\right)^{\frac{1}{q}} \\ &\leq 2C||f||_{L^1_{d\mu}} = 2C < \infty.\end{aligned}$$

Secondly, suppose z is not an atom for μ. If we let $\epsilon > 0$, and $f(t) = [1/\mu(z, z+\epsilon)]\chi_{(z,z+\epsilon)}(t)$, then for every $x \in [z+\epsilon, \infty)$, we have $I_\mu f(x) = 1$ and hence

$$\begin{aligned}\left(\int_{z+\epsilon}^{\infty} d\nu\right)^{\frac{1}{q}} &\leq \left(\left\{x : I_\mu f(x) > \frac{1}{2}\right\}\right)^{\frac{1}{q}} \\ &\leq 2C||f||_{L^1_{d\mu}} = 2C < \infty.\end{aligned}$$

As $\epsilon \to 0^+$, we have $\left(\int_E d\nu\right)^{\frac{1}{q}} < \infty$, and (2.2) holds.

If $E = (-\infty, \infty)$, then we do the same discussion as above on the interval $[z, \infty)$ and then let $z \to -\infty$, and this completes the proof of Theorem 2.2. $\square$

Now let us consider the weighted weak type inequality for the classical Hardy operator I (with $c = 0$ for convenience). We make use of some of the techniques in [17]. Notice that in Theorem 2.2, the conclusion for I_μ is independent of the relation between the indices q and $p = 1$. The operator I is a little bit more subtle. It does matter whether $q < 1$ or $q \geq 1$.

Definition 2.1. *For a non-negative function u, define $\underline{u}$ by*

$$\underline{u}(x) = \operatorname*{essinf}_{0<t<x} u(t).$$

It is easy to see that $\underline{u}$ is non-increasing and $\underline{u} \leq u$ almost everywhere.

Lemma 2.3. *Suppose that $0 < q < \infty$ and that $k(x,t)$ is a non-negative kernel which is non-increasing in t for each x. Suppose u and v are non-negative functions. The best constant in the weighted weak type inequality*

$$\left(v\left\{ x : \int_0^\infty k(x,t) f(t) dt > \lambda \right\} \right)^{\frac{1}{q}} \leq \frac{C}{\lambda} \int_0^\infty f u \quad \text{for } f \geq 0,$$

is unchanged when u is replaced by $\underline{u}$.

Proof. Let C be the best constant in the above inequality and let $\underline{C}$ be the best constant in the above inequality with u replaced by $\underline{u}$. Since $\underline{u} \leq u$ almost everywhere, $C \leq \underline{C}$. To prove the reverse inequality it is enough to show that

$$\left(v\left\{ x : \int_{\underline{x}}^\infty k(x,t) f(t) dt > \lambda \right\} \right)^{\frac{1}{q}} \leq \frac{C}{\lambda} \int_{\underline{x}}^\infty f \underline{u} \tag{2.4}$$

for all non-negative $f \in L^1(\underline{x}, \infty)$, where $\underline{x} = \inf\{x \geq 0 : \underline{u}(x) < \infty\}$. The proof of Theorem 3.2 in [17] shows that for every non-negative $f \in L^1(\underline{x}, \infty)$

and any $\epsilon > 0$, there exists an f_ϵ such that

$$\int_{\underline{x}}^{\infty} f_\epsilon u \leq \int_{\underline{x}}^{\infty} f \underline{u} + 2\epsilon \int_{\underline{x}}^{\infty} f,$$

and

$$\int_{\underline{x}}^{\infty} k(x,t) f(t) dt \leq \liminf_{\epsilon \to 0+} \int_{\underline{x}}^{\infty} k(x,t) f_\epsilon(t) dt.$$

If $\liminf_{\epsilon \to 0+} \int_{\underline{x}}^{\infty} k(x,t) f_\epsilon(t) dt > \lambda$, then $\int_{\underline{x}}^{\infty} k(x,t) f_\epsilon(t) dt > \lambda$ for all sufficiently small $\epsilon > 0$. Thus, for all $x \geq \underline{x}$ and all $\lambda > 0$,

$$\chi_{\left\{x : \liminf_{\epsilon \to 0+} \int_{\underline{x}}^{\infty} k(x,t) f_\epsilon(t) dt > \lambda\right\}}(x) \leq \liminf_{\epsilon \to 0+} \chi_{\left\{x : \int_{\underline{x}}^{\infty} k(x,t) f_\epsilon(t) dt > \lambda\right\}}(x).$$

We use these estimates to obtain

$$\begin{aligned}
v & \left\{x : \int_{\underline{x}}^{\infty} k(x,t) f(t) dt > \lambda\right\} \\
& \leq v \left\{x : \liminf_{\epsilon \to 0+} \int_{\underline{x}}^{\infty} k(x,t) f_\epsilon(t) dt > \lambda\right\} \\
& = \int_0^{\infty} \chi_{\left\{x : \liminf_{\epsilon \to 0+} \int_{\underline{x}}^{\infty} k(x,t) f_\epsilon(t) dt > \lambda\right\}}(x) v(x) dx \\
& \leq \int_0^{\infty} \liminf_{\epsilon \to 0+} \chi_{\left\{x : \int_{\underline{x}}^{\infty} k(x,t) f_\epsilon(t) dt > \lambda\right\}}(x) v(x) dx \\
& \leq \liminf_{\epsilon \to 0+} \int_0^{\infty} \chi_{\left\{x : \int_{\underline{x}}^{\infty} k(x,t) f_\epsilon(t) dt > \lambda\right\}}(x) v(x) dx
\end{aligned}$$

$$
\begin{aligned}
&= \liminf_{\epsilon\to 0+} v\left\{x : \int_{\underline{x}}^{\infty} k(x,t) f_\epsilon(t)dt > \lambda\right\} \\
&\le \liminf_{\epsilon\to 0+} C^q \lambda^{-q} \left(\int_{\underline{x}}^{\infty} f_\epsilon u\right)^q \\
&\le C^q \lambda^{-q} \liminf_{\epsilon\to 0+} \left(\int_{\underline{x}}^{\infty} f\underline{u} + 2\epsilon \int_{\underline{x}}^{\infty} f\right)^q \\
&= C^q \lambda^{-q} \left(\int_{\underline{x}}^{\infty} f\underline{u}\right)^q,
\end{aligned}
$$

which gives (2.4) and completes the proof. □

Theorem 2.4. *Suppose $0 < q < \infty$, and u, v are non-negative functions on $\mathbb{R}$. Then the weak type inequality for the classical Hardy operator $If(x) = \int_0^x f(t)dt$,*

$$
(v\{x : If(x) > \lambda\})^{\frac{1}{q}} \le \frac{C}{\lambda} \int_0^{\infty} f(t)u(t)dt, \tag{2.5}
$$

holds for $f \ge 0$ if and only if

$$
\sup_{y>0} v(y,\infty)^{\frac{1}{q}} (\underline{u}(y))^{-1} = A < \infty. \tag{2.6}
$$

Moreover, $C = A$ is the best constant in (2.5).

Proof. Since $If(x) = \int_0^\infty \chi_{(0,x)}(t)f(t)dt$, the kernel $\chi_{(0,x)}(t)$ satisfies the hypotheses of Lemma 2.3. By Lemma 2.3, we only need to show that A is the best constant in

$$(v\{x : If(x) > \lambda\})^{\frac{1}{q}} \leq \frac{C}{\lambda}\int_0^\infty f\underline{u}. \tag{2.7}$$

We first consider the case $\underline{u} = \int_x^\infty b$ for some b satisfying

$$\int_x^\infty b < \infty \quad \text{for all } x > 0, \quad \text{and} \int_0^\infty b = \infty. \tag{2.8}$$

Then the right hand side of (2.7) becomes

$$\begin{aligned} \frac{C}{\lambda}\int_0^\infty f(t)\underline{u}(t)dt &= \frac{C}{\lambda}\int_0^\infty f(t)\left(\int_t^\infty b(x)dx\right)dt \\ &= \frac{C}{\lambda}\int_0^\infty \left(\int_0^x f\right)b(x)dx. \end{aligned}$$

Since any non-negative, non-decreasing function F is the limit of an increasing sequence of functions of the form $\int_0^x f$ with $f \geq 0$, it is sufficient to show that $C = A$ is also the best constant in the following inequality

$$v\{x : F(x) > \lambda\}^{\frac{1}{q}} \leq \frac{C}{\lambda}\int_0^\infty Fb, \quad \text{for } F \geq 0, \text{ and } F \text{ non-decreasing.} \tag{2.9}$$

Suppose that $A < \infty$ and F is non-decreasing, then $\{x : F(x) > \lambda\}$ is an interval of the form (y, ∞) or $[y, \infty)$. Since the left end point y does not change the integral, we have

$$\begin{aligned} v\{x : F(x) > \lambda\}^{\frac{1}{q}} &= v(y,\infty)^{\frac{1}{q}} \\ &\leq A\underline{u}(y) = A\int_y^\infty b \leq A\int_y^\infty \frac{F(x)}{\lambda}b = \frac{A}{\lambda}\int_y^\infty Fb, \end{aligned}$$

which gives (2.9) with the constant A.

Now suppose (2.9) holds. Fix $y > 0$. For a given $\epsilon > 0$, let $\lambda = 1 - \epsilon$, and $F(x) = \chi_{(y,\infty)}(x)$, then

$$v(y,\infty)^{\frac{1}{q}} = v\{x : F(x) > \lambda\}^{\frac{1}{q}} \le \frac{C}{\lambda}\int_0^\infty Fb = \frac{C}{1-\epsilon}\int_y^\infty b = \frac{C}{1-\epsilon}\underline{u}(y).$$

Letting $\epsilon \to 0^+$, we get

$$v(y,\infty)^{\frac{1}{q}}\underline{u}(y)^{-1} \le C.$$

In the case $\underline{u}(y) = 0$, we use the convention $0 \cdot \infty = 0$. Then we obtain $A \le C$, and also get that A is the best constant in (2.9).

Next we consider the case of general $\underline{u}$. We can assume that $\underline{u}(x) < \infty$ for all x, since if $\underline{u} = \infty$ on some interval $(0, x)$ then we translate $\underline{u}$ to the left to get a smaller $\underline{u}$ and reduce the problem to one in which this does not happen. Then for each $n > 0$, the function $\underline{u}\chi_{(0,n)}$ is finite, non-increasing and tends to 0 at ∞. We can approximate it from above by functions of the form $\int_x^\infty b$ with b satisfying (2.8). Let $\{u_m\}$ be a non-increasing sequence of such functions that converges to $\underline{u}\chi_{(0,n)}$ pointwise almost everywhere. Let $v_n = v\chi_{(0,n)}$, then the first part of the proof gives

$$v_n\left\{x : \int_0^x f(t)dt > \lambda\right\}^{\frac{1}{q}} \le \frac{1}{\lambda}\sup_{y>0} v_n(y,\infty)^{\frac{1}{q}} u_m(y)^{-1}\int_0^\infty f(t)u_m(t)dt, \quad f \ge 0,$$

which implies

$$v_n\left\{x:\int_0^x gu_m^{-1}>\lambda\right\}^{\frac{1}{q}}\le\frac{1}{\lambda}\sup_{y>0}v_n(y,\infty)^{\frac{1}{q}}u_m(y)^{-1}\int_0^\infty g,\quad g\ge 0.$$

The Monotone Convergence Theorem, and the fact $u_m(y)^{-1}<\underline{u}(y)^{-1}$ when $y\in(0,n)$ give

$$v_n\left\{x:\int_0^x g\underline{u}^{-1}>\lambda\right\}^{\frac{1}{q}}\le\frac{1}{\lambda}\sup_{y>0}v_n(y,\infty)^{\frac{1}{q}}\underline{u}(y)^{-1}\int_0^\infty g,\quad g\ge 0.$$

Let $f=g\underline{u}^{-1}$ to get

$$\begin{aligned}v_n\left\{x:\int_0^x f>\lambda\right\}^{\frac{1}{q}}&\le\frac{1}{\lambda}\sup_{y>0}v_n(y,\infty)^{\frac{1}{q}}\underline{u}(y)^{-1}\int_0^\infty f\underline{u}\\&\le\frac{A}{\lambda}\int_0^\infty f\underline{u},\quad f\ge 0.\end{aligned}$$

Let $n\to\infty$, we get (2.7) with the constant $C=A$.

Conversely, suppose (2.7) holds for some constant C. Since $v_n\le v$, then

$$\left(v_n\{x:I(f\chi_{(0,n)})(x)>\lambda\}\right)^{\frac{1}{q}}\le\frac{C}{\lambda}\int_0^\infty f\chi_{(0,n)}\underline{u}.$$

Note that $\underline{u}\chi_{(0,n)}\le u_m$, then we have

$$\left(v_n\{x:If(x)>\lambda\}\right)^{\frac{1}{q}}\le\frac{C}{\lambda}\int_0^\infty fu_m.$$

The first part of the proof gives

$$\sup_{y>0} v_n(y,\infty)^{\frac{1}{q}} u_m(y)^{-1} \leq C.$$

Then for every $y > 0$,

$$v_n(y,\infty)^{\frac{1}{q}} u_m(y)^{-1} \leq C,$$

which gives

$$v_n(y,\infty)^{\frac{1}{q}} \underline{u}(y)^{-1} \leq C,$$

when $m \to \infty$. Thus

$$\sup_{y>0} v(y,\infty)^{\frac{1}{q}} \underline{u}(y)^{-1} \leq C,$$

which is $A \leq C$. Since A itself is a constant such that (2.7) holds, A is the best constant in (2.7). Theorem 2.4 is proved. □

Remark 2.1. *Theorem 2.4 characterizes the weighted weak type inequality for the classical Hardy operator in the case $p = 1$. The theorem imposes no restriction on q, except that q is a positive number. In fact, different q reveals different information on the equivalence of the weak and strong type inequalities. Recall that when $0 < q < p = 1$, the weight characterization of the strong type inequality for I is (see [17])*

$$\int_0^\infty \underline{u}(x)^{q/(q-1)} \left(\int_x^\infty v\right)^{\frac{q}{1-q}} v(x)dx < \infty.$$

This condition is stronger than the condition (2.6) in general. For example, if we set $u(x) = x^{(\alpha+1)/q}$ and $v(x) = x^\alpha$ for some $\alpha < -1$, then the condition

(2.6) is satisfied but the above condition for the strong type inequality does not hold.

For the case $1 = p \leq q < \infty$, it is known that the weak and strong type inequalities for the operator I are equivalent. This conclusion can also be confirmed by 2.4. Recall that when $1 = p \leq q < \infty$, the necessary and sufficient condition of the strong type inequality for I is (see [2])

$$\sup_{r>0} \left(\int_r^\infty v \right)^{\frac{1}{q}} ||u^{-1}\chi_{(0,r)}||_{L^\infty} < \infty.$$

It is easy to see that $||u^{-1}\chi_{(0,r)}||_{L^\infty}$ coincides with $\underline{u}(r)^{-1}$ and hence we get (2.6).

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