

# Journal of Inequalities in Pure and Applied Mathematics

## ASYMPTOTIC BEHAVIOUR OF SOME EQUATIONS IN ORLICZ SPACES

D. MESKINE AND A. ELMAHI

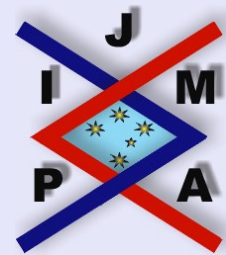
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Abstract

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## Abstract

In this paper, we prove an existence and uniqueness result for solutions of some bilateral problems of the form

$$\begin{cases} \langle Au, v - u \rangle \geq \langle f, v - u \rangle, \forall v \in K \\ u \in K \end{cases}$$

where  $A$  is a standard Leray-Lions operator defined on  $W_0^1 L_M(\Omega)$ , with  $M$  an N-function which satisfies the  $\Delta_2$ -condition, and where  $K$  is a convex subset of  $W_0^1 L_M(\Omega)$  with obstacles depending on some Carathéodory function  $g(x, u)$ . We consider first, the case  $f \in W^{-1} E_{\overline{M}}(\Omega)$  and secondly where  $f \in L^1(\Omega)$ . Our method deals with the study of the limit of the sequence of solutions  $u_n$  of some approximate problem with nonlinearity term of the form  $|g(x, u_n)|^{n-1} g(x, u_n) \times M(|\nabla u_n|)$ .

**2000 Mathematics Subject Classification:** 35J25, 35J60.

**Key words:** Strongly nonlinear elliptic equations, Natural growth, Truncations, Variational inequalities, Bilateral problems.



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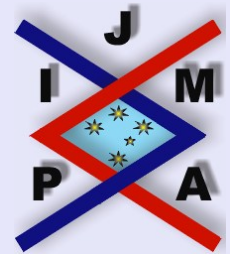
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# 1. Introduction

Let  $\Omega$  be an open bounded subset of  $\mathbb{R}^N$ ,  $N \geq 2$ , with the segment property. Consider the following obstacle problem:

$$(P) \quad \begin{cases} \langle Au, v - u \rangle \geq \langle f, v - u \rangle, \quad \forall v \in K, \\ u \in K, \end{cases}$$

where  $A(u) = -\operatorname{div}(a(x, u, \nabla u))$  is a Leray-Lions operator defined on  $W_0^1 L_M(\Omega)$ , with  $M$  being an  $N$ -function which satisfies the  $\Delta_2$ -condition and where  $K$  is a convex subset of  $W_0^1 L_M(\Omega)$ .

In the variational case (i.e. where  $f \in W^{-1} E_{\overline{M}}(\Omega)$ ), it is well known that problem (P) has been already studied by Gossez and Mustonen in [10].

In this paper, we consider a recent approach of penalization in order to prove an existence theorem for solutions of some bilateral problems of (P) type.

We recall that L. Boccardo and F. Murat, see [6], have approximated the model variational inequality:

$$\begin{cases} \langle -\Delta_p u, v - u \rangle \geq \langle f, v - u \rangle, \quad \forall v \in K \\ u \in K = \{v \in W_0^{1,p}(\Omega) : |v(x)| \leq 1 \text{ a.e. in } \Omega\}, \end{cases}$$

with  $f \in W^{-1,p'}(\Omega)$  and  $-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2} \nabla u)$ , by the sequence of problems:

$$\begin{cases} -\Delta_p u_n + |u_n|^{n-1} u_n = f \text{ in } \mathcal{D}'(\Omega) \\ u_n \in W_0^{1,p}(\Omega) \cap L^n(\Omega). \end{cases}$$



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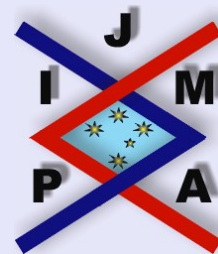
In [7], A. Dall’aglio and L. Orsina generalized this result by taking increasing powers depending also on some Carathéodory function  $g$  satisfying the sign condition and some hypothesis of integrability. Following this idea, we have studied in [5] the sequence of problems:

$$\begin{cases} -\Delta_p u_n + |g(x, u_n)|^{n-1} g(x, u_n) |\nabla u_n|^p = f \text{ in } \mathcal{D}'(\Omega) \\ u_n \in W_0^{1,p}(\Omega), |g(x, u_n)|^n |\nabla u_n|^p \in L^1(\Omega) \end{cases}$$

Here, we introduce the general sequence of equations in the setting of Orlicz-Sobolev spaces

$$\begin{cases} Au_n + |g(x, u_n)|^{n-1} g(x, u_n) M(|\nabla u_n|) = f \text{ in } \mathcal{D}'(\Omega) \\ u_n \in W_0^1 L_M(\Omega), |g(x, u_n)|^n M(|\nabla u_n|) \in L^1(\Omega). \end{cases}$$

We are interested throughout the paper in studying the limit of the sequence  $u_n$ . We prove that this limit satisfies some bilateral problem of the ( $\mathcal{P}$ ) form under some conditions on  $g$ . In the first we take  $f \in W^{-1} E_{\overline{M}}(\Omega)$  and next in  $L^1(\Omega)$ .



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## 2. Preliminaries

### 2.1. $N$ -Functions

Let  $M : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be an  $N$ -function, i.e.  $M$  is continuous, convex, with  $M(t) > 0$  for  $t > 0$ ,  $\frac{M(t)}{t} \rightarrow 0$  as  $t \rightarrow 0$  and  $\frac{M(t)}{t} \rightarrow \infty$  as  $t \rightarrow \infty$ .

Equivalently,  $M$  admits the representation:  $M(t) = \int_0^t a(s)ds$ , where  $a : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is nondecreasing, right continuous, with  $a(0) = 0$ ,  $a(t) > 0$  for  $t > 0$  and  $a(t)$  tends to  $\infty$  as  $t \rightarrow \infty$ .

The  $N$ -function  $\bar{M}$  conjugate to  $M$  is defined by  $\bar{M}(t) = \int_0^t \bar{a}(s)ds$ , where  $\bar{a} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is given by  $\bar{a}(t) = \sup\{s : a(s) \leq t\}$  (see [1]).

The  $N$ -function is said to satisfy the  $\Delta_2$  condition, denoted by  $M \in \Delta_2$ , if for some  $k > 0$ :

$$(2.1) \quad M(2t) \leq kM(t) \quad \forall t \geq 0;$$

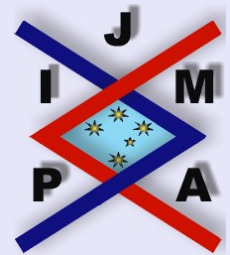
when (2.1) holds only for  $t \geq$  some  $t_0 > 0$  then  $M$  is said to satisfy the  $\Delta_2$  condition near infinity.

We will extend these  $N$ -functions into even functions on all  $\mathbb{R}$ .

Let  $P$  and  $Q$  be two  $N$ -functions.  $P \ll Q$  means that  $P$  grows essentially less rapidly than  $Q$ , i.e. for each  $\epsilon > 0$ ,  $\frac{P(t)}{Q(kt)} \rightarrow 0$  as  $t \rightarrow \infty$ . This is the case if and only if  $\lim_{t \rightarrow \infty} \frac{Q^{-1}(t)}{P^{-1}(t)} = 0$ .

### 2.2. Orlicz spaces

Let  $\Omega$  be an open subset of  $\mathbb{R}^N$ . The Orlicz class  $K_M(\Omega)$  (resp. the Orlicz space  $L_M(\Omega)$ ) is defined as the set of (equivalence classes of) real valued measurable



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functions  $u$  on  $\Omega$  such that:

$$\int_{\Omega} M(u(x))dx < +\infty \quad \left( \text{resp. } \int_{\Omega} M\left(\frac{u(x)}{\lambda}\right)dx < +\infty \text{ for some } \lambda > 0 \right).$$

$L_M(\Omega)$  is a Banach space under the norm

$$\|u\|_{M,\Omega} = \inf \left\{ \lambda > 0 : \int_{\Omega} M\left(\frac{u(x)}{\lambda}\right)dx \leq 1 \right\}$$

and  $K_M(\Omega)$  is a convex subset of  $L_M(\Omega)$ .

The closure in  $L_M(\Omega)$  of the set of bounded measurable functions with compact support in  $\bar{\Omega}$  is denoted by  $E_M(\Omega)$ .

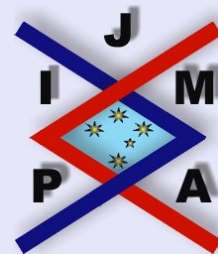
The equality  $E_M(\Omega) = L_M(\Omega)$  holds if only if  $M$  satisfies the  $\Delta_2$  condition, for all  $t$  or for  $t$  large according to whether  $\Omega$  has infinite measure or not.

The dual of  $E_M(\Omega)$  can be identified with  $L_{\bar{M}}(\Omega)$  by means of the pairing  $\int_{\Omega} uvdx$ , and the dual norm of  $L_{\bar{M}}(\Omega)$  is equivalent to  $\|\cdot\|_{\bar{M},\Omega}$ .

The space  $L_M(\Omega)$  is reflexive if and only if  $M$  and  $\bar{M}$  satisfy the  $\Delta_2$  condition, for all  $t$  or for  $t$  large, according to whether  $\Omega$  has infinite measure or not.

### 2.3. Orlicz-Sobolev spaces

We now turn to the Orlicz-Sobolev space,  $W^1L_M(\Omega)$  (resp.  $W^1E_M(\Omega)$ ) is the space of all functions  $u$  such that  $u$  and its distributional derivatives up to order 1 lie in  $L_M(\Omega)$  (resp.  $E_M(\Omega)$ ). It is a Banach space



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under the norm

$$\|u\|_{1,M} = \sum_{|\alpha| \leq 1} \|D^\alpha u\|_M.$$

Thus,  $W^1 L_M(\Omega)$  and  $W^1 E_M(\Omega)$  can be identified with subspaces of product of  $N + 1$  copies of  $L_M(\Omega)$ . Denoting this product by  $\prod L_M$ , we will use the weak topologies  $\sigma(\prod L_M, \prod E_{\overline{M}})$  and  $\sigma(\prod L_M, \prod L_{\overline{M}})$ .

The space  $W_0^1 E_M(\Omega)$  is defined as the (norm) closure of the Schwarz space  $D(\Omega)$  in  $W^1 E_M(\Omega)$  and the space  $W_0^1 L_M(\Omega)$  as the  $\sigma(\prod L_M, \prod E_{\overline{M}})$  closure of  $D(\Omega)$  in  $W^1 L_M(\Omega)$ .

We say that  $u_n$  converges to  $u$  for the modular convergence in  $W^1 L_M(\Omega)$  if for some  $\lambda > 0$

$$\int_{\Omega} M \left( \frac{D^\alpha u_n - D^\alpha u}{\lambda} \right) dx \rightarrow 0 \text{ for all } |\alpha| \leq 1.$$

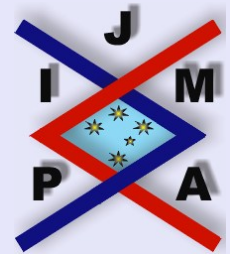
This implies convergence for  $\sigma(\prod L_M, \prod L_{\overline{M}})$ .

If  $M$  satisfies the  $\Delta_2$ -condition on  $\mathbb{R}^+$ , then modular convergence coincides with norm convergence.

## 2.4. The spaces $W^{-1} L_{\overline{M}}(\Omega)$ and $W^{-1} E_{\overline{M}}(\Omega)$

Let  $W^{-1} L_{\overline{M}}(\Omega)$  (resp.  $W^{-1} E_{\overline{M}}(\Omega)$ ) denote the space of distributions on  $\Omega$  which can be written as sums of derivatives of order  $\leq 1$  of functions in  $L_{\overline{M}}$  (resp.  $E_{\overline{M}}(\Omega)$ ). It is a Banach space under the usual quotient norm.

If the open set  $\Omega$  has the segment property then the space  $D(\Omega)$  is dense in  $W_0^{-1} L_{\overline{M}}(\Omega)$  for the modular convergence and thus for the topology  $\sigma(\prod L_M, \prod L_{\overline{M}})$



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(cf. [8, 9]). Consequently, the action of a distribution in  $W^{-1}L_{\overline{M}}(\Omega)$  on an element of  $W_0^1L_M(\Omega)$  is well defined.

## 2.5. Lemmas related to the Nemytskii operators in Orlicz spaces

We recall some lemmas introduced in [3] which will be used in this paper.

**Lemma 2.1.** *Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be uniformly Lipschitzian, with  $F(0) = 0$ . Let  $M$  be an  $N$ -function and let  $u \in W^1L_M(\Omega)$  (resp.  $W^1E_M(\Omega)$ ). Then  $F(u) \in W^1L_M(\Omega)$  (resp.  $W^1E_M(\Omega)$ ). Moreover, if the set  $D$  of discontinuity points of  $F'$  is finite, then*

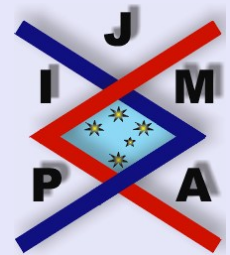
$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u) \frac{\partial}{\partial x_i} u & \text{a.e. in } \{x \in \Omega : u(x) \notin D\}, \\ 0 & \text{a.e. in } \{x \in \Omega : u(x) \in D\}. \end{cases}$$

**Lemma 2.2.** *Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be uniformly Lipschitzian, with  $F(0) = 0$ . We suppose that the set of discontinuity points of  $F'$  is finite. Let  $M$  be an  $N$ -function, then the mapping  $F : W^1L_M(\Omega) \rightarrow W^1L_M(\Omega)$  is sequentially continuous with respect to the weak\* topology  $\sigma(\prod L_M, \prod E_{\overline{M}})$ .*

## 2.6. Abstract lemma applied to the truncation operators

We now give the following lemma which concerns operators of the Nemytskii type in Orlicz spaces (see [3]).

**Lemma 2.3.** *Let  $\Omega$  be an open subset of  $\mathbb{R}^N$  with finite measure.*



Let  $M, P$  and  $Q$  be  $N$ -functions such that  $Q \ll P$ , and let  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  be a Carathéodory function such that a.e.  $x \in \Omega$  and all  $s \in \mathbb{R}$ :

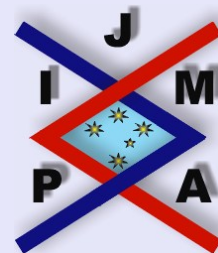
$$|f(x, s)| \leq c(x) + k_1 P^{-1} M(k_2 |s|),$$

where  $k_1, k_2$  are real constants and  $c(x) \in E_Q(\Omega)$ .

Then the Nemytskii operator  $N_f$  defined by  $N_f(u)(x) = f(x, u(x))$  is strongly continuous from

$$\mathcal{P} \left( E_M(\Omega), \frac{1}{k_2} \right) = \left\{ u \in L_M(\Omega) : d(u, E_M(\Omega)) < \frac{1}{k_2} \right\}$$

into  $E_Q(\Omega)$ .



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### 3. The Main Result

Let  $\Omega$  be an open bounded subset of  $\mathbb{R}^N$ ,  $N \geq 2$ , with the segment property.

Let  $M$  be an  $N$ -function satisfying the  $\Delta_2$ -condition near infinity.

Let  $A(u) = -\operatorname{div}(a(x, \nabla u))$  be a Leray-Lions operator defined on  $W_0^1 L_M(\Omega)$  into

$W^{-1} L_{\overline{M}}(\Omega)$ , where  $a : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$  is a Carathéodory function satisfying for a.e.  $x \in \Omega$  and for all  $\zeta, \zeta' \in \mathbb{R}^N$ , ( $\zeta \neq \zeta'$ ):

$$(3.1) \quad |a(x, \zeta)| \leq h(x) + \overline{M}^{-1} M(k_1 |\zeta|)$$

$$(3.2) \quad (a(x, \zeta) - a(x, \zeta'))(\zeta - \zeta') > 0$$

$$(3.3) \quad a(x, \zeta)\zeta \geq \alpha M\left(\frac{|\zeta|}{\lambda}\right)$$

with  $\alpha, \lambda > 0$ ,  $k_1 \geq 0$ ,  $h \in E_{\overline{M}}(\Omega)$ .

Furthermore, let  $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  be a Carathéodory function such that for a.e.  $x \in \Omega$  and for all  $s \in \mathbb{R}$ :

$$(3.4) \quad g(x, s)s \geq 0$$

$$(3.5) \quad |g(x, s)| \leq b(|s|)$$



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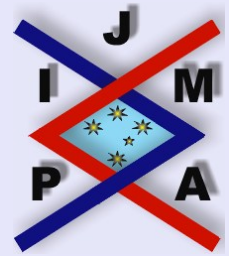


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$$(3.6) \quad \left\{ \begin{array}{l} \text{for almost } x \in \Omega \setminus \Omega_+^\infty \text{ there exists } \epsilon = \epsilon(x) > 0 \text{ such that:} \\ g(x, s) > 1, \forall s \in ]q_+(x), q_+(x) + \epsilon[; \\ \text{for almost } x \in \Omega \setminus \Omega_-^\infty \text{ there exists } \epsilon = \epsilon(x) > 0 \text{ such that:} \\ g(x, s) < -1, \forall s \in ]q_-(x) - \epsilon, q_-(x)[, \end{array} \right.$$

where  $b : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a continuous and nondecreasing function, with  $b(0) = 0$  and where

$$\begin{aligned} q_+(x) &= \inf\{s > 0 : g(x, s) \geq 1\} \\ q_-(x) &= \sup\{s < 0 : g(x, s) \leq -1\} \\ \Omega_+^\infty &= \{x \in \Omega : q_+(x) = +\infty\} \\ \Omega_-^\infty &= \{x \in \Omega : q_-(x) = -\infty\}. \end{aligned}$$

We define for  $s$  and  $k$  in  $\mathbb{R}$ ,  $k \geq 0$ ,  $T_k(s) = \max(-k, \min(k, s))$ .

**Theorem 3.1.** *Let  $f \in W^{-1}E_{\overline{M}}(\Omega)$ . Assume that (3.1) – (3.6) hold true and that the function  $s \rightarrow g(x, s)$  is nondecreasing for a.e.  $x \in \Omega$ . Then, for any real number  $\mu > 0$ , the problem*

$$(P_n) \quad \left\{ \begin{array}{l} A(u_n) + |g(x, u_n)|^{n-1} g(x, u_n) M \left( \frac{|\nabla u_n|}{\mu} \right) = f \text{ in } \mathcal{D}'(\Omega) \\ u_n \in W_0^1 L_M(\Omega), |g(x, u_n)|^n M \left( \frac{|\nabla u_n|}{\mu} \right) \in L^1(\Omega) \end{array} \right.$$

admits at least one solution  $u_n$  such that:

$$(3.7) \quad \forall k > 0 \quad T_k(u_n) \rightarrow T_k(u) \text{ for modular convergence in } W_0^1 L_M(\Omega)$$

where  $u$  is the unique solution of the following bilateral problem

$$(P) \quad \begin{cases} \langle Au, v - u \rangle \geq \langle f, v - u \rangle, \forall v \in K \\ u \in K = \{v \in W_0^1 L_M(\Omega) : q_- \leq v \leq q_+ \text{ a.e.}\}, \end{cases}$$

**Remark 3.1.** If the function  $s \rightarrow g(x, s)$  is strictly nondecreasing for a.e.  $x \in \Omega$  then the assumption (3.6) holds true.

*Proof.*

**Step 1:** A priori estimates.

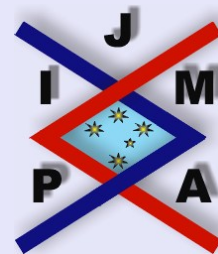
The existence of  $u_n$  is given by Theorem 3.1 of [3]. Choosing  $v = u_n$  as a test function in  $(P_n)$ , and using the sign condition (3.4), we get

$$\langle Au_n, u_n \rangle \leq \langle f, u_n \rangle.$$

By Proposition 5 of [11] one has:

$$(3.8) \quad \int_{\Omega} M \left( \frac{|\nabla u_n|}{\lambda} \right) dx \leq C, \text{ and } \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n dx \leq C,$$

$$(3.9) \quad (a(x, u_n, \nabla u_n)) \text{ is bounded in } (L_{\overline{M}}(\Omega))^N,$$



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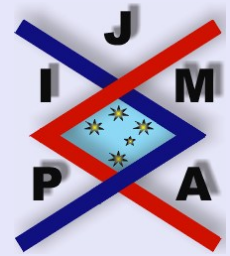


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$$(3.10) \quad \int_{\Omega} |g(x, u_n)|^{n-1} g(x, u_n) M \left( \frac{|\nabla u_n|}{\mu} \right) u_n dx \leq C.$$

We then deduce

$$\int_{\{|u_n|>k\}} |g(x, u_n)|^n M \left( \frac{|\nabla u_n|}{\mu} \right) dx \leq C, \quad \text{for all } k > 0.$$

Since  $b$  is continuous and since  $b(0) = 0$  there exists  $\delta > 0$  such that

$$b(|s|) \leq 1 \quad \text{for all } |s| \leq \delta.$$

On the other hand, by the  $\Delta_2$  condition there exist two positive constants  $K$  and  $K'$  such that

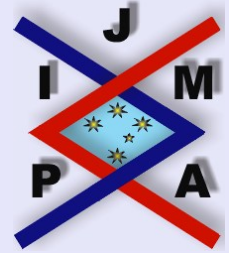
$$M \left( \frac{t}{\mu} \right) \leq KM \left( \frac{t}{\lambda} \right) + K' \quad \text{for all } t \geq 0,$$

which implies

$$\begin{aligned} \int_{\{|u_n| \leq \delta\}} |g(x, u_n)|^n M \left( \frac{|\nabla u_n|}{\mu} \right) dx \\ \leq \int_{\{|u_n| \leq \delta\}} \left( K' + KM \left( \frac{|\nabla u_n|}{\lambda} \right) \right) dx. \end{aligned}$$

Consequently from (3.8)

$$(3.11) \quad \int_{\Omega} |g(x, u_n)|^n M \left( \frac{|\nabla u_n|}{\mu} \right) dx \leq C, \quad \text{for all } n.$$



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**Step 2:** Almost everywhere convergence of the gradients.

Since  $(u_n)$  is a bounded sequence in  $W_0^1 L_M(\Omega)$  there exist some  $u \in W_0^1 L_M(\Omega)$  such that (for a subsequence still denoted by  $u_n$ )

$$(3.12) \quad u_n \rightharpoonup u \text{ weakly in } W_0^1 L_M(\Omega) \text{ for } \sigma \left( \prod L_M, \prod E_M \right),$$

strongly in  $E_M(\Omega)$  and a.e. in  $\Omega$ .

Furthermore, if we have

$$Au_n = f - |g(x, u_n)|^{n-1} g(x, u_n) M \left( \frac{|\nabla u_n|}{\mu} \right)$$

with  $|g(x, u_n)|^{n-1} g(x, u_n) M \left( \frac{|\nabla u_n|}{\mu} \right)$  being bounded in  $L^1(\Omega)$  then as in [2], one can show that

$$(3.13) \quad \nabla u_n \rightarrow \nabla u \text{ a.e. in } \Omega.$$

**Step 3:**  $u \in K = \{v \in W_0^1 L_M(\Omega) : q_- \leq v \leq q_+ \text{ a.e. in } \Omega\}$ .

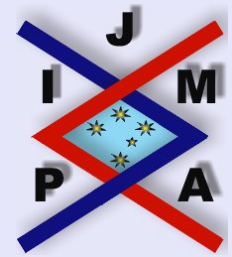
Since  $s \rightarrow g(x, s)$  is nondecreasing, then in view of (3.6), we have:

$$\{s \in \mathbb{R} : |g(x, s)| \leq 1 \text{ a.e. in } \Omega\} = \{s \in \mathbb{R} : q_- \leq s \leq q_+ \text{ a.e. in } \Omega\}.$$

It suffices to verify that  $|g(x, u)| \leq 1$  a.e.

We have

$$\int_{\Omega} |g(x, u_n)|^n M \left( \frac{|\nabla u_n|}{\mu} \right) dx \leq C,$$



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which gives

$$\int_{\{|g(x,u_n)|>k\}} |g(x, u_n)|^n M\left(\frac{|\nabla u_n|}{\mu}\right) dx \leq C$$

and

$$\int_{\{|g(x,u_n)|>k\}} M\left(\frac{|\nabla u_n|}{\mu}\right) dx \leq \frac{C}{k^n}$$

where  $k > 1$ . Letting  $n \rightarrow +\infty$  for  $k$  fixed, we deduce by using Fatou's lemma

$$\int_{\{|g(x,u)|>k\}} M\left(\frac{|\nabla u|}{\mu}\right) dx = 0$$

and so that,

$$|g(x, u)| \leq 1 \text{ a.e. in } \Omega.$$

#### Step 4: Strong convergence of the truncations.

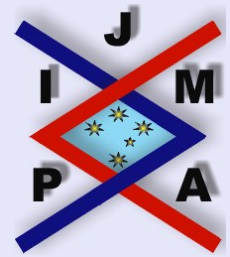
Let  $\phi(s) = s \exp(\gamma s^2)$ , where  $\gamma$  is chosen such that  $\gamma \geq (\frac{1}{\alpha})^2$ .

It is well known that  $\phi'(s) - \frac{2K}{\alpha} |\phi(s)| \geq \frac{1}{2}, \forall s \in \mathbb{R}$ , where  $K$  is a constant which will be used later. The use of the test function  $v_n = \phi(z_n)$  in  $(P_n)$  where  $z_n = T_k(u_n) - T_k(u)$  gives

$$\langle Au_n, \phi(z_n) \rangle + \int_{\Omega} |g(x, u_n)|^{n-1} g(x, u_n) M\left(\frac{|\nabla u_n|}{\mu}\right) \phi(z_n) dx = \langle f, \phi(z_n) \rangle$$

which implies, by using the fact that  $g(x, u_n)\phi(z_n) \geq 0$  on  $\{x \in \Omega :$





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$$\{|u_n| > k\},$$

$$\begin{aligned} \langle Au_n, \phi(z_n) \rangle + \int_{\{0 \leq u_n \leq T_k(u)\} \cap \{|u_n| \leq k\}} |g(x, u_n)|^{n-1} \\ \times g(x, u_n) M\left(\frac{|\nabla u_n|}{\mu}\right) \phi(z_n) dx \\ + \int_{\{T_k(u) \leq u_n \leq 0\} \cap \{|u_n| \leq k\}} |g(x, u_n)|^{n-1} \\ \times g(x, u_n) M\left(\frac{|\nabla u_n|}{\mu}\right) \phi(z_n) dx \leq \langle f, \phi(z_n) \rangle. \end{aligned}$$

The second and the third terms of the last inequality will be denoted respectively by  $I_{n,k}^1$  and  $I_{n,k}^2$  and  $\epsilon_i(n)$  denote various sequences of real numbers which tend to 0 as  $n \rightarrow +\infty$ .

On the one hand we have

$$\begin{aligned} |I_{n,k}^1| &\leq \int_{\{0 \leq u_n \leq T_k(u)\} \cap \{|u_n| \leq k\}} |g(x, u_n)|^n M\left(\frac{|\nabla u_n|}{\mu}\right) |\phi(z_n)| dx \\ &\leq \int_{\{0 \leq u_n \leq u\} \cap \{|u_n| \leq k\}} |g(x, u_n)|^n M\left(\frac{|\nabla u_n|}{\mu}\right) |\phi(z_n)| dx, \end{aligned}$$

but since  $|g(x, u_n)| \leq 1$  on  $\{x \in \Omega : 0 \leq u_n \leq u\}$ , then we have

$$|I_{n,k}^1| \leq \int_{\{|u_n| \leq k\}} M\left(\frac{|\nabla u_n|}{\mu}\right) |\phi(z_n)| dx.$$



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By using the fact that

$$M\left(\frac{|\nabla u_n|}{\mu}\right) \leq K' + KM\left(\frac{|\nabla u_n|}{\lambda}\right)$$

we obtain

$$|I_{n,k}^1| \leq \int_{\Omega} K' |\phi(z_n)| dx + \frac{K}{\alpha} \int_{\Omega} a(x, \nabla T_k(u_n)) \nabla T_k(u_n) |\phi(z_n)| dx,$$

which gives

$$(3.14) \quad |I_{n,k}^1| \leq \epsilon_1(n) + \frac{K}{\alpha} \int_{\Omega} a(x, \nabla T_k(u_n)) \nabla T_k(u_n) |\phi(z_n)| dx.$$

Similarly,

$$(3.15) \quad |I_{n,k}^2| \leq \int_{\{|u_n| \leq k\}} M\left(\frac{|\nabla u_n|}{\mu}\right) |\phi(z_n)| dx \\ \leq \epsilon_1(n) + \frac{K}{\alpha} \int_{\Omega} a(x, \nabla T_k(u_n)) \nabla T_k(u_n) |\phi(z_n)| dx.$$

The first term on the left hand side of the last inequality can be written as:

$$(3.16) \quad \int_{\Omega} a(x, \nabla u_n) [\nabla T_k(u_n) - \nabla T_k(u)] \phi'(z_n) dx \\ = \int_{\{|u_n| \leq k\}} a(x, \nabla u_n) [\nabla T_k(u_n) - \nabla T_k(u)] \phi'(z_n) dx \\ - \int_{\{|u_n| > k\}} a(x, \nabla u_n) \nabla T_k(u) \phi'(z_n) dx.$$

For the second term on the right hand side of the last equality, we have

$$\left| \int_{\{|u_n|>k\}} a(x, \nabla u_n) \nabla T_k(u) \phi'(z_n) dx \right| \leq C_k \int_{\Omega} |a(x, \nabla u_n)| |\nabla T_k(u)| \chi_{\{|u_n|>k\}} dx.$$

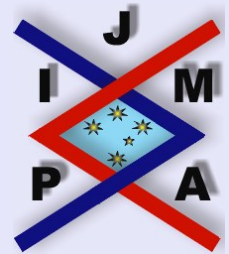
The right hand side of the last inequality tends to 0 as  $n$  tends to infinity. Indeed, the sequence  $(a(x, \nabla u_n))_n$  is bounded in  $(L_{\overline{M}}(\Omega))^N$  while  $\nabla T_k(u) \chi_{\{|u_n|>k\}}$  tends to 0 strongly in  $(E_M(\Omega))^N$ .

We define for every  $s > 0$ ,  $\Omega_s = \{x \in \Omega : |\nabla T_k(u(x))| \leq s\}$  and we denote by  $\chi_s$  its characteristic function. For the first term of the right hand side of (3.16), we can write

$$\begin{aligned} (3.17) \quad & \int_{\{|u_n| \leq k\}} a(x, \nabla u_n) [\nabla T_k(u_n) - \nabla T_k(u)] \phi'(z_n) dx \\ &= \int_{\Omega} [a(x, \nabla T_k(u_n)) - a(x, \nabla T_k(u) \chi_s)] [\nabla T_k(u_n) - \nabla T_k(u) \chi_s] \phi'(z_n) dx \\ & \quad + \int_{\Omega} a(x, \nabla T_k(u) \chi_s) [\nabla T_k(u_n) - \nabla T_k(u) \chi_s] \phi'(z_n) dx \\ & \quad - \int_{\Omega} a(x, \nabla T_k(u_n)) \nabla T_k(u) \chi_{\Omega \setminus \Omega_s} \phi'(z_n) dx. \end{aligned}$$

The second term of the right hand side of (3.17) tends to 0 since

$$a(x, \nabla T_k(u_n) \chi_s) \phi'(z_n) \rightarrow a(x, \nabla T_k(u) \chi_s) \text{ strongly in } (E_{\overline{M}}(\Omega))^N$$



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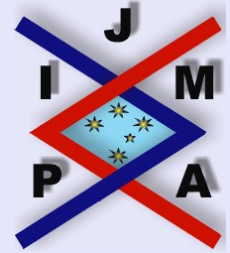


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by Lemma 2.3 and

$$\nabla T_k(u_n) \rightharpoonup \nabla T_k(u) \text{ weakly in } (L_M(\Omega))^N \text{ for } \sigma \left( \prod L_M(\Omega), \prod E_{\overline{M}}(\Omega) \right).$$

The third term of (3.17) tends to  $-\int_{\Omega} a(x, \nabla T_k(u)) \nabla T_k(u) \chi_{\Omega \setminus \Omega_s} dx$  as  $n \rightarrow \infty$  since

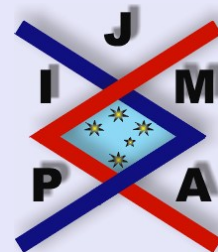
$$a(x, \nabla T_k(u_n)) \rightharpoonup a(x, \nabla T_k(u)) \text{ weakly for } \sigma \left( \prod E_{\overline{M}}(\Omega), \prod L_M(\Omega) \right).$$

Consequently, from (3.16) we have

$$\begin{aligned} (3.18) \quad & \int_{\Omega} a(x, \nabla u_n) [\nabla T_k(u_n) - \nabla T_k(u)] \phi'(z_n) dx \\ &= \int_{\Omega} [a(x, \nabla T_k(u_n)) - a(x, \nabla T_k(u) \chi_s)] \\ & \quad \times [\nabla T_k(u_n) - \nabla T_k(u) \chi_s] \phi'(z_n) dx + \epsilon_2(n). \end{aligned}$$

We deduce that, in view of (3.17) and (3.18),

$$\begin{aligned} & \int_{\Omega} [a(x, \nabla T_k(u_n)) - a(x, \nabla T_k(u) \chi_s)] \\ & \quad \times [\nabla T_k(u_n) - \nabla T_k(u) \chi_s] \left( \phi'(z_n) - \frac{2K}{\alpha} |\phi(z_n)| \right) dx \\ & \leq \epsilon_3(n) + \int_{\Omega} a(x, \nabla T_k(u)) \nabla T_k(u) \chi_{\Omega \setminus \Omega_s} dx, \end{aligned}$$



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and so

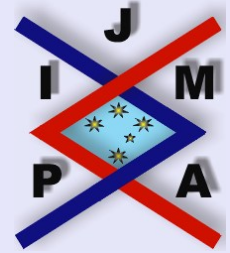
$$\begin{aligned} \int_{\Omega} [a(x, \nabla T_k(u_n)) - a(x, \nabla T_k(u)\chi_s)] [\nabla T_k(u_n) - \nabla T_k(u)\chi_s] dx \\ \leq 2\epsilon_3(n) + 2 \int_{\Omega} a(x, \nabla T_k(u)) \nabla T_k(u) \chi_{\Omega \setminus \Omega_s} dx. \end{aligned}$$

Hence

$$\begin{aligned} \int_{\Omega} a(x, \nabla T_k(u_n)) \nabla T_k(u_n) dx \\ \leq \int_{\Omega} a(x, \nabla T_k(u_n)) \nabla T_k(u) \chi_s dx \\ + \int_{\Omega} a(x, \nabla T_k(u) \chi_s) [\nabla T_k(u_n) - \nabla T_k(u) \chi_s] dx \\ + 2\epsilon_3(n) + 2 \int_{\Omega} a(x, \nabla T_k(u)) \nabla T_k(u) \chi_{\Omega \setminus \Omega_s} dx. \end{aligned}$$

Now considering the limit sup over  $n$ , one has

$$\begin{aligned} (3.19) \quad \limsup_{n \rightarrow +\infty} \int_{\Omega} a(x, \nabla T_k(u_n)) \nabla T_k(u_n) dx \\ \leq \limsup_{n \rightarrow +\infty} \int_{\Omega} a(x, \nabla T_k(u_n)) \nabla T_k(u) \chi_s dx \\ + \limsup_{n \rightarrow +\infty} \int_{\Omega} a(x, \nabla T_k(u) \chi_s) [\nabla T_k(u_n) - \nabla T_k(u) \chi_s] dx \\ + 2 \int_{\Omega} a(x, \nabla T_k(u)) \nabla T_k(u) \chi_{\Omega \setminus \Omega_s} dx. \end{aligned}$$



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The second term of the right hand side of the inequality (3.19) tends to 0, since

$$a(x, \nabla T_k(u_n)\chi_s) \rightarrow a(x, \nabla T_k(u)\chi_s) \text{ strongly in } E_{\overline{M}}(\Omega),$$

while  $\nabla T_k(u_n)$  tends weakly to  $\nabla T_k(u)$ .

The first term of the right hand side of (3.19) tends to  $\int_{\Omega} a(x, \nabla T_k(u)) \nabla T_k(u)\chi_s dx$  since

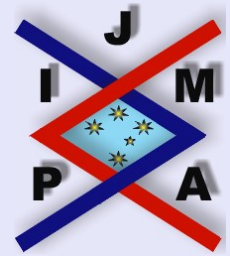
$$a(x, \nabla T_k(u_n)) \rightarrow a(x, \nabla T_k(u)) \text{ weakly in } (L_{\overline{M}}(\Omega))^N$$

for  $\sigma(\prod L_{\overline{M}}, \prod E_M)$  while  $\nabla T_k(u)\chi_s \in E_M(\Omega)$ . We deduce then

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \int_{\Omega} a(x, \nabla T_k(u_n)) \nabla T_k(u_n) dx \\ \leq \int_{\Omega} a(x, \nabla T_k(u)) \nabla T_k(u)\chi_s dx \\ + 2 \int_{\Omega} a(x, \nabla T_k(u)) \nabla T_k(u)\chi_{\Omega \setminus \Omega_s} dx, \end{aligned}$$

by using the fact that  $a(x, \nabla T_k(u)) \nabla T_k(u) \in L^1(\Omega)$  and letting  $s \rightarrow \infty$  we get, since  $meas(\Omega \setminus \Omega_s) \rightarrow 0$

$$\limsup_{n \rightarrow +\infty} \int_{\Omega} a(x, \nabla T_k(u_n)) \nabla T_k(u_n) dx \leq \int_{\Omega} a(x, \nabla T_k(u)) \nabla T_k(u) dx$$



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which gives, by using Fatou's lemma,

$$(3.20) \quad \lim_{n \rightarrow +\infty} \int_{\Omega} a(x, \nabla T_k(u_n)) \nabla T_k(u_n) dx = \int_{\Omega} a(x, \nabla T_k(u)) \nabla T_k(u) dx.$$

On the other hand, we have

$$M \left( \frac{|\nabla T_k(u_n)|}{\mu} \right) \leq K' + \frac{K}{\alpha} \int_{\Omega} a(x, \nabla T_k(u_n)) \nabla T_k(u_n) dx,$$

then by using (3.20) and Vitali's theorem, one easily has

$$(3.21) \quad M \left( \frac{|\nabla T_k(u_n)|}{\mu} \right) \rightarrow M \left( \frac{|\nabla T_k(u)|}{\mu} \right) \text{ strongly in } L^1(\Omega).$$

By writing

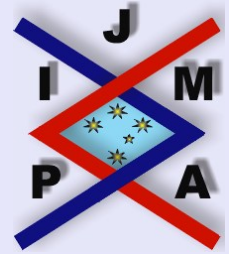
$$(3.22) \quad M \left( \frac{|\nabla T_k(u_n) - \nabla T_k(u)|}{2\mu} \right) \leq \frac{M \left( \frac{|\nabla T_k(u_n)|}{\mu} \right)}{2} + \frac{M \left( \frac{|\nabla T_k(u)|}{\mu} \right)}{2}$$

one has, by (3.21) and Vitali's theorem again,

$$(3.23) \quad T_k(u_n) \rightarrow T_k(u) \text{ for modular convergence in } W_0^1 L_M(\Omega).$$

**Step 5:**  $u$  is the solution of the variational inequality (P).

Choosing  $w = T_k(u_n - \theta T_m(v))$  as a test function in  $(P_n)$ , where  $v \in K$



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and  $0 < \theta < 1$ , gives

$$\begin{aligned} & \langle Au_n, T_k(u_n - \theta T_m(v)) \rangle \\ & + \int_{\Omega} |g(x, u_n)|^{n-1} g(x, u_n) M \left( \frac{|\nabla u_n|}{\mu} \right) T_k(u_n - \theta T_m(v)) dx \\ & = \langle f, T_k(u_n - \theta T_m(v)) \rangle, \end{aligned}$$

since  $g(x, u_n) T_k(u_n - \theta T_m(v)) \geq 0$  on

$$\{x \in \Omega : u_n \geq 0 \text{ and } u_n \geq \theta T_m(v)\} \cup \{x \in \Omega : u_n \leq 0 \text{ and } u_n \leq \theta T_m(v)\}$$

we have

$$\begin{aligned} & \int_{\Omega} |g(x, u_n)|^{n-1} g(x, u_n) M \left( \frac{|\nabla u_n|}{\mu} \right) T_k(u_n - \theta T_m(v)) dx \\ & \geq \int_{\{0 \leq u_n \leq \theta T_m(v)\}} |g(x, u_n)|^{n-1} g(x, u_n) M \left( \frac{|\nabla u_n|}{\mu} \right) T_k(u_n - \theta T_m(v)) dx \\ & + \int_{\{\theta T_m(v) \leq u_n \leq 0\}} |g(x, u_n)|^{n-1} g(x, u_n) M \left( \frac{|\nabla u_n|}{\mu} \right) T_k(u_n - \theta T_m(v)) dx. \end{aligned}$$

The first and the second terms in the right hand side of the last inequality will be denoted respectively by  $J_{n,m}^1$  and  $J_{n,m}^2$ .

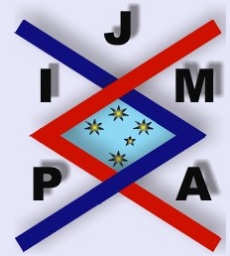
Defining

$$\delta_{1,m}(x) = \sup_{0 \leq s \leq \theta T_m(v)} g(x, s)$$

we get  $0 \leq \delta_{1,m}(x) < 1$  a.e. and

$$|J_{n,m}^1| \leq k \int_{\{0 \leq u_n \leq \theta T_m(v)\}} (\delta_{1,m}(x))^n M \left( \frac{|\nabla u_n|}{\mu} \right) dx.$$





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Since

$$\left| (\delta_{1,m}(x))^n M \left( \frac{|\nabla u_n|}{\mu} \right) \chi_{\{|u_n| \leq m\}} \right| \leq M \left( \frac{|\nabla T_m(u_n)|}{\mu} \right),$$

we have then by using (3.23) and Lebesgue's theorem

$$J_{n,m}^1 \longrightarrow 0 \text{ as } n \rightarrow +\infty.$$

Similarly

$$|J_{n,m}^2| \leq k \int_{\{|u_n| \leq m\}} |\delta_{2,m}(x)|^n M \left( \frac{|\nabla T_m(u_n)|}{\mu} \right) dx \rightarrow 0 \text{ as } n \rightarrow +\infty,$$

where

$$\delta_{2,m}(x) = \inf_{\theta T_m(v) \leq s \leq 0} g(x, s).$$

On the other hand, by using Fatou's lemma and the fact that

$$a(x, \nabla u_n) \rightarrow a(x, \nabla u) \text{ weakly in } (L_{\overline{M}}(\Omega))^N \text{ for } \sigma(\Pi L_{\overline{M}}, \Pi E_M),$$

one easily has

$$\liminf_{n \rightarrow +\infty} \langle Au_n, T_k(u_n - \theta T_m(v)) \rangle \leq \langle Au, T_k(u - \theta T_m(v)) \rangle.$$

Consequently

$$\langle Au, T_k(u - \theta T_m(v)) \rangle \leq \langle f, T_k(u - \theta T_m(v)) \rangle,$$

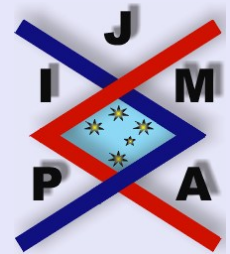
this implies that by letting  $k \rightarrow +\infty$ , since  $T_k(u - \theta T_m(v)) \rightarrow u - \theta T_m(v)$  for modular convergence in  $W_0^1 L_M(\Omega)$ ,

$$\langle Au, u - \theta T_m(v) \rangle \leq \langle f, u - \theta T_m(v) \rangle,$$

in which we can easily pass to the limit as  $\theta \rightarrow 1$  and  $m \rightarrow +\infty$  to obtain

$$\langle Au, u - v \rangle \leq \langle f, u - v \rangle.$$

□




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## 4. The $L^1$ Case

In this section, we study the same problems as before but we assume that  $q_-$  and  $q_+$  are bounded.

**Theorem 4.1.** *Let  $f \in L^1(\Omega)$ . Assume that the hypotheses are as in Theorem 3.1,  $q_-$  and  $q_+$  belong to  $L^\infty(\Omega)$ . Then the problem  $(P_n)$  admits at least one solution  $u_n$  such that:*

$$u_n \rightarrow u \text{ for modular convergence in } W_0^1 L_M(\Omega),$$

where  $u$  is the unique solution of the bilateral problem:

$$(Q) \quad \begin{cases} \langle Au, v - u \rangle \geq \int_{\Omega} f(v - u) dx, \forall v \in K \\ u \in K = \{v \in W_0^1 L_M(\Omega) : q_- \leq v \leq q_+ \text{ a.e.}\}. \end{cases}$$

*Proof.* We sketch the proof since the steps are similar to those in Section 3.

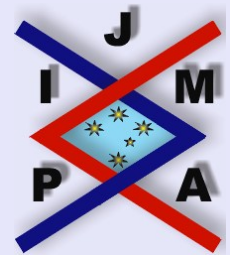
The existence of  $u_n$  is given by Theorem 1 of [4]. Indeed, it is easy to see that  $|g(x, s)| \geq 1$  on  $\{|s| \geq \gamma\}$ , where  $\gamma = \max\{\text{supess } q_+, -\text{infess } q_-\}$  and so that

$$|g(x, s)|^n M\left(\frac{|\zeta|}{\mu}\right) \geq M\left(\frac{|\zeta|}{\mu}\right) \text{ for } |s| \geq \gamma.$$

**Step 1:** A priori estimates.

Choosing  $v = T_\gamma(u_n)$ , as a test function in  $(P_n)$ , and using the sign condition (3.4), we obtain

$$(4.1) \quad \alpha \int_{\Omega} M\left(\frac{|\nabla T_\gamma(u_n)|}{\lambda}\right) dx \leq \gamma \|f\|_1$$



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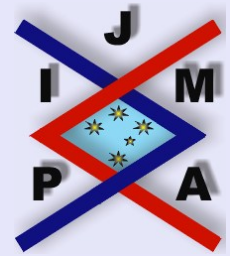


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and

$$\int_{\{|u_n|>\gamma\}} |g(x, u_n)|^n M \left( \frac{|\nabla u_n|}{\mu} \right) dx \leq \|f\|_1,$$

which gives

$$\int_{\{|u_n|>\gamma\}} M \left( \frac{|\nabla u_n|}{\mu} \right) dx \leq C$$

and finally

$$(4.2) \quad \int_{\Omega} M \left( \frac{|\nabla u_n|}{\max\{\lambda, \mu\}} \right) dx \leq C.$$

On the other hand, as in Section 3, we have

$$(4.3) \quad \int_{\Omega} |g(x, u_n)|^n M \left( \frac{|\nabla u_n|}{\mu} \right) dx \leq C.$$

**Step 2:** Almost everywhere convergence of the gradients.

Due to (4.2), there exists some  $u \in W_0^1 L_M(\Omega)$  such that (for a subsequence)

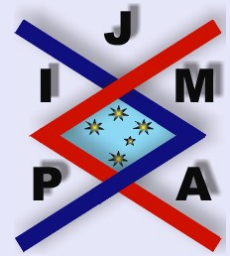
$$u_n \rightharpoonup u \text{ weakly in } W_0^1 L_M(\Omega) \text{ for } \sigma(\Pi L_M, \Pi E_{\overline{M}}).$$

Write

$$Au_n = f - |g(x, u_n)|^{n-1} g(x, u_n) M \left( \frac{|\nabla u_n|}{\mu} \right)$$

and remark that, by (4.2), the second hand side is uniformly bounded in  $L^1(\Omega)$ . Then as in Section 3

$$\nabla u_n \rightarrow \nabla u \text{ a.e. in } \Omega.$$



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**Step 3:**  $u \in K = \{v \in W_0^1 L_M(\Omega) : q_- \leq v \leq q_+ \text{ a.e. in } \Omega\}$ .

Similarly, as in the proof of Theorem 3.1, one can prove this step with the aid of property (4.3).

**Step 4:** Strong convergence of the truncations.

It is easy to see that the proof is the same as in Section 3.

**Step 5:**  $u$  is the solution of the bilateral problem  $(Q)$ .

Let  $v \in K$  and  $0 < \theta < 1$ . Taking  $v_n = T_k(u_n - \theta v)$ ,  $k > 0$  as a test function in  $(P_n)$ , one can see that the proof is the same by replacing  $T_m(v)$  with  $v$  in Section 3. We remark that  $K \subset L^\infty(\Omega)$ .

**Step 6:**  $u_n \rightarrow u$  for modular convergence in  $W_0^1 L_M(\Omega)$ .

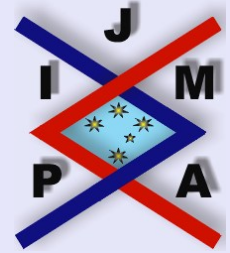
We shall prove that  $\nabla u_n \rightarrow \nabla u$  in  $(L_M(\Omega))^N$  for the modular convergence by using Vitali's theorem.

Let  $E$  be a measurable subset of  $\Omega$ , we have for any  $k > 0$

$$\begin{aligned} \int_E M\left(\frac{|\nabla u_n|}{\mu}\right) dx &\leq \int_{E \cap \{|u_n| \leq k\}} M\left(\frac{|\nabla u_n|}{\mu}\right) dx + \int_{E \cap \{|u_n| > k\}} M\left(\frac{|\nabla u_n|}{\mu}\right) dx. \end{aligned}$$

Let  $\epsilon > 0$ . By virtue of the modular convergence of the truncates, there exists some  $\eta(\epsilon, k)$  such that for any  $E$  measurable

$$(4.4) \quad |E| < \eta(\epsilon, k) \Rightarrow \int_{E \cap \{|u_n| \leq k\}} M\left(\frac{|\nabla u_n|}{\mu}\right) dx < \frac{\epsilon}{2}, \quad \forall n.$$



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Choosing  $T_1(u_n - T_k(u_n))$ , with  $k > 0$  a test function in  $(P_n)$  we obtain:

$$\begin{aligned} & \langle Au_n, T_1(u_n - T_k(u_n)) \rangle \\ & + \int_{\Omega} |g(x, u_n)|^{n-1} g(x, u_n) M \left( \frac{|\nabla u_n|}{\mu} \right) T_1(u_n - T_k(u_n)) dx \\ & = \int_{\Omega} f T_1(u_n - T_k(u_n)) dx, \end{aligned}$$

which implies

$$\int_{\{|u_n|>k+1\}} |g(x, u_n)|^n M \left( \frac{|\nabla u_n|}{\mu} \right) dx \leq \int_{\{|u_n|>k\}} |f| dx.$$

Note that  $meas\{x \in \Omega : |u_n(x)| > k\} \rightarrow 0$  uniformly on  $n$  when  $k \rightarrow \infty$ .  
 We deduce then that there exists  $k = k(\epsilon)$  such that

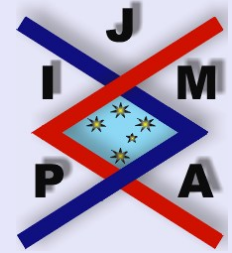
$$\int_{\{|u_n|>k\}} |f| dx < \frac{\epsilon}{2}, \quad \forall n,$$

which gives

$$\int_{\{|u_n|>k+1\}} |g(x, u_n)|^n M \left( \frac{|\nabla u_n|}{\mu} \right) dx < \frac{\epsilon}{2}, \quad \forall n.$$

By setting  $t(\epsilon) = \max\{k + 1, \gamma\}$  we obtain

$$(4.5) \quad \int_{\{|u_n|>t(\epsilon)\}} M \left( \frac{|\nabla u_n|}{\mu} \right) dx < \frac{\epsilon}{2}, \quad \forall n.$$



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Combining (4.4) and (4.5) we deduce that there exists  $\eta > 0$  such that

$$\int_E M\left(\frac{|\nabla u_n|}{\mu}\right) < \epsilon, \quad \forall n \text{ when } |E| < \eta, \quad E \text{ measurable,}$$

which shows the equi-integrability of  $M\left(\frac{|\nabla u_n|}{\mu}\right)$  in  $L^1(\Omega)$ , and therefore we have

$$M\left(\frac{|\nabla u_n|}{\mu}\right) \rightarrow M\left(\frac{|\nabla u|}{\mu}\right) \text{ strongly in } L^1(\Omega).$$

By remarking that

$$M\left(\frac{|\nabla u_n - \nabla u|}{2\mu}\right) \leq \frac{1}{2} \left[ M\left(\frac{|\nabla u_n|}{\mu}\right) + M\left(\frac{|\nabla u|}{\mu}\right) \right]$$

one easily has, by using the Lebesgue theorem

$$\int_{\Omega} M\left(\frac{|\nabla u_n - \nabla u|}{2\mu}\right) dx \rightarrow 0 \text{ as } n \rightarrow +\infty,$$

which completes the proof. □

**Remark 4.1.** *The condition  $b(0) = 0$  is not necessary. Indeed, taking  $\theta_h(u_n)$ ,  $h > 0$ , as a test function in  $(P_n)$  with*

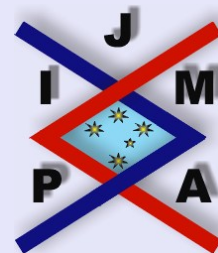
$$\theta_h(s) = \begin{cases} hs & \text{if } |s| \leq \frac{1}{h} \\ \text{sgn}(s) & \text{if } |s| \geq \frac{1}{h}, \end{cases}$$

we obtain

$$\int_{\Omega} |g(x, u_n)|^{n-1} g(x, u_n) M\left(\frac{|\nabla u_n|}{\mu}\right) \theta_h(u_n) dx \leq \int_{\Omega} f \theta_h(u_n) dx.$$

and then, by letting  $h \rightarrow +\infty$ ,

$$\int_{\Omega} |g(x, u_n)|^n M\left(\frac{|\nabla u_n|}{\mu}\right) dx \leq C.$$



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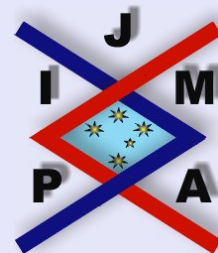
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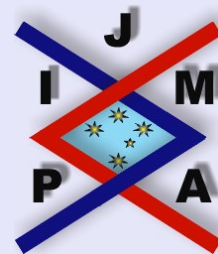
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