Journal of Inequalities in Pure and Applied Mathematics

ASYMPTOTIC BEHAVIOUR OF SOME EQUATIONS IN ORLICZ SPACES

D. MESKINE AND A. ELMAHI

Département de Mathématiques et Informatique
Faculté des Sciences Dhar-Mahraz
B.P 1796 Atlas-Fès,Fès Maroc.
EMail: meskinedriss@hotmail.com

C.P.R de Fès, B.P 49, Fès, Maroc.

ISSN (electronic): 1443-5756
040-03

volume 4, issue 5, article 98, 2003.

Received 26 March, 2003; accepted 05 August, 2003.

Communicated by: A. Fiorenza

Abstract

In this paper, we prove an existence and uniqueness result for solutions of some bilateral problems of the form

$$\begin{cases} \langle Au, v-u\rangle \geq \langle f, v-u\rangle,\ \forall v \in K \\ u \in K \end{cases}$$

where A is a standard Leray-Lions operator defined on $W_0^1 L_M(\Omega)$, with M an N-function which satisfies the Δ_2-condition, and where K is a convex subset of $W_0^1 L_M(\Omega)$ with obstacles depending on some Carathéodory function $g(x,u)$. We consider first, the case $f \in W^{-1}E_{\overline{M}}(\Omega)$ and secondly where $f \in L^1(\Omega)$. Our method deals with the study of the limit of the sequence of solutions u_n of some approximate problem with nonlinearity term of the form $|g(x,u_n)|^{n-1} g(x,u_n) \times M(|\nabla u_n|)$.

2000 Mathematics Subject Classification: 35J25, 35J60.
Key words: Strongly nonlinear elliptic equations, Natural growth, Truncations, Variational inequalities, Bilateral problems.

Asymptotic Behaviour of Some Equations in Orlicz Spaces

D. Meskine and A. Elmahi

Title Page

Contents

Go Back

Close

Quit

Contents

1. Introduction

Let Ω be an open bounded subset of $\mathbb{R}^N$, $N \geq 2$, with the segment property. Consider the following obstacle problem:

$$(\mathcal{P}) \qquad \begin{cases} \langle Au, v-u\rangle \geq \langle f, v-u\rangle, \ \forall v \in K, \\ u \in K, \end{cases}$$

where $A(u) = -div(a(x,u,\nabla u))$ is a Leray-Lions operator defined on $W_0^1 L_M(\Omega)$, with M being an N-function which satisfies the Δ_2-condition and where K is a convex subset of $W_0^1 L_M(\Omega)$.

In the variational case (i.e. where $f \in W^{-1}E_{\overline{M}}(\Omega)$), it is well known that problem ($\mathcal{P}$) has been already studied by Gossez and Mustonen in [10].

In this paper, we consider a recent approach of penalization in order to prove an existence theorem for solutions of some bilateral problems of ($\mathcal{P}$) type.

We recall that L. Boccardo and F. Murat, see [6], have approximated the model variational inequality:

$$\begin{cases} \langle -\Delta_p u, v-u\rangle \geq \langle f, v-u\rangle, \ \forall v \in K \\ u \in K = \{v \in W_0^{1,p}(\Omega) : |v(x)| \leq 1 \text{ a.e. in } \Omega\}, \end{cases}$$

with $f \in W^{-1,p'}(\Omega)$ and $-\Delta_p u = -div(|\nabla u|^{p-2}\nabla u)$, by the sequence of problems:

$$\begin{cases} -\Delta_p u_n + |u_n|^{n-1}u_n = f \text{ in } \mathcal{D}'(\Omega) \\ u_n \in W_0^{1,p}(\Omega) \cap L^n(\Omega). \end{cases}$$

In [7], A. Dall'aglio and L. Orsina generalized this result by taking increasing powers depending also on some Carathéodory function g satisfying the sign condition and some hypothesis of integrability. Following this idea, we have studied in [5] the sequence of problems:

$$\begin{cases} -\Delta_p u_n + |g(x,u_n)|^{n-1} g(x,u_n)|\nabla u_n|^p = f \text{ in } \mathcal{D}'(\Omega) \\ u_n \in W_0^{1,p}(\Omega), \ |g(x,u_n)|^n |\nabla u_n|^p \in L^1(\Omega) \end{cases}$$

Here, we introduce the general sequence of equations in the setting of Orlicz-Sobolev spaces

$$\begin{cases} Au_n + |g(x,u_n)|^{n-1} g(x,u_n)M(|\nabla u_n|) = f \text{ in } \mathcal{D}'(\Omega) \\ u_n \in W_0^1 L_M(\Omega), \ |g(x,u_n)|^n M(|\nabla u_n|) \in L^1(\Omega). \end{cases}$$

We are interested throughout the paper in studying the limit of the sequence u_n. We prove that this limit satisfies some bilateral problem of the ($\mathcal{P}$) form under some conditions on g. In the first we take $f \in W^{-1}E_{\overline{M}}(\Omega)$ and next in $L^1(\Omega)$.

2. Preliminaries

2.1. $N-$Functions

Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an N-function, i.e. M is continuous, convex, with $M(t) > 0$ for $t > 0$, $\frac{M(t)}{t} \to 0$ as $t \to 0$ and $\frac{M(t)}{t} \to \infty$ as $t \to \infty$.

Equivalently, M admits the representation: $M(t) = \int_0^t a(s)ds$, where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing, right continuous, with $a(0) = 0$, $a(t) > 0$ for $t > 0$ and $a(t)$ tends to ∞ as $t \to \infty$.

The N-function $\overline{M}$ conjugate to M is defined by $\overline{M}(t) = \int_0^t \bar{a}(s)ds$, where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is given by $\bar{a}(t) = \sup\{s : a(s) \leq t\}$ (see [1]).

The N-function is said to satisfy the Δ_2 condition, denoted by $M \in \Delta_2$, if for some $k > 0$:

$$M(2t) \leq kM(t) \ \ \forall t \geq 0; \tag{2.1}$$

when (2.1) holds only for $t \geq$ some $t_0 > 0$ then M is said to satisfy the Δ_2 condition near infinity.

We will extend these N-functions into even functions on all $\mathbb{R}$.

Let P and Q be two N-functions. $P \ll Q$ means that P grows essentially less rapidly than Q, i.e. for each $\epsilon > 0$, $\frac{P(t)}{Q(\epsilon t)} \to 0$ as $t \to \infty$. This is the case if and only if $\lim_{t\to\infty} \frac{Q^{-1}(t)}{P^{-1}(t)} = 0$.

2.2. Orlicz spaces

Let Ω be an open subset of $\mathbb{R}^N$. The Orlicz class $K_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real valued measurable functions u on Ω such that:

$$\int_{\Omega} M(u(x))dx < +\infty \quad \left(\text{resp. } \int_{\Omega} M(\frac{u(x)}{\lambda})dx < +\infty \text{ for some } \lambda > 0\right).$$

$L_M(\Omega)$ is a Banach space under the norm

$$\|u\|_{M,\Omega} = \inf\left\{\lambda > 0 : \int_{\Omega} M(\frac{u(x)}{\lambda})dx \leq 1\right\}$$

and $K_M(\Omega)$ is a convex subset of $L_M(\Omega)$.

The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$.

The equality $E_M(\Omega) = L_M(\Omega)$ holds if only if M satisfies the Δ_2 condition, for all t or for t large according to whether Ω has infinite measure or not.

The dual of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_{\Omega} uv dx$, and the dual norm of $L_{\overline{M}}(\Omega)$ is equivalent to $\|\cdot\|_{\overline{M},\Omega}$.

The space $L_M(\Omega)$ is reflexive if and only if M and $\overline{M}$ satisfy the Δ_2 condition, for all t or for t large, according to whether Ω has infinite measure or not.

2.3. Orlicz-Sobolev spaces

We now turn to the Orlicz-Sobolev space, $W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$) is the space of all functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp. $E_M(\Omega)$). It is a Banach space under the norm

$$\|u\|_{1,M} = \sum_{|\alpha|\le 1} \|D^\alpha u\|_M.$$

Thus, $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of product of $N+1$ copies of $L_M(\Omega)$. Denoting this product by $\prod L_M$, we will use the weak topologies $\sigma\left(\prod L_M, \prod E_{\overline{M}}\right)$ and $\sigma\left(\prod L_M, \prod L_{\overline{M}}\right)$.

The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwarz space $D(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma\left(\prod L_M, \prod E_{\overline{M}}\right)$ closure of $D(\Omega)$ in $W^1L_M(\Omega)$.

We say that u_n converges to u for the modular convergence in $W^1L_M(\Omega)$ if for some $\lambda > 0$

$$\int_\Omega M\left(\frac{D^\alpha u_n - D^\alpha u}{\lambda}\right) dx \to 0 \text{ for all } |\alpha| \le 1.$$

This implies convergence for $\sigma\left(\prod L_M, \prod L_{\overline{M}}\right)$.

If M satisfies the Δ_2-condition on $\mathbb{R}^+$, then modular convergence coincides with norm convergence.

2.4. The spaces $W^{-1}L_{\bar{M}}(\Omega)$ and $W^{-1}E_{\bar{M}}(\Omega)$

Let $W^{-1}L_{\overline{M}}(\Omega)$ (resp. $W^{-1}E_{\overline{M}}(\Omega)$) denote the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\overline{M}}$ (resp. $E_{\overline{M}}(\Omega)$). It is a Banach space under the usual quotient norm.

If the open set Ω has the segment property then the space $D(\Omega)$ is dense in $W_0^1L_M(\Omega)$ for the modular convergence and thus for the topology $\sigma\left(\prod L_M, \prod L_{\overline{M}}\right)$

(cf. [8, 9]). Consequently, the action of a distribution in $W^{-1}L_{\overline{M}}(\Omega)$ on an element of $W_0^1 L_M(\Omega)$ is well defined.

2.5. Lemmas related to the Nemytskii operators in Orlicz spaces

We recall some lemmas introduced in [3] which will be used in this paper.

Lemma 2.1. *Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let M be an $N-$function and let $u \in W^1 L_M(\Omega)$ (resp. $W^1 E_M(\Omega)$). Then $F(u) \in W^1 L_M(\Omega)$ (resp. $W^1 E_M(\Omega)$). Moreover, if the set D of discontinuity points of F' is finite, then*

$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u)\frac{\partial}{\partial x_i} u & \text{a.e. in } \{x \in \Omega : u(x) \notin D\}, \\ 0 & \text{a.e. in } \{x \in \Omega : u(x) \notin D\}. \end{cases}$$

Lemma 2.2. *Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. We suppose that the set of discontinuity points of F' is finite. Let M be an N-function, then the mapping $F : W^1 L_M(\Omega) \to W^1 L_M(\Omega)$ is sequentially continuous with respect to the weak* topology $\sigma\left(\prod L_M, \prod E_{\overline{M}}\right)$.*

2.6. Abstract lemma applied to the truncation operators

We now give the following lemma which concerns operators of the Nemytskii type in Orlicz spaces (see [3]).

Lemma 2.3. *Let Ω be an open subset of $\mathbb{R}^N$ with finite measure.*

Let M, P and Q be N-functions such that $Q \ll P$, and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that a.e. $x \in \Omega$ and all $s \in \mathbb{R}$:

$$|f(x,s)| \leq c(x) + k_1 P^{-1} M(k_2|s|),$$

where k_1, k_2 are real constants and $c(x) \in E_Q(\Omega)$.

Then the Nemytskii operator N_f defined by $N_f(u)(x) = f(x, u(x))$ is strongly continuous from

$$\mathcal{P}\left(E_M(\Omega), \frac{1}{k_2}\right) = \left\{u \in L_M(\Omega) : d(u, E_M(\Omega)) < \frac{1}{k_2}\right\}$$

into $E_Q(\Omega)$.

3. The Main Result

Let Ω be an open bounded subset of $\mathbb{R}^N$, $N \geq 2$, with the segment property.

Let M be an N-function satisfying the Δ_2-condition near infinity.

Let $A(u) = -div(a(x, \nabla u))$ be a Leray-Lions operator defined on $W_0^1 L_M(\Omega)$ into
$W^{-1}L_{\overline{M}}(\Omega)$, where $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying for a.e. $x \in \Omega$ and for all $\zeta, \zeta' \in \mathbb{R}^N$, $(\zeta \neq \zeta')$:

$$|a(x,\zeta)| \leq h(x) + \overline{M}^{-1} M(k_1|\zeta|) \tag{3.1}$$

$$(a(x,\zeta) - a(x,\zeta^{'}))(\zeta - \zeta') > 0 \tag{3.2}$$

$$a(x,\zeta)\zeta \geq \alpha M\left(\frac{|\zeta|}{\lambda}\right) \tag{3.3}$$

with $\alpha, \lambda > 0,\ k_1 \geq 0,\ h \in E_{\overline{M}}(\Omega)$.

Furthermore, let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$:

$$g(x,s)s \geq 0 \tag{3.4}$$

$$|g(x,s)| \leq b(|s|) \tag{3.5}$$

$$
(3.6)\quad \begin{cases} \text{for almost } x\in\Omega\backslash\Omega_+^\infty \text{ there exists } \epsilon=\epsilon(x)>0 \text{ such that:} \\ g(x,s)>1,\ \forall s\in]q_+(x),q_+(x)+\epsilon[; \\ \\ \text{for almost } x\in\Omega\backslash\Omega_-^\infty \text{ there exists } \epsilon=\epsilon(x)>0 \text{ such that:} \\ g(x,s)<-1,\ \forall s\in]q_-(x)-\epsilon,q_-(x)[, \end{cases}
$$

where $b:\mathbb{R}_+\to\mathbb{R}_+$ is a continuous and nondecreasing function, with $b(0)=0$ and where

$$
\begin{aligned}
q_+(x) &= \inf\{s>0 : g(x,s)\geq 1\} \\
q_-(x) &= \sup\{s<0 : g(x,s)\leq -1\} \\
\Omega_+^\infty &= \{x\in\Omega : q_+(x)=+\infty\} \\
\Omega_-^\infty &= \{x\in\Omega : q_-(x)=-\infty\}.
\end{aligned}
$$

We define for s and k in $\mathbb{R}$, $k\geq 0$, $T_k(s)=\max(-k,\min(k,s))$.

Theorem 3.1. *Let $f\in W^{-1}E_{\overline{M}}(\Omega)$. Assume that (3.1) – (3.6) hold true and that the function $s\to g(x,s)$ is nondecreasing for a.e. $x\in\Omega$. Then, for any real number $\mu>0$, the problem*

$$
(P_n)\quad \begin{cases} A(u_n)+|g(x,u_n)|^{n-1}g(x,u_n)M\left(\frac{|\nabla u_n|}{\mu}\right)=f \text{ in } \mathcal{D}'(\Omega) \\ u_n\in W_0^1L_M(\Omega), |g(x,u_n)|^n M\left(\frac{|\nabla u_n|}{\mu}\right)\in L^1(\Omega) \end{cases}
$$

admits at least one solution u_n *such that:*

$$\forall k > 0 \qquad T_k(u_n) \to T_k(u) \text{ for modular convergence in } W_0^1 L_M(\Omega) \tag{3.7}$$

where u *is the unique solution of the following bilateral problem*

$$(P) \qquad \begin{cases} \langle Au, v-u \rangle \geq \langle f, v-u \rangle, \ \forall v \in K \\ u \in K = \{v \in W_0^1 L_M(\Omega) : q_- \leq v \leq q_+ \ \textit{a.e.}\}, \end{cases}$$

Remark 3.1. *If the function* $s \to g(x,s)$ *is strictly nondecreasing for a.e.* $x \in \Omega$ *then the assumption (3.6) holds true.*

Proof.

Step 1: A priori estimates.
The existence of u_n is given by Theorem 3.1 of [3]. Choosing $v = u_n$ as a test function in (P_n), and using the sign condition (3.4), we get

$$\langle Au_n, u_n \rangle \leq \langle f, u_n \rangle.$$

By Proposition 5 of [11] one has:

$$\int_\Omega M\left(\frac{|\nabla u_n|}{\lambda}\right) dx \leq C, \text{ and } \int_\Omega a(x, u_n, \nabla u_n) \nabla u_n dx \leq C, \tag{3.8}$$

$$(a(x, u_n, \nabla u_n)) \text{ is bounded in } (L_{\overline{M}}(\Omega))^N, \tag{3.9}$$

$$\int_\Omega |g(x,u_n)|^{n-1} g(x,u_n) M\left(\frac{|\nabla u_n|}{\mu}\right) u_n dx \leq C. \tag{3.10}$$

We then deduce

$$\int_{\{|u_n|>k\}} |g(x,u_n)|^n M\left(\frac{|\nabla u_n|}{\mu}\right) dx \leq C, \quad \text{for all} \quad k > 0.$$

Since b is continuous and since $b(0) = 0$ there exists $\delta > 0$ such that

$$b(|s|) \leq 1 \text{ for all } |s| \leq \delta.$$

On the other hand, by the Δ_2 condition there exist two positive constants K and K' such that

$$M\left(\frac{t}{\mu}\right) \leq KM\left(\frac{t}{\lambda}\right) + K' \quad \text{for all} \quad t \geq 0,$$

which implies

$$\int_{\{|u_n|\leq\delta\}} |g(x,u_n)|^n M\left(\frac{|\nabla u_n|}{\mu}\right) dx \leq \int_{\{|u_n|\leq\delta\}} \left(K' + KM\left(\frac{|\nabla u_n|}{\lambda}\right)\right) dx.$$

Consequently from (3.8)

$$\int_\Omega |g(x,u_n)|^n M\left(\frac{|\nabla u_n|}{\mu}\right) dx \leq C, \quad \text{for all} \quad n. \tag{3.11}$$

Step 2: Almost everywhere convergence of the gradients.

Since (u_n) is a bounded sequence in $W_0^1 L_M(\Omega)$ there exist some $u \in W_0^1 L_M(\Omega)$ such that (for a subsequence still denoted by u_n)

$$u_n \rightharpoonup u \text{ weakly in } W_0^1 L_M(\Omega) \text{ for } \sigma\left(\prod L_M, \prod E_{\overline{M}}\right), \quad \text{strongly in } E_M(\Omega) \text{ and a.e. in } \Omega. \tag{3.12}$$

Furthermore, if we have

$$Au_n = f - |g(x,u_n)|^{n-1} g(x,u_n) M\left(\frac{|\nabla u_n|}{\mu}\right)$$

with $|g(x,u_n)|^{n-1} g(x,u_n) M\left(\frac{|\nabla u_n|}{\mu}\right)$ being bounded in $L^1(\Omega)$ then as in [2], one can show that

$$\nabla u_n \to \nabla u \text{ a.e. in } \Omega. \tag{3.13}$$

Step 3: $u \in K = \{v \in W_0^1 L_M(\Omega) : q_- \le v \le q_+ \text{ a.e. in } \Omega\}$.

Since $s \to g(x,s)$ is nondecreasing, then in view of (3.6), we have:

$$\{s \in \mathbb{R} : |g(x,s)| \le 1 \text{ a.e. in } \Omega\} = \{s \in \mathbb{R} : q_- \le s \le q_+ \text{ a.e. in } \Omega\}.$$

It suffices to verify that $|g(x,u)| \le 1$ a.e.

We have

$$\int_\Omega |g(x,u_n)|^n M\left(\frac{|\nabla u_n|}{\mu}\right) dx \le C,$$

which gives

$$\int_{\{|g(x,u_n)|>k\}} |g(x,u_n)|^n M\left(\frac{|\nabla u_n|}{\mu}\right) |dx \leq C$$

and

$$\int_{\{|g(x,u_n)|>k\}} M\left(\frac{|\nabla u_n|}{\mu}\right) dx \leq \frac{C}{k^n}$$

where $k > 1$. Letting $n \to +\infty$ for k fixed, we deduce by using Fatou's lemma

$$\int_{\{|g(x,u)|>k\}} M\left(\frac{|\nabla u_n|}{\mu}\right) dx = 0$$

and so that,

$$|g(x,u)| \leq 1 \text{ a.e. in } \Omega.$$

Step 4: Strong convergence of the truncations.
Let $\phi(s) = s\exp(\gamma s^2)$, where γ is chosen such that $\gamma \geq \left(\frac{1}{\alpha}\right)^2$.
It is well known that $\phi^{'}(s) - \frac{2K}{\alpha}|\phi(s)| \geq \frac{1}{2}, \forall s \in \mathbb{R}$, where K is a constant which will be used later. The use of the test function $v_n = \phi(z_n)$ in (P_n) where $z_n = T_k(u_n) - T_k(u)$ gives

$$\langle Au_n, \phi(z_n)\rangle + \int_\Omega |g(x,u_n)|^{n-1} g(x,u_n) M\left(\frac{|\nabla u_n|}{\mu}\right)\phi(z_n)dx = \langle f, \phi(z_n)\rangle$$

which implies, by using the fact that $g(x,u_n)\phi(z_n) \geq 0$ on $\{x \in \Omega :$

$|u_n| > k\}$,

$$\langle Au_n, \phi(z_n)\rangle + \int_{\{0\le u_n\le T_k(u)\}\cap\{|u_n|\le k\}} |g(x,u_n)|^{n-1} \times g(x,u_n) M\left(\frac{|\nabla u_n|}{\mu}\right)\phi(z_n)dx + \int_{\{T_k(u)\le u_n\le 0\}\cap\{|u_n|\le k\}} |g(x,u_n)|^{n-1} \times g(x,u_n) M\left(\frac{|\nabla u_n|}{\mu}\right)\phi(z_n)dx \le \langle f, \phi(z_n)\rangle.$$

The second and the third terms of the last inequality will be denoted respectively by $I^1_{n,k}$ and $I^2_{n,k}$ and $\epsilon_i(n)$ denote various sequences of real numbers which tend to 0 as $n \to +\infty$.
On the one hand we have

$$\left|I^1_{n,k}\right| \le \int_{\{0\le u_n\le T_k(u)\}\cap\{|u_n|\le k\}} |g(x,u_n)|^n M\left(\frac{|\nabla u_n|}{\mu}\right)|\phi(z_n)|dx \le \int_{\{0\le u_n\le u\}\cap\{|u_n|\le k\}} |g(x,u_n)|^n M\left(\frac{|\nabla u_n|}{\mu}\right)|\phi(z_n)|dx,$$

but since $|g(x,u_n)| \le 1$ on $\{x\in\Omega : 0\le u_n\le u\}$, then we have

$$\left|I^1_{n,k}\right| \le \int_{\{|u_n|\le k\}} M\left(\frac{|\nabla u_n|}{\mu}\right)|\phi(z_n)|dx.$$

By using the fact that

$$M\left(\frac{|\nabla u_n|}{\mu}\right) \leq K' + KM\left(\frac{|\nabla u_n|}{\lambda}\right)$$

we obtain

$$|I^1_{n,k}| \leq \int_\Omega K'|\phi(z_n)|dx + \frac{K}{\alpha}\int_\Omega a(x,\nabla T_k(u_n))\nabla T_k(u_n)|\phi(z_n)|dx,$$

which gives

$$|I^1_{n,k}| \leq \epsilon_1(n) + \frac{K}{\alpha}\int_\Omega a(x,\nabla T_k(u_n))\nabla T_k(u_n)|\phi(z_n)|dx. \tag{3.14}$$

Similarly,

$$\begin{aligned} |I^2_{n,k}| &\leq \int_{\{|u_n|\leq k\}} M\left(\frac{|\nabla u_n|}{\mu}\right)|\phi(z_n)|dx \\ &\leq \epsilon_1(n) + \frac{K}{\alpha}\int_\Omega a(x,\nabla T_k(u_n))\nabla T_k(u_n)|\phi(z_n)|dx. \end{aligned} \tag{3.15}$$

The first term on the left hand side of the last inequality can be written as:

$$\begin{aligned} &\int_\Omega a(x,\nabla u_n)[\nabla T_k(u_n) - \nabla T_k(u)]\phi'(z_n)dx \\ &\quad = \int_{\{|u_n|\leq k\}} a(x,\nabla u_n)[\nabla T_k(u_n) - \nabla T_k(u)]\phi'(z_n)dx \\ &\qquad\qquad - \int_{\{|u_n|>k\}} a(x,\nabla u_n)\nabla T_k(u)\phi'(z_n)dx. \end{aligned} \tag{3.16}$$

For the second term on the right hand side of the last equality, we have

$$\left|\int_{\{|u_n|>k\}} a(x,\nabla u_n)\nabla T_k(u)\phi'(z_n)dx\right| \\ \leq C_k \int_\Omega |a(x,\nabla u_n)||\nabla T_k(u)|\chi_{\{|u_n|>k\}}dx.$$

The right hand side of the last inequality tends to 0 as n tends to infinity. Indeed, the sequence $(a(x,\nabla u_n))_n$ is bounded in $(L_{\overline{M}}(\Omega))^N$ while $\nabla T_k(u)\chi_{\{|u_n|>k\}}$ tends to 0 strongly in $(E_M(\Omega))^N$.

We define for every $s>0$, $\Omega_s = \{x\in\Omega : |\nabla T_k(u(x))| \leq s\}$ and we denote by χ_s its characteristic function. For the first term of the right hand side of (3.16), we can write

$$\begin{aligned}(3.17)\quad &\int_{\{|u_n|\leq k\}} a(x,\nabla u_n)[\nabla T_k(u_n)-\nabla T_k(u)]\phi'(z_n)dx \\ &= \int_\Omega [a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u)\chi_s)][\nabla T_k(u_n)-\nabla T_k(u)\chi_s]\phi'(z_n)dx \\ &\quad + \int_\Omega a(x,\nabla T_k(u)\chi_s)[\nabla T_k(u_n)-\nabla T_k(u)\chi_s]\phi'(z_n)dx \\ &\quad - \int_\Omega a(x,\nabla T_k(u_n))\nabla T_k(u)\chi_{\Omega\setminus\Omega_s}\phi'(z_n)dx.\end{aligned}$$

The second term of the right hand side of (3.17) tends to 0 since

$$a(x,\nabla T_k(u_n)\chi_s)\phi'(z_n)\to a(x,\nabla T_k(u)\chi_s) \text{ strongly in } (E_{\overline{M}}(\Omega))^N$$

by Lemma 2.3 and

$$\nabla T_k(u_n) \rightharpoonup \nabla T_k(u) \text{ weakly in } (L_M(\Omega))^N \text{ for } \sigma\left(\prod L_M(\Omega), \prod E_{\overline{M}}(\Omega)\right).$$

The third term of (3.17) tends to $-\int_\Omega a(x,\nabla T_k(u))\nabla T_k(u)\chi_{\Omega\setminus\Omega_s}dx$ as $n \to \infty$ since

$$a(x,\nabla T_k(u_n)) \rightharpoonup a(x,\nabla T_k(u)) \text{ weakly for } \sigma\left(\prod E_{\overline{M}}(\Omega), \prod L_M(\Omega)\right).$$

Consequently, from (3.16) we have

$$\begin{aligned}(3.18)\quad \int_\Omega a(x,\nabla u_n)[\nabla T_k(u_n) - \nabla T_k(u)]\phi'(z_n)dx \\ = \int_\Omega [a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u)\chi_s)] \\ \times [\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\phi'(z_n)dx + \epsilon_2(n).\end{aligned}$$

We deduce that, in view of (3.17) and (3.18),

$$\begin{aligned}\int_\Omega [a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u)\chi_s)] \\ \times [\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\left(\phi'(z_n) - \frac{2K}{\alpha}|\phi(z_n)|\right)dx \\ \le \epsilon_3(n) + \int_\Omega a(x,\nabla T_k(u))\nabla T_k(u)\chi_{\Omega\setminus\Omega_s}dx,\end{aligned}$$

and so

$$\int_\Omega [a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s]dx$$
$$\leq 2\epsilon_3(n) + 2\int_\Omega a(x,\nabla T_k(u))\nabla T_k(u)\chi_{\Omega\setminus\Omega_s}dx.$$

Hence

$$\begin{aligned}\int_\Omega a(x,\nabla T_k(u_n))\nabla T_k(u_n)dx \\ \leq \int_\Omega a(x,\nabla T_k(u_n))\nabla T_k(u)\chi_s dx \\ + \int_\Omega a(x,\nabla T_k(u)\chi_s)[\nabla T_k(u_n) - \nabla T_k(u)\chi_s]dx \\ + 2\epsilon_3(n) + 2\int_\Omega a(x,\nabla T_k(u))\nabla T_k(u)\chi_{\Omega\setminus\Omega_s}dx.\end{aligned}$$

Now considering the limit sup over n, one has

$$\begin{aligned}\limsup_{n\to+\infty}\int_\Omega a(x,\nabla T_k(u_n))\nabla T_k(u_n)dx \\ \leq \limsup_{n\to+\infty}\int_\Omega a(x,\nabla T_k(u_n))\nabla T_k(u)\chi_s dx \\ + \limsup_{n\to+\infty}\int_\Omega a(x,\nabla T_k(u)\chi_s)[\nabla T_k(u_n) - \nabla T_k(u)\chi_s]dx \\ + 2\int_\Omega a(x,\nabla T_k(u))\nabla T_k(u)\chi_{\Omega\setminus\Omega_s}dx.\end{aligned} \tag{3.19}$$

The second term of the right hand side of the inequality (3.19) tends to 0, since

$$a(x, \nabla T_k(u_n)\chi_s) \to a(x, \nabla T_k(u)\chi_s) \text{ strongly in } E_{\overline{M}}(\Omega),$$

while $\nabla T_k(u_n)$ tends weakly to $\nabla T_k(u)$.

The first term of the right hand side of (3.19) tends to $\int_\Omega a(x, \nabla T_k(u)) \nabla T_k(u)\chi_s dx$ since

$$a(x, \nabla T_k(u_n)) \rightharpoonup a(x, \nabla T_k(u)) \text{ weakly in } (L_{\overline{M}}(\Omega))^N$$

for $\sigma\left(\prod L_{\overline{M}}, \prod E_M\right)$ while $\nabla T_k(u)\chi_s \in E_M(\Omega)$. We deduce then

$$\begin{aligned}\limsup_{n\to+\infty} \int_\Omega a(x, \nabla T_k(u_n))\nabla T_k(u_n)dx \\ \leq \int_\Omega a(x, \nabla T_k(u))\nabla T_k(u)\chi_s dx \\ + 2\int_\Omega a(x, \nabla T_k(u))\nabla T_k(u)\chi_{\Omega\backslash\Omega_s} dx,\end{aligned}$$

by using the fact that $a(x, \nabla T_k(u))\nabla T_k(u) \in L^1(\Omega)$ and letting $s \to \infty$ we get, since $meas(\Omega\backslash\Omega_s) \to 0$

$$\limsup_{n\to+\infty} \int_\Omega a(x, \nabla T_k(u_n))\nabla T_k(u_n)dx \leq \int_\Omega a(x, \nabla T_k(u))\nabla T_k(u)dx$$

which gives, by using Fatou's lemma,

$$\lim_{n\to+\infty}\int_\Omega a(x,\nabla T_k(u_n))\nabla T_k(u_n)dx = \int_\Omega a(x,\nabla T_k(u))\nabla T_k(u)dx. \tag{3.20}$$

On the other hand, we have

$$M\left(\frac{|\nabla T_k(u_n)|}{\mu}\right) \le K' + \frac{K}{\alpha}\int_\Omega a(x,\nabla T_k(u_n))\nabla T_k(u_n)dx,$$

then by using (3.20) and Vitali's theorem, one easily has

$$M\left(\frac{|\nabla T_k(u_n)|}{\mu}\right) \to M\left(\frac{|\nabla T_k(u)|}{\mu}\right) \text{ strongly in } L^1(\Omega). \tag{3.21}$$

By writing

$$M\left(\frac{|\nabla T_k(u_n)-\nabla T_k(u)|}{2\mu}\right) \le \frac{M\left(\frac{|\nabla T_k(u_n)|}{\mu}\right)}{2} + \frac{M\left(\frac{|\nabla T_k(u_n)|}{\mu}\right)}{2} \tag{3.22}$$

one has, by (3.21) and Vitali's theorem again,

$$T_k(u_n) \to T_k(u) \text{ for modular convergence in } W_0^1L_M(\Omega). \tag{3.23}$$

Step 5: u is the solution of the variational inequality (P).
Choosing $w = T_k(u_n - \theta T_m(v))$ as a test function in (P_n), where $v \in K$

and $0 < \theta < 1$, gives

$$\langle Au_n, T_k(u_n - \theta T_m(v))\rangle + \int_\Omega |g(x,u_n)|^{n-1} g(x,u_n) M\left(\frac{|\nabla u_n|}{\mu}\right) T_k(u_n - \theta T_m(v))dx = \langle f, T_k(u_n - \theta T_m(v))\rangle,$$

since $g(x,u_n)T_k(u_n - \theta T_m(v)) \geq 0$ on

$\{x \in \Omega : u_n \geq 0 \text{ and } u_n \geq \theta T_m(v)\} \cup \{x \in \Omega : u_n \leq 0 \text{ and } u_n \leq \theta T_m(v)\}$

we have

$$\int_\Omega |g(x,u_n)|^{n-1} g(x,u_n) M\left(\frac{|\nabla u_n|}{\mu}\right) T_k(u_n - \theta T_m(v))dx$$
$$\geq \int_{\{0\leq u_n \leq \theta T_m(v)\}} |g(x,u_n)|^{n-1} g(x,u_n) M\left(\frac{|\nabla u_n|}{\mu}\right) T_k(u_n - \theta T_m(v))dx$$
$$+ \int_{\{\theta T_m(v)\leq u_n \leq 0\}} |g(x,u_n)|^{n-1} g(x,u_n) M\left(\frac{|\nabla u_n|}{\mu}\right) T_k(u_n - \theta T_m(v))dx.$$

The first and the second terms in the right hand side of the last inequality will be denoted respectively by $J^1_{n,m}$ and $J^2_{n,m}$.
Defining

$$\delta_{1,m}(x) = \sup_{0\leq s\leq \theta T_m(v)} g(x,s)$$

we get $0 \leq \delta_{1,m}(x) < 1$ a.e. and

$$|J^1_{n,m}| \leq k \int_{\{0\leq u_n \leq \theta T_m(v)\}} (\delta_{1,m}(x))^n M\left(\frac{|\nabla u_n|}{\mu}\right) dx.$$

Since

$$\left|(\delta_{1,m}(x))^n M\left(\frac{|\nabla u_n|}{\mu}\right)\chi_{\{|u_n|\le m\}}\right| \le M\left(\frac{|\nabla T_m(u_n)|}{\mu}\right),$$

we have then by using (3.23) and Lebesgue's theorem

$$J^1_{n,m} \longrightarrow 0 \text{ as } n \to +\infty.$$

Similarly

$$|J^2_{n,m}| \le k\int_{\{|u_n|\le m\}} |\delta_{2,m}(x)|^n M\left(\frac{|\nabla T_m(u_n)|}{\mu}\right)dx \to 0 \text{ as } n \to +\infty,$$

where

$$\delta_{2,m}(x) = \inf_{\theta T_m(v)\le s\le 0} g(x,s).$$

On the other hand , by using Fatou's lemma and the fact that

$$a(x,\nabla u_n) \to a(x,\nabla u) \text{ weakly in } (L_{\overline{M}}(\Omega))^N \text{ for } \sigma(\Pi L_{\overline{M}}, \Pi E_M),$$

one easily has

$$\liminf_{n\to+\infty}\langle Au_n, T_k(u_n-\theta T_m(v))\rangle \le \langle Au, T_k(u-\theta T_m(v))\rangle.$$

Consequently

$$\langle Au, T_k(u-\theta T_m(v))\rangle \le \langle f, T_k(u-\theta T_m(v))\rangle,$$

this implies that by letting $k \to +\infty$, since $T_k(u - \theta T_m(v)) \to u - \theta T_m(v)$ for modular convergence in $W_0^1 L_M(\Omega)$,

$$\langle Au, u - \theta T_m(v) \rangle \leq \langle f, u - \theta T_m(v) \rangle,$$

in which we can easily pass to the limit as $\theta \to 1$ and $m \to +\infty$ to obtain

$$\langle Au, u - v \rangle \rangle \leq \langle f, u - v \rangle.$$

$\square$

4. The L^1 Case

In this section, we study the same problems as before but we assume that q_- and q_+ are bounded.

Theorem 4.1. *Let $f \in L^1(\Omega)$. Assume that the hypotheses are as in Theorem 3.1, q_- and q_+ belong to $L^\infty(\Omega)$. Then the problem (P_n) admits at least one solution u_n such that:*

$$u_n \to u \text{ for modular convergence in } W_0^1 L_M(\Omega),$$

where u is the unique solution of the bilateral problem:

$$(Q) \qquad \begin{cases} \langle Au, v-u\rangle \geq \int_\Omega f(v-u)dx, \forall v \in K \\ u \in K = \{v \in W_0^1 L_M(\Omega) : q_- \leq v \leq q_+ \textit{ a.e.}\}. \end{cases}$$

Proof. We sketch the proof since the steps are similar to those in Section 3. The existence of u_n is given by Theorem 1 of [4]. Indeed, it is easy to see that $|g(x,s)| \geq 1$ on $\{|s| \geq \gamma\}$, where $\gamma = \max\{\operatorname{supess} q_+, -\operatorname{infess} q_-\}$ and so that

$$|g(x,s)|^n M\left(\frac{|\zeta|}{\mu}\right) \geq M\left(\frac{|\zeta|}{\mu}\right) \text{ for } |s| \geq \gamma.$$

Step 1: A priori estimates.
Choosing $v = T_\gamma(u_n)$, as a test function in (P_n), and using the sign condition (3.4), we obtain

$$\alpha \int_\Omega M\left(\frac{|\nabla T_\gamma(u_n)|}{\lambda}\right) dx \leq \gamma \|f\|_1 \tag{4.1}$$

and

$$\int_{\{|u_n|>\gamma\}} |g(x,u_n)|^n M\left(\frac{|\nabla u_n|}{\mu}\right) dx \leq \|f\|_1,$$

which gives

$$\int_{\{|u_n|>\gamma\}} M\left(\frac{|\nabla u_n|}{\mu}\right) dx \leq C$$

and finally

$$\int_\Omega M\left(\frac{|\nabla u_n|}{\max\{\lambda,\mu\}}\right) dx \leq C. \tag{4.2}$$

On the other hand, as in Section 3, we have

$$\int_\Omega |g(x,u_n)|^n M\left(\frac{|\nabla u_n|}{\mu}\right) dx \leq C. \tag{4.3}$$

Step 2: Almost everywhere convergence of the gradients.
Due to (4.2), there exists some $u \in W_0^1 L_M(\Omega)$ such that (for a subsequence)

$$u_n \rightharpoonup u \text{ weakly in } W_0^1 L_M(\Omega) \text{ for } \sigma(\Pi L_M, \Pi E_{\overline{M}}).$$

Write

$$Au_n = f - |g(x,u_n)|^{n-1} g(x,u_n) M\left(\frac{|\nabla u_n|}{\mu}\right)$$

and remark that, by (4.2), the second hand side is uniformly bounded in $L^1(\Omega)$. Then as in Section 3

$$\nabla u_n \to \nabla u \text{ a.e. in } \Omega.$$

Step 3: $u \in K = \{v \in W_0^1 L_M(\Omega) : q_- \leq v \leq q_+ \text{ a.e. in } \Omega\}$.
Similarly, as in the proof of Theorem 3.1, one can prove this step with the aid of property (4.3).

Step 4: Strong convergence of the truncations.
It is easy to see that the proof is the same as in Section 3.

Step 5: u is the solution of the bilateral problem (Q).
Let $v \in K$ and $0 < \theta < 1$. Taking $v_n = T_k(u_n - \theta v), k > 0$ as a test function in (P_n), one can see that the proof is the same by replacing $T_m(v)$ with v in Section 3. We remark that $K \subset L^\infty(\Omega)$.

Step 6: $u_n \to u$ for modular convergence in $W_0^1 L_M(\Omega)$.
We shall prove that $\nabla u_n \to \nabla u$ in $(L_M(\Omega))^N$ for the modular convergence by using Vitali's theorem.
Let E be a measurable subset of Ω, we have for any $k > 0$

$$\int_E M\left(\frac{|\nabla u_n|}{\mu}\right) dx \leq \int_{E\cap\{|u_n|\leq k\}} M\left(\frac{|\nabla u_n|}{\mu}\right) dx + \int_{E\cap\{|u_n|>k\}} M\left(\frac{|\nabla u_n|}{\mu}\right) dx.$$

Let $\epsilon > 0$. By virtue of the modular convergence of the truncates, there exists some $\eta(\epsilon, k)$ such that for any E measurable

$$|E| < \eta(\epsilon, k) \Rightarrow \int_{E\cap\{|u_n|\leq k\}} M\left(\frac{|\nabla u_n|}{\mu}\right) dx < \frac{\epsilon}{2}, \quad \forall n. \tag{4.4}$$

Choosing $T_1(u_n - T_k(u_n))$, with $k > 0$ a test function in (P_n) we obtain:

$$\langle Au_n, T_1(u_n - T_k(u_n))\rangle + \int_\Omega |g(x,u_n)|^{n-1} g(x,u_n) M\left(\frac{|\nabla u_n|}{\mu}\right) T_1(u_n - T_k(u_n))dx = \int_\Omega f T_1(u_n - T_k(u_n))dx,$$

which implies

$$\int_{\{|u_n|>k+1\}} |g(x,u_n)|^n M\left(\frac{|\nabla u_n|}{\mu}\right) dx \le \int_{\{|u_n|>k\}} |f|dx.$$

Note that $meas\{x \in \Omega : |u_n(x)| > k\} \to 0$ uniformly on n when $k \to \infty$. We deduce then that there exists $k = k(\epsilon)$ such that

$$\int_{\{|u_n|>k\}} |f|dx < \frac{\epsilon}{2}, \quad \forall n,$$

which gives

$$\int_{\{|u_n|>k+1\}} |g(x,u_n)|^n M\left(\frac{|\nabla u_n|}{\mu}\right) dx < \frac{\epsilon}{2}, \quad \forall n.$$

By setting $t(\epsilon) = \max\{k+1, \gamma\}$ we obtain

$$\int_{\{|u_n|>t(\epsilon)\}} M\left(\frac{|\nabla u_n|}{\mu}\right) dx < \frac{\epsilon}{2}, \quad \forall n. \tag{4.5}$$

Combining (4.4) and (4.5) we deduce that there exists $\eta > 0$ such that

$$\int_E M\left(\frac{|\nabla u_n|}{\mu}\right) < \epsilon, \ \ \forall n \text{ when } |E| < \eta, \ E \text{ measurable},$$

which shows the equi-integrability of $M\left(\frac{|\nabla u_n|}{\mu}\right)$ in $L^1(\Omega)$, and therefore we have

$$M\left(\frac{|\nabla u_n|}{\mu}\right) \to M\left(\frac{|\nabla u|}{\mu}\right) \text{ strongly in } L^1(\Omega).$$

By remarking that

$$M\left(\frac{|\nabla u_n - \nabla u|}{2\mu}\right) \leq \frac{1}{2}\left[M\left(\frac{|\nabla u_n|}{\mu}\right) + M\left(\frac{|\nabla u|}{\mu}\right)\right]$$

one easily has, by using the Lebesgue theorem

$$\int_\Omega M\left(\frac{|\nabla u_n - \nabla u|}{2\mu}\right) dx \to 0 \text{ as } n \to +\infty,$$

which completes the proof.

□

Remark 4.1. *The condition* $b(0) = 0$ *is not necessary. Indeed, taking* $\theta_h(u_n), \ h > 0$, *as a test function in* (P_n) *with*

$$\theta_h(s) = \begin{cases} hs & \textit{if} \quad |s| \leq \frac{1}{h} \\ \operatorname{sgn}(s) & \textit{if} \quad |s| \geq \frac{1}{h}, \end{cases}$$

we obtain

$$\int_\Omega |g(x,u_n)|^{n-1} g(x,u_n) M\left(\frac{|\nabla u_n|}{\mu}\right)\theta_h(u_n)dx \leq \int_\Omega f\theta_h(u_n)dx.$$

and then, by letting $h \to +\infty$,

$$\int_\Omega |g(x,u_n)|^n M\left(\frac{|\nabla u_n|}{\mu}\right) dx \leq C.$$

References

[1] R. ADAMS, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] A. BENKIRANE AND A. ELMAHI, Almost everywhere convergence of the gradients of solutions to elliptic equations in Orlicz spaces and application, *Nonlinear Anal. T.M.A.*, **28** (11) (1997), 1769–1784.

[3] A. BENKIRANE AND A. ELMAHI, An existence theorem for a strongly nonlinear elliptic problem in Orlicz spaces, *Nonlinear Anal. T.M.A.*, **36** (1999), 11–24.

[4] A. BENKIRANE AND A. ELMAHI, A strongly nonlinear elliptic equation having natural growth terms and L^1 data, *Nonlinear Anal. T.M.A.*, **39** (2000), 403–411.

[5] A. BENKIRANE, A. ELMAHI AND D. MESKINE, On the limit of some nonlinear elliptic problems, *Archives of Inequalities and Applications*, **1** (2003), 207–220.

[6] L. BOCCARDO AND F. MURAT, Increase of power leads to bilateral problems, in *Composite Media and Homogenization Theory*, G. Dal Maso and G. F. Dell'Antonio (Eds.), World Scientific, Singapore, 1995, pp. 113–123.

[7] A. DALL'AGLIO AND L. ORSINA, On the limit of some nonlinear elliptic equations involving increasing powers, *Asympt. Anal.*, **14** (1997), 49–71.

[8] J.-P. GOSSEZ, Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients, *Trans. Amer. Math. Soc.*, **190** (1974), 163–205.

[9] J.-P. GOSSEZ, Some approximation properties in Orlicz-Sobolev spaces, *Studia Math.*, **74** (1982), 17–24.

[10] J.-P. GOSSEZ, A strongly nonlinear elliptic problem in Orlicz-Sobolev spaces, *Proc. A.M.S. Symp. Pure Math.*, **45** (1986), 455–462.

[11] J.-P. GOSSEZ AND V. MUSTONEN, Variational inequalities in Orlicz-Sobolev spaces, *Nonlinear Anal. T.M.A.*, **11** (1987), 379–392.