Journal of Integer Sequences, Vol. 10 (2007),
Article 07.5.4

Algorithms for Bernoulli and Allied Polynomials

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Abstract

We investigate some algorithms that produce Bernoulli, Euler and Genocchi polynomials. We also give closed formulas for Bernoulli, Euler and Genocchi polynomials in terms of weighted Stirling numbers of the second kind, which are extensions of known formulas for Bernoulli, Euler and Genocchi numbers involving Stirling numbers of the second kind.

1 Introduction and Primary Concepts

Bernoulli numbers B_n, $n \in \mathbb{Z}$, $n \geqslant 0$, originally arise in the study of finite sums of a given power of consecutive integers. They are given by $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30, \ldots$, with $B_n = 0$ for odd $n > 1$, and

$$B_n = -\frac{1}{n+1} \sum_{m=0}^{n-1} \binom{n+1}{m} B_m$$

for all even $n \geqslant 2$. In the symbolic notation, Bernoulli numbers are given recursively by

$$(B+1)^n - B_n = \begin{cases} 1, & \text{if } n = 1; \\ 0, & \text{if } n > 1. \end{cases}$$

with the usual convention about replacing B^j by B_j after expansion. The Bernoulli polynomials $B_n(x)$, $n \in \mathbb{Z}$, $n \geqslant 0$, can be expressed in the form

$$B_n(x) = (B+x)^n = \sum_{m=0}^{n} \binom{n}{m} B_m x^{n-m}. \tag{1}$$

The generating functions of Bernoulli numbers and polynomials are respectively given by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!},$$

and

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!},$$

for $|t| < 2\pi$ ([2]). The Euler polynomials $E_n(x)$, $n \in \mathbb{Z}$, $n \geqslant 0$, may be defined by the generating function [23, 24]

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \tag{2}$$

for $|t| < \pi$. The Euler numbers E_n are defined by

$$E_n = 2^n E_n\left(\frac{1}{2}\right), \; n \geqslant 0.$$

The Genocchi numbers and polynomials, G_n and $G_n(x)$, $n \in \mathbb{Z}$, $n \geqslant 0$, are defined respectively as follows [8, p.49]:

$$\frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, \tag{3}$$

and

$$\frac{2te^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!},$$

for $|t| < \pi$, which have several combinatorial interpretations in terms of certain surjective maps of finite sets [10, 12, 13]. The well-known identity

$$G_n = 2\left(1 - 2^n\right) B_n, \; n \geqslant 0,$$

shows the relation between Genocchi and Bernoulli numbers. From (2) and (3), it is easy to see that

$$G_n = nE_{n-1}(0), \; n \geqslant 0,$$

with $E_{-1} := 0$.

Several arithmetical properties of Bernoulli, Euler and Genocchi polynomials can be obtained from the generating functions. We list here three of them, which will be used in the next section.

$$B_n(1+x) = B_n(x) + nx^{n-1}, \; n \geqslant 0, \tag{4}$$

$$E_n(1+x) = -E_n(x) + 2x^n, \; n \geqslant 0, \tag{5}$$

$$G_n(1+x) = -G_n(x) + 2nx^{n-1}, \; n \geqslant 0. \tag{6}$$

Bernoulli, Euler and Genocchi numbers can be directly computed from generating functions by dividing numerator to denominator (after expanding e^t) and comparing the coefficients on either sides. Besides this method, many recurrence formulas for Bernoulli, Euler and Genocchi numbers are obtained [3, 6, 8, 9, 16, 18, 19, 22]. Alternative methods for computing these numbers are developed as well. In this paper, we concern two of them, the Euler-Seidel matrices and the Akiyama-Tanigawa algorithm.

Definition 1.1. *([11]) Let a_0, a_1, a_2,...,a_n,... be an initial sequence. The Euler-Seidel matrix corresponding to the sequence $\{a_n\}$ is determined by the sequence $\left\{a_n^k\right\}$, whose elements are recursively given by*

$$\begin{aligned} a_n^0 &= a_n \qquad (n \geqslant 0)\,, \\ a_n^k &= a_n^{k-1} + a_{n+1}^{k-1} \qquad (n \geqslant 0, k \geqslant 1)\,. \end{aligned} \tag{7}$$

The sequence $\{a_n^0\}$ is the first row and the sequence $\left\{a_0^k\right\}$ is the first column of the matrix. From the recurrence relation (7), it can be seen that

$$a_n^k = \sum_{i=0}^{k} \binom{k}{i} a_{n+i}^0,\ (n \geqslant 0, k \geqslant 1)\,. \tag{8}$$

The first row and column can be determined from (8) as follows:

$$\begin{aligned} a_0^n &= \sum_{i=0}^{n} \binom{n}{i} a_i^0,\ n \geqslant 0, \\ a_n^0 &= \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i}\, a_0^i,\ n \geqslant 0. \end{aligned} \tag{9}$$

Thus, as the first row is given, the first column can be found and vice versa. Next two propositions show the relationship between the generating functions of a_n^0 and a_0^n.

Proposition 1.1. *Let*

$$a\,(t) = \sum_{n=0}^{\infty} a_n^0 t^n$$

be the generating function of the initial sequence $\{a_n^0\}$. Then the generating function of the sequence $\{a_0^n\}$ is given by

$$\overline{a}\,(t) = \sum_{n=0}^{\infty} a_0^n t^n = \frac{1}{1-t} a \left(\frac{t}{1-t}\right).$$

The proof of this proposition is given by Euler [14].

Proposition 1.2. *Let*

$$A\,(t) = \sum_{n=0}^{\infty} a_n^0 \frac{t^n}{n!}$$

be the exponential generating function of the initial sequence $\{a_n^0\}$. *Then the exponential generating function of the sequence* $\{a_0^n\}$ *is given by*

$$\overline{A}(t) = \sum_{n=0}^{\infty} a_0^n \frac{t^n}{n!} = e^t A(t).$$

The proof of this proposition is given by Seidel [25].

Of notable interest of Euler-Seidel matrices is the examples for Bernoulli and allied numbers. Let

$$A(t) = \frac{t}{e^t - 1}.$$

Then

$$\overline{A}(t) = \frac{te^t}{e^t - 1} = t + \frac{t}{e^t - 1},$$

and thus

$$\overline{A}(t) = t + A(t).$$

Therefore,

$$a_0^0 = B_0 = 1,\ a_0^1 = \frac{1}{2},\ \text{and } a_n^0 = a_0^n \text{ for } n \geqslant 2,$$

and the corresponding Euler-Seidel matrix is

$$\begin{pmatrix} 1 & -\frac{1}{2} & \frac{1}{6} & 0 & -\frac{1}{30} & \cdots \\ \frac{1}{2} & -\frac{1}{3} & \frac{1}{6} & -\frac{1}{30} & \cdots & \\ \frac{1}{6} & -\frac{1}{6} & \frac{2}{15} & \cdots & & \\ 0 & -\frac{1}{30} & \cdots & & & \\ -\frac{1}{30} & \cdots & & & & \\ \cdots & & & & & \end{pmatrix}$$

For

$$A(t) = \frac{2t}{e^t + 1},$$

the initial sequence is

$$a_0^0 = 0,\ a_1^0 = 1,\ a_{2n+1}^0 = 0 \text{ and } a_{2n}^0 = G_{2n} \text{ for } n \geqslant 1.$$

Since

$$\overline{A}(t) = \frac{2te^t}{e^t + 1} = 2t - \frac{2t}{e^t + 1} = 2t - A(t),$$

it is seen that

$$a_0^0 = 0,\ a_0^1 = 1,\ a_0^{2n+1} = 0 \text{ and } a_0^{2n} = -G_{2n} \text{ for } n \geqslant 1.$$

Thus, the corresponding Euler-Seidel matrix is

$$\begin{pmatrix}
0 & 1 & -1 & 0 & 1 & 0 & -3 & 0 & 17 & \cdots \\
1 & 0 & -1 & 1 & 1 & -3 & -3 & 17 & \cdots & \\
1 & -1 & 0 & 2 & -2 & -6 & 14 & \cdots & & \\
0 & -1 & 2 & 0 & -8 & 8 & \cdots & & & \\
-1 & 1 & 2 & -8 & 0 & \cdots & & & & \\
0 & 3 & -6 & -8 & \cdots & & & & & \\
3 & -3 & -14 & \cdots & & & & & & \\
0 & -17 & \cdots & & & & & & & \\
-17 & \cdots & & & & & & & & \\
\cdots & & & & & & & & &
\end{pmatrix}$$

In their study of values at nonpositive integer arguments of multiple zeta functions, Akiyama and Tanigawa [1] found an algorithm for computing Bernoulli numbers in a manner similar to Pascal's triangle for binomial coefficients. The algorithm is as follows: Starting with the sequence $\{1/n\}$, $n \geqslant 1$, as the 0-th row, the m-th number in the $(n+1)$-st row a_m^{n+1} is determined recursively by

$$a_m^{n+1} = (m+1)\left(a_m^n - a_{m+1}^n\right),$$

for $m = 0, 1, 2, \ldots$ and $n = 0, 1, 2, \ldots$. Then, a_0^n of each row is the n-th Bernoulli number B_n with $B_1 = 1/2$. Kaneko [20] reformulated the Akiyama-Tanigawa algorithm as follows:

Proposition 1.3. *Given an initial sequence a_m^0 with $m = 0, 1, 2, \ldots$ define sequence a_m^n for $n \geqslant 1$ recursively by*

$$a_m^n = (m+1)\left(a_m^{n-1} - a_{m+1}^{n-1}\right).$$

Then the leading elements are given by

$$a_0^n = \sum_{m=0}^{n} (-1)^m m! S(n+1, m+1)\, a_m^0.$$

Here, $S(n,m)$ denotes the Stirling numbers of the second kind which are defined as follows [8, Chap. 5]:

$$\frac{(e^t - 1)^m}{m!} = \sum_{n=m}^{\infty} S(n,m) \frac{t^n}{n!}. \tag{10}$$

For $1 \leqslant m \leqslant n$, $S(n,m) > 0$ and for $1 \leqslant n < m$, $S(n,m) = 0$. The Stirling numbers of the second kind $S(n,m)$ satisfy the recurrence relation

$$S(n,m) = S(n-1, m-1) + mS(n-1, m) \tag{11}$$

for $n, m \geqslant 1$ with $S(n,0) = S(0,m) = 0$, except $S(0,0) = 1$.

If the initial sequence is $a_m^0 = 1/(m+1)$, $m \geqslant 0$, in Proposition 1.3, then the leading elements are Bernoulli numbers with $B_1 = 1/2$. By replacing the initial sequence $a_m^0 =$

$1/(m+1)$ by $a_m^0 = 1/(m+1)^k$, $m \geqslant 0$, and applying same algorithm, Kaneko also derived the resulting sequence as poly-Bernoulli numbers [20].

Chen [7] changed the recursive step in Proposition 1.3 to

$$a_m^n = m a_m^{n-1} - (m+1)\, a_{m+1}^{n-1} \qquad (n \geqslant 1, m \geqslant 0)\,,$$

and proved the following:

Proposition 1.4. *Given an initial sequence a_m^0 with $m = 0, 1, 2, \ldots$ define sequence a_m^n for $n \geqslant 1$ recursively by*

$$a_m^n = m a_m^{n-1} - (m+1)\, a_{m+1}^{n-1}.$$

Then

$$a_0^n = \sum_{m=0}^{n} (-1)^m\, m! S(n,m)\, a_m^0.$$

Given initial sequences $1/(m+1)$, $m \geqslant 0$, $1/2^m$, $m \geqslant 0$, and $(-1)^{[m/4]}\, 2^{-[m/2]}\, (1 - \delta_{4,m+1})$, $m \geqslant 0$, Chen obtained the leading elements respectively as Bernoulli, Euler and tangent numbers. Here, $[x]$ is the greatest integer $\leqslant x$ and $\delta_{4,i} = 1$ if $4 \mid i$ and $\delta_{4,i} = 0$ otherwise. Tangent numbers are defined as follows (cf. [24, p. 35]):

$$\mathrm{tg} t = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\, T_{2n+1}}{(2n+1)!} t^{2n+1},$$

with $T_0 = 1$. At this point, we note that by choosing the initial sequence $a_m^0 = 1/2^m$, $m \geqslant 0$, we obtain the leading elements a_0^{n-1} as G_n/n for $n \geqslant 1$, which is not obtained in [7]. For other studies on the Akiyama-Tanigawa algorithm, see [17, 21].

In this paper, we give the Euler-Seidel matrices and the Akiyama-Tanigawa algorithms for Bernoulli, Euler and Genocchi polynomials. These matrices and algorithms are polynomial extensions of the corresponding matrices and algorithms for Bernoulli, Euler and Genocchi numbers. In particular, the Akiyama-Tanigawa algorithms for these polynomials lead new closed formulas for Bernoulli, Euler and Genocchi polynomials in terms of weighted Stirling numbers of the second kind.

2 Euler-Seidel Matrices for Bernoulli, Euler and Genocchi Polynomials

We start with the polynomial extension of Definition 1.1.

Definition 2.1. *Let $a_0(x)$, $a_1(x)$, $a_2(x)$,…,$a_n(x)$,… be initial sequence of polynomials in x. The Euler-Seidel matrix corresponding to the sequence $\{a_n(x)\}$ is determined by the sequence $\left\{a_n^k(x)\right\}$ for $n \geqslant 0$ and $k \geqslant 1$, whose elements are recursively given by*

$$\begin{aligned} a_n^0(x) &= a_n(x)\,, \\ a_n^k(x) &= a_n^{k-1}(x) + a_{n+1}^{k-1}(x)\,. \end{aligned}$$

From the recurrence relation above, we have

$$\begin{aligned}
a_n^k(x) &= \sum_{i=0}^{k} \binom{k}{i} a_{n+i}^0(x),\ n \geqslant 0, \\
a_0^n(x) &= \sum_{i=0}^{n} \binom{n}{i} a_i^0(x),\ n \geqslant 0, \\
a_n^0(x) &= \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} a_0^i(x),\ n \geqslant 0.
\end{aligned} \tag{12}$$

Let $A(x,t)$ and $\overline{A}(x,t)$ be the exponential generating functions of $a_n^0(x)$ and $a_0^n(x)$, respectively. Then, we have

$$\overline{A}(x,t) = e^t A(x,t).$$

Note that for $x = 0$, the above equalities reduce to Dumont's.

Now, we construct the Euler-Seidel matrices corresponding to Bernoulli, Euler and Genocchi polynomials.

Let $a_n^0(x) = B_n(x)$, $n \geqslant 0$. Then

$$A(x,t) = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = \frac{te^{xt}}{e^t - 1},$$

and

$$\overline{A}(x,t) = e^t A(x,t) = \frac{te^{(x+1)t}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x+1) \frac{t^n}{n!}. \tag{13}$$

Using (4), we obtain

$$\overline{A}(x,t) = A(x,t) + te^{xt},$$

and

$$a_0^n(x) = B_n(x) + nx^{n-1},\ n \geqslant 0.$$

Therefore, given Bernoulli polynomials as the first row, then the first column of the corresponding Euler-Seidel matrix is again Bernoulli polynomials except the opposite sign for the coefficient of the second highest power. This is because

$$\begin{aligned}
a_0^n(x) &= B_n(x) + nx^{n-1} \\
&= \sum_{m=0}^{n} \binom{n}{m} B_m x^{n-m} + nx^{n-1} \\
&= B_0 x^n + B_1 x^{n-1} + nx^{n-1} + \sum_{m=2}^{n} \binom{n}{m} B_m x^{n-m} \\
&= B_0 x^n - \frac{n}{2} x^{n-1} + nx^{n-1} + \sum_{m=2}^{n} \binom{n}{m} B_m x^{n-m} \\
&= B_0 x^n + \frac{n}{2} x^{n-1} + \sum_{m=2}^{n} \binom{n}{m} B_m x^{n-m},
\end{aligned}$$

by using (1) and $B_1 = -1/2$. The Euler-Seidel matrix corresponding to Bernoulli polynomials is as follows:

$$\begin{pmatrix} 1 & x-\frac{1}{2} & x^2-x+\frac{1}{6} & x^3-\frac{3}{2}x^2+\frac{1}{2}x & \cdots \\ x+\frac{1}{2} & x^2-\frac{1}{3} & x^3-\frac{1}{2}x^2-\frac{1}{2}x+\frac{1}{6} & \cdots & \\ x^2+x+\frac{1}{6} & x^3+\frac{1}{2}x^2-\frac{1}{2}x-\frac{1}{6} & \cdots & & \\ x^3+\frac{3}{2}x^2+\frac{1}{2}x & \cdots & & & \\ \cdots & & & & \end{pmatrix}$$

Now let $a_n^0(x) = E_n(x)$, $n \geqslant 0$. Then

$$A(x,t) = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} = \frac{2e^{xt}}{e^t+1},$$

and

$$\overline{A}(x,t) = e^t A(x,t) = \frac{2e^{(x+1)t}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x+1) \frac{t^n}{n!}. \tag{14}$$

Using (5), we get

$$\overline{A}(x,t) = -A(x,t) + 2e^{xt},$$

and

$$a_0^n(x) = -E_n(x) + 2x^n,\ n \geqslant 0.$$

Thus, if Euler polynomials are taken as the first row, then the first column of the corresponding Euler-Seidel matrix is negative Euler polynomials except the coefficient of the highest power. The matrix is

$$\begin{pmatrix} 1 & x-\frac{1}{2} & x^2-x & x^3-\frac{3}{2}x^2+\frac{1}{4} & \cdots \\ x+\frac{1}{2} & x^2-\frac{1}{2} & x^3-\frac{1}{2}x^2-x+\frac{1}{4} & \cdots & \\ x^2+x & x^3+\frac{1}{2}x^2-x-\frac{1}{4} & \cdots & & \\ x^3+\frac{3}{2}x^2-\frac{1}{4} & \cdots & & & \\ \cdots & & & & \end{pmatrix}$$

Finally, let $a_n^0(x) = G_n(x)$, $n \geqslant 0$. Then

$$A(x,t) = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!} = \frac{2te^{xt}}{e^t+1},$$

and

$$\overline{A}(x,t) = e^t A(x,t) = \frac{2te^{(x+1)t}}{e^t+1} = \sum_{n=0}^{\infty} G_n(x+1) \frac{t^n}{n!}. \tag{15}$$

Using (6) yields

$$\overline{A}(x,t) = -A(x,t) + 2te^{xt},$$

and

$$a_0^n(x) = -G_n(x) + 2nx^{n-1},\ n \geqslant 0.$$

The corresponding matrix is

$$\begin{pmatrix} 0 & 1 & 2x-1 & 3x^2-3x & \cdots \\ 1 & 2x & 3x^2-x-1 & \cdots & \\ 2x+1 & 3x^2+x-1 & \cdots & & \\ 3x^2+3x & \cdots & & & \\ \cdots & & & & \end{pmatrix}$$

We conclude this section by giving alternative proofs of the equations

$$\begin{aligned} B_n(x+1) &= \sum_{m=0}^{n} \binom{n}{m} B_m(x), \; n \geqslant 0, \\ E_n(x+1) &= \sum_{m=0}^{n} \binom{n}{m} E_m(x), \; n \geqslant 0, \\ G_n(x+1) &= \sum_{m=0}^{n} \binom{n}{m} G_m(x), \; n \geqslant 0. \end{aligned}$$

From (13), we see that

$$\sum_{n=0}^{\infty} a_0^n(x) \frac{t^n}{n!} = \overline{A}(x,t) = \sum_{n=0}^{\infty} B_n(x+1) \frac{t^n}{n!}.$$

Thus, by comparing the coefficients of power series on both sides, we get $a_0^n(x) = B_n(x+1)$. By (12), we have

$$B_n(x+1) = a_0^n(x) = \sum_{m=0}^{n} \binom{n}{m} a_m^0(x) = \sum_{m=0}^{n} \binom{n}{m} B_m(x).$$

Other two equalities can be proved in the same way by using appropriate relations (14) or (15).

3 The Akiyama-Tanigawa Algorithms for Bernoulli, Euler and Genocchi Polynomials

In this section, we derive the Akiyama-Tanigawa algorithms for Bernoulli, Euler and Genocchi polynomials and give closed formulas for these polynomials in terms of weighted Stirling numbers of the second kind.

The weighted Stirling numbers of the second kind, $S(n,m,x)$, are defined as follows [4, 5]:

$$\frac{e^{xt}(e^t-1)^m}{m!} = \sum_{n=m}^{\infty} S(n,m,x) \frac{t^n}{n!}. \tag{16}$$

When $x = 0$, $S(n,m,0) = S(n,m)$ are the Stirling numbers of the second kind. For $n = m = 0$, $S(0,0,x) = 1$, for $1 \leqslant m < n$, $S(n,m,x) = 0$ and for other values of m and n, $S(n,m,x)$ are determined by the following recurrence formula:

$$S(n+1,m,x) = (m+x) S(n,m,x) + S(n,m-1,x). \tag{17}$$

The relationship between $S(n,m,x)$ and $S(n,m)$ can be given by the following equation:

Lemma 3.1. *We have*

$$S(n,m,x) = \sum_{k=0}^{n-m} \binom{n}{k} x^k S(n-k,m).$$

Proof. By (16) and (10), we have

$$\begin{aligned} \sum_{n=m}^{\infty} S(n,m,x) \frac{t^n}{n!} &= \frac{e^{xt}(e^t-1)^m}{m!} = e^{xt} \sum_{n=m}^{\infty} S(n,m) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} x^n \frac{t^n}{n!} \sum_{n=0}^{\infty} S(n+m,m) \frac{t^{n+m}}{(n+m)!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{x^k}{k!} \frac{S(n-k+m,m)}{(n-k+m)!} \right) t^{n+m}. \end{aligned}$$

Comparing the coefficients of power series yields the result. $\square$

Given an initial sequence a_m^0 with $m = 0, 1, 2, \ldots$, let the sequence a_m^n for $n \geqslant 1$ be given recursively by

$$a_m^n = m a_m^{n-1} - (m+1) a_{m+1}^{n-1} \tag{18}$$

as in Proposition 1.4. Let $g_n(t)$ be the generating function of a_m^n,

$$g_n(t) = \sum_{m=0}^{\infty} a_m^n t^m. \tag{19}$$

We define $a_m^n(x)$ and $g_n(x,t)$ by means of

$$a_m^n(x) = (x + a_m)^n = \sum_{k=0}^{n} \binom{n}{k} x^k a_m^{n-k},\ n \geqslant 0, \tag{20}$$

$$g_n(x,t) = (x + g(t))^n = \sum_{k=0}^{n} \binom{n}{k} x^k g_{n-k}(t),\ n \geqslant 0, \tag{21}$$

where we use the usual convention about replacing $(a_m)^j$ by a_m^j and $(g(t))^j$ by $g_j(t)$ in the binomial expansions. Note that for $x = 0$, (20) reduces to (18) and (21) reduces to (19).

Proposition 3.1. *Given an initial sequence $a_{0,m}$ with $m = 0, 1, 2, \ldots$, let the sequence a_m^n for $n \geqslant 1$ be given as (18) and $a_m^n(x)$ be given as (20). Then*

$$a_0^n(x) = \sum_{m=0}^{n} (-1)^m m! S(n, m, x) a_m^0, \tag{22}$$

where $S(n, m, x)$ are the weighted Stirling numbers of the second kind.

Proof. From (21) and (19), we have

$$g_n(x,t) = \sum_{k=0}^{n} \binom{n}{k} x^k g_{n-k}(t) = \sum_{k=0}^{n} \binom{n}{k} x^k \sum_{m=0}^{\infty} a_m^{n-k} t^m.$$

By the recurrence formula (18), we get

$$\begin{aligned}
g_n(x,t) &= \sum_{k=0}^{n} \binom{n}{k} x^k \sum_{m=0}^{\infty} \left\{ m a_m^{n-k-1} - (m+1) a_{m+1}^{n-k-1} \right\} t^m \\
&= \sum_{k=0}^{n} \binom{n}{k} x^k \sum_{m=0}^{\infty} (m+1) a_{m+1}^{n-k-1} t^{m+1} \\
&\quad - \sum_{k=0}^{n} \binom{n}{k} x^k (m+1) a_{m+1}^{n-k-1} t^m \\
&= \sum_{k=0}^{n} \binom{n}{k} x^k (t-1) \sum_{m=0}^{\infty} (m+1) a_{m+1}^{n-k-1} t^m \\
&= \sum_{k=0}^{n} \binom{n}{k} x^k (t-1) \frac{d}{dt} (g_{n-k-1}(t)) \\
&= \sum_{k=0}^{n} \binom{n}{k} x^k \left((t-1) \frac{d}{dt} \right)^{n-k} g_0(t).
\end{aligned}$$

Applying the formula (cf. [15, p. 310])

$$\left((t-1) \frac{d}{dt} \right)^{n-k} = \sum_{m=0}^{n-k} S(n-k, m) (t-1)^m \left(\frac{d}{dt} \right)^m,$$

we have

$$g_n(x,t) = \sum_{k=0}^{n} \binom{n}{k} x^k \sum_{m=0}^{n-k} S(n-k, m) (t-1)^m \left(\frac{d}{dt} \right)^m g_0(t).$$

Setting $t = 0$ and interchanging summations, we obtain

$$a_0^n(x) = \sum_{m=0}^{n} (-1)^m m! a_m^0 \sum_{k=0}^{n-m} \binom{n}{k} x^k S(n-k, m).$$

Using Lemma 3.1, the result follows. □

Theorem 3.1. *Let $a_m^0 = 1/(m+1)$, $m \geqslant 0$, in (22). Then the leading polynomials $a_0^n(x)$, $n \geqslant 0$, are given by*

$$a_0^n(x) = \sum_{m=0}^{n} (-1)^m m! \frac{S(n,m,x)}{m+1} = B_n(x).$$

Proof. We have

$$\begin{aligned}
&\sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} (-1)^m m! \frac{S(n,m,x)}{m+1} \right) \frac{t^n}{n!} \\
&\quad = \sum_{m=0}^{\infty} \frac{(-1)^m m!}{m+1} \sum_{n=m}^{\infty} S(n,m,x) \frac{t^n}{n!} \\
&\quad = \sum_{m=0}^{\infty} \frac{(-1)^m m!}{m+1} \frac{e^{xt}(e^t-1)^m}{m!} \\
&\quad = \frac{e^{xt}}{1-e^t} \sum_{m=1}^{\infty} \frac{(1-e^t)^m}{m} \\
&\quad = \frac{e^{xt}}{1-e^t} \left(-\log\left(1-\left(1-e^t\right)\right)\right) = \frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.
\end{aligned}$$

This proves the theorem. □

Theorem 3.2. *Let $a_m^0 = 1/2^m$, $m \geqslant 0$, in (22). Then the leading polynomials $a_0^n(x)$, $n \geqslant 0$, are given by*

$$a_0^n(x) = \sum_{m=0}^{n} (-1)^m m! \frac{S(n,m,x)}{2^m} = E_n(x).$$

Proof. We have

$$\begin{aligned}
&\sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} (-1)^m m! \frac{S(n,m,x)}{2^m} \right) \frac{t^n}{n!} \\
&\quad = \sum_{m=0}^{\infty} \frac{(-1)^m m!}{2^m} \sum_{n=m}^{\infty} S(n,m,x) \frac{t^n}{n!} \\
&\quad = \sum_{m=0}^{\infty} \frac{(-1)^m m!}{2^m} \frac{e^{xt}(e^t-1)^m}{m!} \\
&\quad = e^{xt} \sum_{m=0}^{\infty} \left(\frac{1-e^t}{2} \right)^m = \frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients of power series yields the result. □

Theorem 3.3. *Let $a_m^0 = 1/2^m$, $m \geqslant 0$, as in Theorem 3.2. Then the polynomials $a_0^n(x)$, $n \geqslant 0$, are given by*

$$(n+1)\, a_0^n(x) = \sum_{m=0}^{n} (-1)^{m+1} (m+1)! \frac{S(n+1,m+1,x)}{2^{m+1}} = G_n(x).$$

Proof. The proof follows from Theorem 3.2 and the relations $G_n(x) = nE_{n-1}(x)$, $n \geqslant 0$, and $S(n, 0, x) = 0$. $\square$

4 Acknowledgment

The authors would like to thank the anonymous referee for valuable comments.

This work was supported by Akdeniz University Scientific Research Project Unit.

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2000 *Mathematics Subject Classification*: Primary 11B68.

Keywords: Bernoulli polynomials, Euler polynomials, Genocchi polynomials, weighted Stirling numbers of the second kind, Euler-Seidel matrices, Akiyama-Tanigawa algorithm.

(Concerned with sequences A000110, A000182, A000364, A001898, A027641, A027642, and A100616.)

Received September 18 2006; revised version received May 10 2007. Published in *Journal of Integer Sequences*, May 10 2007.

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