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# Polynomial Points

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**Abstract**

We determine the infinite sequences $(a_k)$ of integers that can be generated by polynomials with integral coefficients, in the sense that for each finite initial segment of length $n$ there is an integral polynomial $f_n(x)$ of degree $< n$ such that $a_k = f_n(k)$ for $k = 0, 1, \ldots, n-1$.

Let $\mathbf{P}$ be the set of such sequences and $\mathbf{\Pi}$ the additive group of all infinite sequences of integers. Then $\mathbf{P}$ is a subgroup of $\mathbf{\Pi}$ and $\mathbf{\Pi}/\mathbf{P} \cong \prod_{n=2}^{\infty} \mathbb{Z}/n!\mathbb{Z}$. The methods and results are applied to familiar families of polynomials such as Chebyshev polynomials and shifted Legendre polynomials.

The results are achieved by extending Lagrange interpolation polynomials to power series, using a special basis for the group of integral polynomials, called the integral root basis.

## 1 Introduction

In [4], we characterized the finite sequences $(a_1, a_2, \ldots, a_n)$ of integers for which there exists a polynomial $f(x) \in \mathbb{Z}[x]$ such that $f(i) = a_i$ for $i = 1, \ldots, n$. Let $P_n$ denote the set of all such sequences, which we call *polynomial sequences* or *polynomial points*, and let $\mathbb{Z}^n$ be the set of all integer sequences of length $n$, where $\mathbb{Z}$ represents the ring of integers. Further, let $\mathbb{Z}[x]$ denote the ring of polynomials over the integers and $\mathbb{Z}[x]_n$ the group of integral polynomials of degree $< n$; let $\mathbb{N}$ denote the natural numbers, and $\mathbb{N}^+$ the positive integers. Finally, let $\mathbb{Q}$ denote the rationals.

The main results of [4] were the following two theorems:

**Theorem 1.1.** *Let* $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$. *Let* $\ell_{\mathbf{a}}(x)$ *be the Lagrange interpolation polynomial for the sequence* $\mathbf{a}$. *Let* $B_n$ *be the* $n \times n$ *rational matrix whose* $(i, j)$*–entry is* $\frac{(-1)^{i+j}}{i!}\binom{i}{j}$ *for* $0 \leq i, j \leq n-1$. *The following are equivalent:*

1. $\mathbf{a} \in P_n$
2. $\ell_{\mathbf{a}}(x) \in \mathbb{Z}[x]_n$
3. $B_n\mathbf{a} \in \mathbb{Z}^n$ □

**Remarks 1.2.**

1. In (3) above, $\mathbf{a}$ is treated as a column vector. Explicitly, $\mathbf{a} \in P_n$ if and only if for all $i = 0, 1, \ldots, n-1$,

$$\sum_{j=0}^{i} \frac{(-1)^{i+j}}{j!(i-j)!} a_{j+1} \text{ is an integer.}$$

2. The matrices $B_n$, as $n$ varies, have the property that $B_m$ is the upper left $m \times m$ corner of $B_n$ for all $m \leq n$.
3. It is easy to see that if a sequence $\mathbf{a}$ is not generated by a unique polynomial of degree $< n$, then it cannot be generated by any polynomial at all.
4. The theorem determines a duality between the coefficients of an integral polynomial and its initial sequence of values.

**Theorem 1.3.** $P_n$ *is a rank* $n$ *subgroup of the free abelian group* $\mathbb{Z}^n$. $P_1 = \mathbb{Z},\ P_2 = \mathbb{Z}^2$ *and for* $n > 2,\ \mathbb{Z}^n/P_n \cong \mathbb{Z}/2!\mathbb{Z} \oplus \mathbb{Z}/3!\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/(n-1)!\mathbb{Z}$. □

The purpose of this paper is to extend these results to the infinite realm by replacing $\mathbb{Z}^n$ by $\mathbf{\Pi} = \prod_{i\in\mathbb{N}} \mathbb{Z}_i,\ \mathbb{Z}_i = \mathbb{Z}$, known as the Baer–Specker group. The three clauses of Theorem 1.1 suggest possibly different ways of extending the theorem by defining a suitable subgroup of $\mathbf{\Pi}$ as the analog of the $P_n$.

To simplify the notation, we index finite and infinite sequences, sums, products and matrices by natural numbers beginning with zero except where indicated. This slight change in notation from that in [4] does not of course make any substantive changes to the results of that paper.

If $M$ is an $n \times n$ or $\omega \times \omega$ matrix over the integers, we write $M = (m_{ij})$ where $m_{ij}$ is an expression representing an integer, $i$ represents the row index and $j$ the column index. In matrix–vector multiplication, the matrix acts on the left with the vector considered as a column. If necessary to render division by non–zero integers meaningful, we consider a torsion–free abelian group to be imbedded in its divisible hull [6, p 107].

The paper is organized as follows:

In Section 2, we define the integral root basis for $\mathbb{Z}[x]$ and use it to find stacked bases for $\mathbb{Z}^n$ and $P_n,\ n \geq 1$. We describe the transition matrices between these bases and the standard bases, and apply them to determine a duality between the coefficients and the values of Lagrange interpolation polynomials. An interesting consequence is that for every positive integer $n$, there are infinitely many polynomials $f(x) \in \mathbb{Z}[x]$ of degree $< n$ whose sequences of values $(f(0), f(1), \ldots, f(n-1))$ consist entirely of primes.

In the short Section 3, we deal in a similar manner with some special classes of integral polynomials, namely Chebyshev polynomials and shifted Legendre polynomials.

The major results of the paper are in Section 4. We expand integral polynomials to integral power series with respect to the integral root basis and extend the duality between coefficients and values mentioned above. We extend our earlier results from free groups of finite rank to $\mathbf{\Pi}$ and its subgroup $\mathbf{P}$, the infinite analog of $P_n$. We then compute the factor group $\mathbf{\Pi}/\mathbf{P}$ and the product of the factor groups $\prod_{n\in\mathbb{N}^+} \mathbb{Z}^n/P_n$.

In Section 5, we apply these results to study various familiar subgroups of the Baer–Specker group, and describe the torsion part of $\mathbf{\Pi}/\mathbf{P}$.

Finally, in Section 6, we consider consider formal and analytic properties of power series with respect to the integral root basis.

# 2 The Integral Root Basis

The methodology in [4] is based upon the *integral root basis* of $\mathbb{Z}[x],\ R = \{\rho_j(x) : j \in \mathbb{N}\}$ where $\rho_0(x) \equiv 1$ and $\rho_{j+1}(x) = \rho_j(x)(x-j)$ for $j \in \mathbb{N}$. Each initial segment of $R,\ R_n = \{\rho_j(x) : j = 0, 1, \ldots, n-1\}$, forms a basis for $\mathbb{Z}[x]_n,\ n \in \mathbb{N}^+$, which we refer to as the integral root basis of $\mathbb{Z}[x]_n$.

Let $\mathbf{a}$ be the initial segment of length $m$ of an integer sequence $\mathbf{b}$ of length $n,\ m \leq n$. The integral root basis has the property that the Lagrange interpolation polynomial for $\mathbf{a}$ is the initial segment of degree $< m$ of the Lagrange interpolation polynomial for $\mathbf{b}$, when the polynomials are expressed as linear combinations of that basis.

The valuation map $v : \mathbb{Z}[x] \to \mathbb{Z}^n$, defined by

$$f(x) \mapsto (f(0),\ f(1), \ldots, f(n-1))$$

has kernel $\rho_n(x)\mathbb{Z}[x]$. When restricted to $\mathbb{Z}[x]_n,\ v$ determines an isomorphism of $\mathbb{Z}[x]_n$ with $P_n$, under which the integral root basis of $\mathbb{Z}[x]_n$ is mapped to a basis of $P_n$ that we call the Gamma basis. Specifically, $\rho_j(x) \mapsto \gamma_j = ((0)_j,\ (1)_j,\ \ldots, (n-1)_j)$ where $(i)_j$ is the *falling factorial* $j!\binom{i}{j}$. Since $\gamma_j$ is divisible by $j!$, we may set $\alpha_j = \gamma_j/j! = \left(\binom{0}{j}, \binom{1}{j}, \ldots, \binom{n-1}{j}\right)$. Consequently, we obtain a basis of $\mathbb{Z}^n$, which we call the Alpha basis. The Alpha and Gamma bases are stacked bases [3; 6, Lemma 15.4] that readily reveal the structure of $\mathbb{Z}^n/P_n$.

The transition matrix from the Alpha basis for $\mathbb{Z}^n$ to the standard basis is Pascal's matrix $A_n$ [5], a lower-triangular matrix, the $(i, j)$–entry of which is $\binom{i}{j},\ i, j = 0, 1, \ldots, n-$

1. $A_n$ is easily seen to be invertible with lower-triangular inverse whose $(i,j)$–entry is $(-1)^{i+j}\binom{i}{j}$, $i,\, j = 0,\, 1, \ldots, n-1$.

As the dimension increases from $n$ to $n+1$, the valuation map induces a canonical injection of $P_n$ into $P_{n+1}$, under which the Gamma basis of $P_n$ is mapped into the first $n$ elements of the Gamma basis of $P_{n+1}$, which, together with the image of $\rho_n(x)$, form the Gamma basis of $P_{n+1}$. Conversely, the Gamma basis of $P_n$ is the canonical projection of the first $n$ elements of the Gamma basis of $P_{n+1}$ into $P_n$. Similarly, the Alpha basis of $\mathbb{Z}^{n+1}$ is derived from the Alpha basis of $\mathbb{Z}^n$, and conversely. Consequently, no confusion arises from using the symbols $\alpha_j$ and $\gamma_j$ without reference to dimension.

The matrix $C_n$ whose columns are the Gamma basis of $P_n$ is not integrally invertible for $n > 2$, but is invertible over $\mathbb{Q}$. If $D_n$ is the diagonal matrix whose $j$th diagonal entry is $j!$, for $j = 0, 1, \ldots, n-1$, then $C_n = A_n D_n$. Moreover the matrix $B_n$ defined in the Introduction is $C_n^{-1} = D_n^{-1} A_n^{-1}$. Note that for $m \le n$, the top left $m \times m$ corner of $A_n$ is $A_m$ and similarly for $C_n$ and $C_m$.

The following result is implicit in [4]. It is convenient to spell it out explicitly, since it provides the motivation for generalizations to the infinite case.

**Proposition 2.1.** *Let $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1}) \in \mathbb{Z}^n$ and let $f(x) = \sum_{i=0}^{n-1} a_i \rho_i(x)$. Then $C_n\mathbf{a}$ is the sequence $(f(0), f(1), \ldots, f(n-1))$.*

*Conversely, let $\mathbf{a} \in \mathbb{Z}^n$ and let $\mathbf{b} = B_n\mathbf{a}$. Then $\mathbf{b}$ is the sequence of (rational) coefficients with respect to the integral root basis of the Lagrange interpolation polynomial $\ell_{\mathbf{a}}(x)$ of $\mathbf{a}$.*

*Proof.* For $j = 0, \ldots, n-1$, the $j$th of column of $C_n$ is the sequence of values $(\rho_j(0), \rho_j(1), \ldots, \rho_j(n-1))$ of the $j$th element of the integral root basis of $\mathbb{Z}[x]_n$. Hence $C_n\mathbf{a}$ is the sequence of values of $f(x)$ at $0, \ldots, n-1$.

Conversely, the coefficients of $\ell_{\mathbf{a}}(x)$, with respect to the integral root basis, are precisely the terms of the sequence $B_n\mathbf{a}$. □

Proposition 2.1 can be expressed in terms of the standard basis $S_n = \{x^i : i = 0, \ldots, n-1\}$ of $\mathbb{Z}[x]_n$. Let $L_n$ be the $n \times n$ matrix

$$(i^j), \; i, j = 0, 1, \ldots, n-1, \text{ with } 0^0 = 1,$$

and let $M_n$ be the $n \times n$ matrix

$$\left( \sum_{k=0}^{n-1} (-1)^{i+k} \begin{bmatrix} k \\ i \end{bmatrix} (k)_j \right), \; i, j = 0, \ldots, n-1,$$

where the $\begin{bmatrix} k \\ i \end{bmatrix}$ are the Stirling cycle numbers, also known as Stirling numbers of the first kind [10, pp 65–68].

**Corollary 2.1.** *Let $f(x) = \sum_{i=0}^{n-1} c_i x^i \in \mathbb{Z}[x]_n$ and let $\mathbf{a}$ be its sequence of values. Then $\mathbf{a} = L_n\mathbf{c}$, where $\mathbf{c} = (c_0, \ldots, c_{n-1})$.*

*Conversely, let $\mathbf{a} \in \mathbb{Z}^n$. Then $\mathbf{c} = M_n\mathbf{a}$ is the (rational) sequence of coefficients of the Lagrange interpolation polynomial for $\mathbf{a}$, with respect to the standard basis of $\mathbb{Z}[x]_n$. Moreover, $\mathbf{a} \in P_n$ if and only if $\mathbf{c} \in \mathbb{Z}^n$.*

*Proof.* The expansion of the integral root basis polynomials $\rho_i(x)$ as polynomials in the standard basis and *vice versa* are computed in [10, p 65, (40) and (41)]. The corresponding transition matrices appear in slightly different form on page 66 of the same work.

The transition matrix from $R_n$ to $S_n$ is $K_n = \left((-1)^{i+j}\begin{bmatrix} j \\ i \end{bmatrix}\right)$. Its inverse is $J_n = \left(\begin{Bmatrix} j \\ i \end{Bmatrix}\right)$ [10, p 67, (43)], where the $\begin{Bmatrix} j \\ i \end{Bmatrix}$ are Stirling subset numbers, also known as Stirling numbers of the second kind.

Then $L_n = B_n J_n$ and $M_n = K_n C_n$, so the result follows from Proposition 2.1. $\square$

The integral root basis has important properties not shared by the standard basis:

1. Its evaluation provides stacked bases for $P_n \subseteq \mathbb{Z}^n$ for every positive integer $n$, as shown above.

2. Every integral coefficient power series, expressed with respect to the integral root basis, converges at every non–negative integer. We exploit this property in Section 4 below.

An interesting consequence of Proposition 2.1 concerns the representation of primes by polynomials, a topic of current interest in number theory. Green and Tao [9] have shown that for every positive integer $n$, there are infinitely many arithmetic progressions of length $n$ consisting entirely of primes. Since every arithmetic progression of length $n$ is a polynomial point, $P_n$ abounds with such sequences. A weaker but still interesting result can be obtained with only the techniques employed here.

**Proposition 2.2.** *For each $n \in \mathbb{N}^+$, there are infinitely many sequences of primes in $P_n$.*

*Proof.* Fix $n \in \mathbb{N}^+$ and let $\mathbf{a} = (a_i : i = 0, \ldots, n-1)$ be a sequence of primes in the arithmetic progression $\{1 + kn! : k \in \mathbb{N}\}$. There are infinitely many choices of such sequences, by Dirichlet's theorem on primes in arithmetic progressions.

Each $a_i = 1 + k_i n!$ for some $k_i \in \mathbb{N}$. Let $\mathbf{k} = (k_i)$ and let $\mathbf{1} = (1, 1, \ldots, 1)^T$ ($n$ terms) so that $B_n\mathbf{a} = B_n\mathbf{1} + n!B_n\mathbf{k}$. By Proposition 2.1, $B_n\mathbf{1} = (1, 0, \ldots, 0)^T$ and by Remark 1.2 (1), $B_n\mathbf{k} \in \mathbb{Z}^n$. Hence $B_n\mathbf{a} \in \mathbb{Z}^n$, so by Theorem 1.1, $\mathbf{a} \in P_n$. Hence $\mathbf{a}$ is $v(f(x))$ for some polynomial $f(x)$ of degree $< n$. $\square$

# 3 Special polynomials

The foregoing techniques also can be used to study familiar families of integral polynomials. For example, let

$$T_n(x) = \frac{n}{2}\sum_{r=0}^{\lfloor n/2 \rfloor} \frac{(-1)^r}{n-r}\binom{n-r}{r}(2x)^{n-2r}$$

denote the $n$th Chebyshev polynomial, a useful tool in approximation theory, solution of differential equations and numerical analysis. Let $T[x]$ denote the subgroup of $\mathbb{Z}[x]$ that these polynomials generate. Let $T[x]_n = T[x] \cap \mathbb{Z}[x]_n$ be the subgroup of the polynomials in $T[x]$ of degree $< n$. For $n \in \mathbb{N}^+$, let $TP_n$ denote the image in $\mathbb{Z}^n$ of $T[x]_n$ under the valuation map $v$.

**Proposition 3.1.** *With the notation above, Chebyshev polynomials have these properties:*

1. $\{1, x, \ldots, 2^{n-1}x^n, \ldots\}$ *is a basis of* $T[x]$.
2. $\mathbb{Z}[x]_n/T[x]_n \cong P_n/TP_n \cong \oplus_{i=2}^{n-1}\mathbb{Z}/2^{i-1}\mathbb{Z}$
3. $\mathbb{Z}^n/TP_n \cong \oplus_{i=2}^{n-1}\mathbb{Z}/2^{i-1}i!\mathbb{Z}$

*Proof.* (1) It is clear that $\{1\}$ and $\{1, x\}$ are bases for $T_1(x)$ and $T_2(x)$, respectively. From this fact and the recurrence satisfied by the Chebyshev polynomials, $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$, the general result follows.

(2) The first isomorphism is induced by $v$, and the second follows from (1).

(3) follows from (2) and Theorem 1.3. □

Similarly, let

$$L_n(x) = (-1)^n \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}(-x)^k$$

denote the $n$th shifted Legendre polynomial, used in the solution of certain differential equations that arise in physics. Let $L[x]$ denote the subgroup of $\mathbb{Z}[x]$ that these polynomials generate, and let $L[x]_n = L[x] \cap \mathbb{Z}[x]_n$ be the subgroup of polynomials in $L[x]$ of degree $< n$. For $n \in \mathbb{N}^+$, let $LP_n$ denote the image in $\mathbb{Z}^n$ of $L[x]_n$ under the valuation map $v$.

**Proposition 3.2.** *With the notation above, Legendre polynomials have these properties:*

1. $\{1, 2x, \ldots, (2n)!/(n!)^2x^n, \ldots\}$ *is a basis of* $L[x]$.
2. $\mathbb{Z}[x]_n/L[x]_n \cong P_n/LP_n \cong \oplus_{i=1}^{n-1}\mathbb{Z}/(2i)!/(i!)^2\mathbb{Z}$
3. $\mathbb{Z}^n/LP_n \cong \oplus_{i=1}^{n-1}\mathbb{Z}/(2i)!/i!\mathbb{Z}$

*Proof.* (1) A straightforward induction argument reveals that the $k$th coefficient of $L_n(x)$ is a multiple of $(2k)!/(k!)^2$.

(2) is clear from (1).

(3) In general, if $S[x]_n$ is a subgroup of $Z[x]_n$ that satisfies $S[x]_n = \oplus_{i=0}^{n-1}\langle k_i x^i\rangle$ for $k_i \in \mathbb{Z}$, $i = 0, \ldots, n-1$, then $\mathbb{Z}^n/v(S[x]_n) \cong \oplus_{i=0}^{n-1}\mathbb{Z}/k_i i!\mathbb{Z}$, where $v$ is the valuation map. □

# 4 Extensions to infinite sequences of integers

The first step in generalizing Theorems 1.1 and 1.3 to the infinite case is to extend the valuation map $v$ to $v^* : \mathbb{Z}[x] \to \mathbf{\Pi}$ by $v^* : f(x) \mapsto (f(n) : n \in \mathbb{N})$. Let $P_\omega$ denote the image of $v^*$ in $\mathbf{\Pi}$ and let $\mathbf{\Sigma}$ denote those sequences in $\mathbf{\Pi}$ that are eventually zero; $\mathbf{\Sigma}$ is a countable free subgroup of $\mathbf{\Pi}$. Further, let $P_*$ denote the pure subgroup of $\mathbf{\Pi}$ generated by $P_\omega$.

Next, extend the Gamma basis elements to elements of $\mathbf{\Pi}$ by the valuation map $v^*$ : $\rho_n(x) \mapsto \gamma_n = ((k)_n : k \in \mathbb{N})$ for all $n \in \mathbb{N}$. Similarly, extend the Alpha basis elements to $\alpha_n = \gamma_n/n! = \left(\binom{k}{n} : k \in \mathbb{N}\right)$. These extensions yield:

**Theorem 4.1.** *1. The valuation map $v^* : f(x) \mapsto (f(n) : n \in \mathbb{N})$ is an isomorphism of $\mathbb{Z}[x]$ onto $P_\omega$.*

*2. $P_\omega$ is a countable free subgroup of $\mathbf{\Pi}$ and the extended Gamma basis is a basis of $P_\omega$.*

*3. $P_*$ also is a countable free subgroup of $\mathbf{\Pi}$ and the extended Alpha basis is a basis of $P_*$.*

*4. The extended Alpha and Gamma bases are stacked bases [3] of $P_*$ and $P_\omega$ and $P_*/P_\omega \cong \oplus_{n\geq 2}\mathbb{Z}/n!\mathbb{Z}$.*

*Proof.* (1) The valuation map $v^*$ is epic by definition of $P_\omega$ and $v^*$ is certainly monic, since only the zero polynomial has infinitely many zeros. Thus $v^*$ is an isomorphism.

(2) Clearly $\mathbb{Z}[x] \cong \mathbf{\Sigma} \cong P_\omega$, so that $P_\omega$ is a countable free subgroup of $\mathbf{\Pi}$. Since $v^*$ is an isomorphism, it carries the integral root basis of $\mathbb{Z}[x]$ to a basis for $P_\omega$, which by definition is the Gamma basis.

(3) $P_*$ is countable because it is the pure subgroup of $\mathbf{\Pi}$ generated by the countable subgroup $P_\omega$ [6, p 116], and it is free because all countable subgroups of $\mathbf{\Pi}$ are free. The independence of the Alpha basis elements of $P_*$ follows from the fact that $\alpha_n = \gamma_n/n!$ for all $n \in \mathbb{N}$ and the $\gamma_n$ are themselves independent.

All that remains is to demonstrate that the $\alpha_n$ span $P_*$. From $n!\alpha_n = \gamma_n$ it follows that each $\alpha_n \in P_*$. Suppose that $mx = j_0\gamma_0 + \cdots + j_k\gamma_k \in P_\omega$ is a linear combination of the $\gamma_n$, so that $x \in P_*$ by purity.

Then $mx = j_0 0!\alpha_0 + \cdots + j_k k!\alpha_k$. From the finite dimensional case, we know that $m$ divides each term $j_n n!$ in this sum, since $\alpha_0, \ldots, \alpha_k$ is a basis of $\mathbb{Z}^{k+1}$. Thus $x = (j_0 0!/m)\alpha_0 + \cdots + (j_k k!/m)\alpha_k$, so that the $\alpha_n$ span $P_*$ and hence form a basis.

(4) The Alpha and Gamma bases are stacked so that $P_*/P_\omega \cong \oplus_{n\in\mathbb{N}}\mathbb{Z}/n!\mathbb{Z}$. Because $0! = 1 = 1!$, the first two factors are degenerate, leaving the required result. $\square$

The $\omega \times \omega$ matrix whose columns are the extended Alpha basis is the infinite Pascal matrix $A = \left(\binom{i}{j} : i, j \in \mathbb{N}\right)$ which has integral inverse $A^{-1} = \left((-1)^{i+j}\binom{i}{j} : i, j \in \mathbb{N}\right)$[1, p 2]. Similarly, the matrices $C_n$ can be extended to an $\omega \times \omega$ matrix $C$, the top left $n \times n$ corner of which, for all $n \in \mathbb{N}^+$, is $C_n$; and the matrices $B_n$ can be extended to an $\omega \times \omega$ matrix $B$, the top left $n \times n$ corner of which, for all $n \in \mathbb{N}^+$, is $B_n$; then $C = B^{-1}$. Likewise, the diagonal matrices $D_n$ can be extended to a diagonal $\omega \times \omega$ matrix $D$, so that $C = AD$. Since $A$, $B$, $C$ and $D$ are row finite, they act by left multiplication on $\mathbf{\Pi}$. $B$ of course is not integral.

We proceed to generalize Theorems 1.1 and 1.3. Let $\mathbf{a} = (a_i : i \in \mathbb{N}) \in \mathbf{\Pi}$ and define the (formal) power series with respect to the integral root basis with coefficient sequence $\mathbf{a}$ to be the expression, $r_{\mathbf{a}}(x) = \sum_{i\in\mathbb{N}} a_i\rho_i(x)$. Let $\mathbb{Z}[[x]]_R$ denote the group under addition of coefficients of all such power series so that $\mathbb{Z}[[x]]_R \cong \mathbf{\Pi}$. Let $\mathbb{Z}[x]_R$ denote those elements of $\mathbb{Z}[[x]]_R$ which have only finitely many non–zero coefficients, i.e., the polynomials in $\mathbb{Z}[[x]]_R$. As defined here, neither $\mathbb{Z}[x]_R$ nor $\mathbb{Z}[[x]]_R$ is a ring.

Note that for each $n \in \mathbb{N}$, $\rho_i(n) = 0$ for all $i > n$, so that $r_{\mathbf{a}}(n)$ is an integer. Thus the sequence of values $(r_{\mathbf{a}}(0), r_{\mathbf{a}}(1), \ldots)$ is well–defined. As a result, the valuation map $\nu : \mathbb{Z}[[x]]_R \to \mathbf{\Pi}$ given by $r_{\mathbf{a}}(x) \mapsto (r_{\mathbf{a}}(0), r_{\mathbf{a}}(1), \ldots)$ is a monomorphism of $\mathbb{Z}[[x]]_R$ into $\mathbf{\Pi}$. Analogously with the finite dimensional case, we call the image of $\nu$ the group of *polynomial* *points* $\mathbf{P}$. Since each $f(x) \in \mathbb{Z}[x]$ can be expressed uniquely as a polynomial in $\mathbb{Z}[x]_R$, and conversely, $\nu$ maps $\mathbb{Z}[x]_R$ isomorphically onto $P_\omega \subset \mathbf{P}$.

We thank the referee for pointing out that $\mathbf{P}$ is the closure of $P_\omega$ with respect to the product topology on $\mathbf{\Pi}$ where $\mathbb{Z}$ has the discrete topology. For discussion and application of this metrizable topology, see [2].

Let $\boldsymbol{\rho}$ denote the sequence of polynomials $(\rho_0(x), \rho_1(x), \dots)$, considered as a row vector, and for all $\mathbf{a} \in \mathbf{\Pi}$, let $\ell_{\mathbf{a}}(x) = \boldsymbol{\rho} B\mathbf{a}$. Since $B$ is not integral, $\ell_{\mathbf{a}}(x)$ need not lie in $\mathbb{Z}[[x]]_R$, but nevertheless $\mathbf{a} = (\ell_{\mathbf{a}}(0), \ell_{\mathbf{a}}(1), \dots)$; i.e., $\ell_{\mathbf{a}}(x)$ is an infinite extension of the Lagrange interpolation polynomials with respect to the integral root basis. The following lemma demonstrates that the definition of $\ell_{\mathbf{a}}(x)$ in the infinite case is consistent with that in the finite case.

**Lemma 4.1.** *Let* $\mathbf{a} = (a_i : i \in \mathbb{N}) \in \mathbf{\Pi}$ *and for all* $n \geq 1$, *let* $\mathbf{a}_n = (a_0, a_1, \dots, a_{n-1})$. *Then the sum of the first* $n$ *terms of* $\ell_{\mathbf{a}}(x)$ *is* $\ell_{\mathbf{a}_n}(x)$.

*Proof.* The sum of the first $n$ terms of $\boldsymbol{\rho} B\mathbf{a}$ is

$$(\rho_0(x), \rho_1(x), \dots, \rho_{n-1}(x))B_n\mathbf{a}_n,$$

which equals $\ell_{\mathbf{a}_n}(x)$ by [4, Lemma 2.2]. □

**Lemma 4.2.** *Let* $\mathbf{a}$, $\mathbf{b} \in \mathbf{\Pi}$. *The following are equivalent:*

1. $C\mathbf{b} = \mathbf{a} = (r_{\mathbf{b}}(0), r_{\mathbf{b}}(1), \dots)$
2. $B\mathbf{a} = \mathbf{b}$

*Proof.* Both parts follow immediately from Corollary 2.1 and Lemma 4.1. □

The following corollary also is immediate.

**Corollary 4.1.** *Let* $\mathbf{a} \in \mathbf{\Pi}$. *Then*

1. $C\mathbf{a} \in \mathbf{P}$, *and* $\mathbf{P} = C\mathbf{\Pi}$
2. $\mathbf{a} \in \mathbf{P}$ *if and only if* $B\mathbf{a} \in \mathbf{\Pi}$ □

The following theorems are the promised infinite analogs of Theorems 1.1 and 1.3.

**Theorem 4.2.** *Let* $\mathbf{a} = (a_i : i \in \mathbb{N}) \in \mathbf{\Pi}$. *The following are equivalent:*

1. $\mathbf{a} \in \mathbf{P}$
2. $\ell_{\mathbf{a}}(x) \in \mathbb{Z}[[x]]_R$
3. $B\mathbf{a} \in \mathbf{\Pi}$

*Proof.* The equivalence of (1) and (3) is just Corollary 4.1 (2).

For $\mathbf{a} \in \mathbf{\Pi}$, $\mathbf{a} = (\ell_{\mathbf{a}}(0), \ell_{\mathbf{a}}(1), \dots)$. For $\mathbf{a} \in \mathbf{P}$, each intial segment $\mathbf{a}_n = (a_0, a_1, \dots, a_{n-1}) \in P_n$, so by Theorem 1.1 (2), $\ell_{\mathbf{a}_n}(x) \in \mathbb{Z}[x]_n$. That all the coefficients of $\ell_{\mathbf{a}}(x)$ are integral follows from Lemma 4.1. □

**Theorem 4.3.** $\mathbf{\Pi}/\mathbf{P} \cong \prod_{n\geq 2} \mathbb{Z}/n!\mathbb{Z}$

*Proof.* Since the lower-triangular matrix $A$ is integrally invertible with columns forming the Alpha basis $\{\alpha_n : n \in \mathbb{N}\}$, $\mathbf{\Pi}$ is the product of the $\alpha_n$'s, $\mathbf{\Pi} = \prod_{n\in\mathbb{N}} \langle \alpha_n \rangle$ [7, p 164, definitions only]. Similarly, by Corollary 4.1 (1), $\mathbf{P} = C\mathbf{\Pi}$ with the columns of the lower-triangular non-singular matrix $C$ forming the Gamma basis $\{\gamma_n : n \in \mathbb{N}\}$, so that $\mathbf{P}$ is also a product, $\mathbf{P} = \prod_{n\in\mathbb{N}} \langle \gamma_n \rangle$. Since $\gamma_n = n!\alpha_n$ for all $n \in \mathbb{N}$, $\mathbf{\Pi}/\mathbf{P} \cong \prod_{n\in\mathbb{N}} \mathbb{Z}/n!\mathbb{Z}$. Because $0! = 1 = 1!$, the first two factors are degenerate, leaving the required result. $\square$

The following lemma enables us to describe the group $\prod_{n\in\mathbb{N}^+} \mathbb{Z}^n/P_n$.

**Lemma 4.3.** *For all $i \in \mathbb{N}^+$, let $G_i$ be an abelian group and $n_i$ a positive integer such that $n_i < n_{i+1}$. Then $\prod_{j\in\mathbb{N}^+} \bigoplus_{i=1}^{n_j} G_i \cong \prod_{j\in\mathbb{N}^+} \prod_{i\in\mathbb{N}^+} G_i$.*

*Proof.* We construct a 1–1 correspondence between the constituent groups $G_i$ on the left and those on the right, in such a way that corresponding groups have the same subscript. The identity maps on the $G_i$'s then induce the desired isomorphism between the group on the left (call it $L$) and the group on the right (call it $R$).

The proof is essentially combinatorial, and we abuse notation for the sake of simplicity. We display $L$ and $R$ schematically:

$$L \quad = \quad \begin{array}{ccccccc} G_1 & \cdots & G_{n_1} & & & & \\ G_1 & \dots & G_{n_1} & \dots & G_{n_2} & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \\ G_1 & \dots & G_{n_1} & \dots & G_{n_2} & \dots & G_{n_k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

and

$$R \quad = \quad \begin{array}{cccccccc} G_1 & \dots & G_{n_1} & \dots & G_{n_2} & \dots & G_{n_k} & \dots \\ G_1 & \dots & G_{n_1} & \dots & G_{n_2} & \dots & G_{n_k} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ G_1 & \dots & G_{n_1} & \dots & G_{n_2} & \dots & G_{n_k} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

Now operate on the array $L$ by "pushing all the columns to the top" to form the array $R$. This establishes the required isomorphism. $\square$

Application of Theorems 1.3 and 4.3 and Lemma 4.3 immediately yields the following result.

**Theorem 4.4.**

$$\prod_{n\in\mathbb{N}^+} \mathbb{Z}^n/P_n \cong \prod_{n\in\mathbb{N}^+} \bigoplus_{k=1}^{n} \mathbb{Z}/(k-1)!\mathbb{Z} \cong \prod_{n\in\mathbb{N}^+} \prod_{k\in\mathbb{N}^+} \mathbb{Z}/(k-1)!\mathbb{Z} \cong \prod_{n\in\mathbb{N}^+} \mathbf{\Pi}/\mathbf{P}$$

$\square$

# 5 Subgroups of the Baer–Specker group

The Baer–Specker group $\mathbf{\Pi}$ is a well known source of examples and counter-examples in abelian group theory; see for example [6, Section 19] and [2]. We now relate some of these results to our theory of polynomial points.

Let $\mathbf{B}$ denote the subgroup of $\mathbf{\Pi}$ consisting of the bounded sequences of integers; $\mathbf{B}$ is a basic subgroup of $\mathbf{\Pi}$ in the sense that it is a pure, free subgroup with divisible quotient [2, p 5771]. Recall that $\mathbf{\Sigma}$ is the subgroup of sequences that are eventually 0; by [2, Theorem 1.5], $\mathbf{\Sigma}$ is a direct summand of $\mathbf{B}$. Furthermore, by [2, p 5770], the quotient $\mathbf{\Pi}/\mathbf{\Sigma}$ is the direct sum of a divisible group and a reduced algebraically compact group, with the reduced summand being isomorphic to the direct product of countably many copies of the groups of $p$-adic integers for all primes $p$.

The following results show the relation between Theorems 4.3 and 1.1 and illustrate the power and utility of the integral root basis. Recall that $C$ is the $\omega \times \omega$ matrix whose columns are the extended Gamma basis.

**Theorem 5.1.** *In the notation above,*

1. $C\mathbf{\Sigma} = P_\omega$
2. *Multiplication by* $C$ *induces an isomorphism* $\mathbf{\Pi}/\mathbf{\Sigma} \to \mathbf{P}/P_\omega$.
3. $\mathbf{B} \cap \mathbf{P}$ *consists of the constant sequences.*
4. $(\mathbf{B} + \mathbf{P})/\mathbf{P}$ *is torsion–free.*

*Proof.* (1) Let $\mathbf{a} \in P_\omega$, say $\mathbf{a} = (f(k) : k \in \mathbb{N})$, $f(x) \in \mathbb{Z}[x]$. Write $f = c_0\rho_0 + \cdots + c_n\rho_n$ with respect to the integral root basis and let $\mathbf{c} = (c_0, \ldots, c_n, 0, \ldots)$. Then $\mathbf{c} \in \mathbf{\Sigma}$ and $C\mathbf{c} = \mathbf{a}$. Thus $P_\omega \subseteq C\mathbf{\Sigma}$.

Conversely, let $\mathbf{c} = (c_0, \ldots, c_n, 0, \ldots) \in \mathbf{\Sigma}$ and define $f(x) \in \mathbb{Z}[x]_{n+1}$ by $f = c_0\rho_0 + \cdots + c_n\rho_n$. Then the values $(f(k))$, $k \in \mathbb{N}$, form the sequence $C\mathbf{c}$. Thus $C\mathbf{\Sigma} \subseteq P_\omega$, and the two sets are equal.

(2) By Corollary 4.1 (1), $\mathbf{P} = C\mathbf{\Pi}$, and from (1) above, $P_\omega = C\mathbf{\Sigma}$ so that $C$ induces an isomorphism $\mathbf{\Pi}/\mathbf{\Sigma} \to \mathbf{P}/P_\omega$.

(3) The constant sequences, being the values of degree zero polynomials, are clearly in $\mathbf{B} \cap \mathbf{P}$. No other sequence occurs there because the values of polynomials of non–zero degree are unbounded.

(4) $\mathbf{B}$ is separable so that the constant sequences $\mathbf{B} \cap \mathbf{P} = (1, 1, \ldots)^T\mathbb{Z}$ are a direct summand, $\mathbf{B} = (\mathbf{B} \cap \mathbf{P}) \oplus \mathbf{B}'$ say. Thus $(\mathbf{B} + \mathbf{P})/\mathbf{P} \cong \mathbf{B}/(\mathbf{B} \cap \mathbf{P}) \cong \mathbf{B}' \subseteq \mathbf{\Pi}$. □

Recall that a group $H$ is said to be *algebraically compact* if $H$ is a direct summand of every group $G$ that contains $H$ as a pure subgroup [6, Section 38]. Every bounded group is algebraically compact. In the $p$–adic topology of a group $G$, $p$ a prime, the subgroups $p^nG$, $n \in \mathbb{N}$, form a base of neighborhoods of 0 [6, p 30]. A group $G$ is cotorsion if every extension of $G$ by a torsion–free group splits; i.e., if $\operatorname{Ext}(J, G) = 0$ for every torsion–free group $J$ [6, Section 54]. An algebraically compact group is cotorsion, but the converse need not be true.

With this terminology, $\mathbf{\Pi}/\mathbf{P}$ and its torsion subgroup can be described as follows:

**Theorem 5.2.** *1.* $\mathbf{\Pi}/\mathbf{P}$ *is a reduced, algebraically compact group and so is of the form* $\mathbf{\Pi}/\mathbf{P} = \prod A_p$*, the product over all primes p, where each* $A_p$ *is complete in its* $p$*–adic topology and is uniquely determined by* $\mathbf{\Pi}/\mathbf{P}$*.*

2. *Let* $T = \oplus T_p$ *be the torsion subgroup of* $\mathbf{\Pi}/\mathbf{P}$*, where* $T_p$ *is the* $p$*–primary component of* $T$ *for all primes* $p$*. Then* $T_p$ *is the torsion subgroup of* $A_p$ *and is isomorphic to the torsion completion of* $\oplus_{k\in\mathbb{N}^+}\left(\oplus_{2^{\aleph_0}}\mathbb{Z}/p^k\mathbb{Z}\right)$*. Thus* $T$ *is not cotorsion and so is not a direct summand of* $\mathbf{\Pi}/\mathbf{P}$*.*

*Proof.* (1) $\mathbf{\Pi}/\mathbf{P}$ is reduced and algebraically compact because all of the product components $\mathbb{Z}/n!\mathbb{Z}$ in Theorem 4.3 are [6, Corollary 38.3]. As a result, it is of the form $\mathbf{\Pi}/\mathbf{P} = \prod A_p$ where each $A_p$ is complete in its $p$–adic topology and is uniquely determined [6, Proposition 40.1].

(2) By [7, Theorem 68.4], each $T_p$ is torsion–complete since it is the torsion part of the algebraically compact group $A_p$. Hence by the results of [7, Section 68], $T_p$ is uniquely determined by any basic subgroup. Thus it remains to determine the Ulm–Kaplansky invariants of $T_p$ for each prime $p$.

For each prime $p$, the $p$–component of $\prod_{n\geq 2}\mathbb{Z}/n!\mathbb{Z}$ consists of those sequences $\mathbf{a} = (a_n + n!\mathbb{Z})$ for which there is a positive integer $k$ such that for all $n$, $p^k a_n$ is a multiple of $n!$. For each fixed $k$, $\prod_{n\geq 2}\mathbb{Z}/n!\mathbb{Z}$ has $2^{\aleph_0}$ independent elements of order $p^k$. For example, let $\{F_\nu : \nu < 2^{\aleph_0}\}$ be a family of $2^{\aleph_0}$ almost disjoint infinite subsets of $\mathbb{N}$; i.e., any two intersect in a finite set. For each $\nu < 2^{\aleph_0}$ let $\mathbf{a}_\nu$ have $n$–component $(n!/p^k) + n!\mathbb{Z}$ for each $n \geq p^k$ in $F_\nu$, and zero otherwise. Then the $\mathbf{a}_\nu$ are independent elements of the product of order $p^k$. Since $|T_p| \leq |\mathbf{\Pi}| = 2^{\aleph_0}$, $T_p$ is the torsion completion of $\oplus_{k\in\mathbb{N}^+}\left(\oplus_{2^{\aleph_0}}\mathbb{Z}/p^k\mathbb{Z}\right)$ [6, Section 40].

Since $T$ is unbounded and reduced, it cannot be algebraically compact [6, Corollary 40.3] or cotorsion [6, Corollary 54.4], so it is not a summand of $\mathbf{\Pi}/\mathbf{P}$, since any direct summand of an algebraically compact group is itself algebraically compact and hence cotorsion [6, p 159]. $\square$

# 6 Analytic properties of the power series and Pascal's matrix

Recall from Section 4 the definition of $\mathbb{Z}[[x]]_R$ as the group of power series with respect to the integral root basis; that is, elements of $\mathbb{Z}[[x]]_R$ are formal expressions of the form $r_{\mathbf{a}}(x) = \sum_{n\in\mathbb{N}} a_n\rho_n(x)$ with $\mathbf{a} = (a_n) \in \mathbf{\Pi}$.

We have seen that the elements of $\mathbb{Z}[[x]]_R$ are the infinite analogs of the integral Lagrange interpolation polynomials. We also have seen that there is a natural map $\mathbf{\Pi} \to \mathbb{Z}[[x]]_R \to \mathbf{\Pi}$ defined by $\mathbf{a} \mapsto r_{\mathbf{a}}(x) \mapsto (r_{\mathbf{a}}(n))$. The image is $\mathbf{P}$ and as in the proof of Theorem 4.3, $\mathbf{\Pi} = \prod_{j\in\mathbb{N}}\langle\alpha_j\rangle$ and $\mathbf{P} = \prod_{j\in\mathbb{N}}\langle j!\alpha_j\rangle$ where the $\alpha_j$ are the elements of the extended Alpha basis which form the columns of the infinite Pascal matrix $A$.

Through use of the observation that $A$ can be obtained from the transpose $U^T$ of the matrix (with respect to the standard basis) of the translation $U : \mathbb{Z}[x] \to \mathbb{Z}[x]$, $x \mapsto x + 1$, computations involving $A$ can be greatly simplified. More precisely, let $U(t)$ denote the matrix of the translation $x \mapsto x+t$. Then $U(t) = (u_{ij}(t))$ where $u_{ij}(t) = \binom{j}{i} t^{j-i}$, i.e.,

$$U(t) = \begin{bmatrix} 1 & t & t^2 & t^3 & \dots \\ 0 & 1 & 2t & 3t^2 & \dots \\ 0 & 0 & 1 & 3t & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \end{bmatrix}$$

so that $U = U(1)$.

Since for any numbers $m$ and $n$, $U(m)U(n) = U(m+n)$, we have for all $n \in \mathbb{Z}$, $U^n = U(n)$. Then defining $A(t) = U(t)^T$, we find for all $n \in \mathbb{Z}$ that $A(n) = U(n)^T = (U^n)^T = (U^T)^n$. In particular, $A^{-1} = U(-1)^T$. Thus computations with $A$ are easier than they first appear.

As with $\mathbb{Z}[[x]]_R$ considered above, there is a natural isomorphism $\mathbf{\Pi} \to \mathbb{Z}[[x]]$ defined by $\mathbf{a} \mapsto \sum_{i\in\mathbb{N}} a_i x^i$. In particular, the image of the Alpha basis can be expressed in terms of the standard basis by $\alpha_j(x) = \sum_{i\in\mathbb{N}} \binom{i}{j} x^i$. It is interesting to note that $\alpha_j(x) = x^j/(1-x)^{j+1}$.

**Proposition 6.1.** *With the notation above, $\alpha_j(x) = x^j/(1-x)^{j+1}$*

*Proof.*

$$\sum_{i\in\mathbb{N}} \binom{i}{j} x^i = \frac{x^j}{j!}\frac{d^j}{dx^j}\sum_{i\in\mathbb{N}} x^i = \frac{x^j}{j!}\frac{d^j}{dx^j}\left(\frac{1}{1-x}\right) = \frac{x^j}{(1-x)^{j+1}}$$

□

We thank the referee for this short proof, which appeared in [8].

**Proposition 6.2.** *Let $\boldsymbol{F}(x) = \sum_{j\in\mathbb{N}} a_j x^j \in \mathbb{Z}[[x]]$. Then formally, $\boldsymbol{F}(x) = 1/(1-x)\sum_{j\in\mathbb{N}} b_j(x/(1-x))^j$, with $\mathbf{a} = (a_j)$ and $\mathbf{b} = (b_j) = A^{-1}\mathbf{a}$.*

*Proof.* We view $\mathbf{a}$ and $\mathbf{b}$ as infinite column vectors. Since $\mathbf{\Pi} = \prod_{j\in\mathbb{N}} \langle \alpha_j \rangle$ with respect to the Alpha basis and $\mathbf{\Pi} \cong \mathbb{Z}[[x]]$, $\mathbb{Z}[[x]] = \prod_{j\in\mathbb{N}} \langle \alpha_j(x) \rangle$, where $\alpha_j(x) = \sum_{i\in\mathbb{N}} \binom{i}{j} x^i$ for all $j \in \mathbb{N}$.

The coefficients of $\mathbf{F}(x)$ with respect to the Alpha basis are given by $A^{-1}\mathbf{a} = \mathbf{b}$, so

$$\mathbf{F}(x) = \sum_{j\in\mathbb{N}} b_j \alpha_j(x) = \sum_{j\in\mathbb{N}} b_j \frac{x^j}{(1-x)^{j+1}} = \frac{1}{1-x}\sum_{j\in\mathbb{N}} b_j \left(\frac{x}{1-x}\right)^j$$

□

**Remark 6.1.** Although the expansion described in Proposition 6.2 is formal, the series may actually converge analytically. In particular, by the ratio test, each $\alpha_j(x) = \sum \binom{i}{j} x^i$ converges on the unit disc $|x| < 1$. Moreover, we can construct a single variable generating function for the entire infinite Pascal matrix as follows:

From the definition of $\alpha_j(x)$ and Proposition 6.1,

1. $\alpha_j(x) = \sum_{i\in\mathbb{N}} \binom{i}{j} x^i = \dfrac{x^j}{(1-x)^{j+1}}$. Sum equations (1) over $j$ to obtain

2. $\sum_{j\in\mathbb{N}} \alpha_j(x) = \sum_{j\in\mathbb{N}} \sum_{i\in\mathbb{N}} \binom{i}{j} x^i = \sum_{j\in\mathbb{N}} \dfrac{x^j}{(1-x)^{j+1}}$. Now reverse the order of summation in the middle term of (2) and factor out $1/(1-x)$ on the right to obtain

3. $\sum_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} \binom{i}{j} x^i = \dfrac{1}{1-x} \sum_{j\in\mathbb{N}} \left(\dfrac{x}{1-x}\right)^j$. From (3), factor out $x^i$ and use $\binom{i}{j} = 0$ for $j > i$ on the left, and the sum of a geometric series on the right to conclude that

4. $\sum_{i\in\mathbb{N}} x^i \sum_{j=0}^{i} \binom{i}{j} = \dfrac{1}{1-x} \left(\dfrac{1}{1-\frac{x}{1-x}}\right) = \dfrac{1}{1-2x}$. Substituting $2^i$ for $\sum_{j=0}^{i} \binom{i}{j}$ yields

5. $\sum_{j\in\mathbb{N}} \alpha_j(x) = \sum_{i\in\mathbb{N}} (2x)^i = \dfrac{1}{1-2x}$ and the series converges on the smaller disc $|x| < 1/2$.

To explicate Pascal's infinite matrix from $1/(1-2x)$, simply reverse the steps: $\dfrac{1}{1-2x} \to \sum_{i\in\mathbb{N}} x^i 2^i \to \sum_{i\in\mathbb{N}} x^i \sum_{j=0}^{i} \binom{i}{j} \to \sum_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} \binom{i}{j} x^i$. With this technique, two variables are not required to generate the matrix; compare [10, p 93, Ex. 12 and solution p 494].

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