



# On Multiple Sums of Products of Lucas Numbers

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## Abstract

This paper studies some sums of products of the Lucas numbers. They are a generalization of the sums of the Lucas numbers, which were studied another authors. These sums are related to the denominator of the generating function of the  $k$ -th powers of the Fibonacci numbers. We considered a special case for an even positive integer  $k$  in the previous paper and now we generalize this result to an arbitrary positive integer  $k$ . These sums are expressed as the sum of the binomial and Fibonomial coefficients. The proofs of the main theorems are based on special inverse formulas.

## 1 Introduction

Generating functions are very helpful in finding of relations for sequences of integers. Some authors found miscellaneous identities for the Fibonacci numbers  $F_n$ , defined by recurrence relation  $F_{n+2} = F_n + F_{n+1}$ , with  $F_0 = 0$ ,  $F_1 = 1$ , and the Lucas numbers  $L_n$ , defined by the same recurrence but with the initial conditions  $L_0 = 2$ ,  $L_1 = 1$ , by manipulation with their generating functions. Our approach is rather different in this paper.

In 1718 DeMoivre found the generating function of the Fibonacci numbers  $F_n$  and used it for deriving the closed form  $F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)$ , with  $\alpha = \frac{1}{2}(1 + \sqrt{5})$  and  $\beta = \frac{1}{2}(1 - \sqrt{5})$  (similarly the formula  $L_n = \alpha^n + \beta^n$  holds for the Lucas numbers). In 1957 S. W. Golomb [2] found the generating function for the square of  $F_n$  and this result started the effort to find a recurrence or a closed form for the generating function  $f_k(x) = \sum_{n=0}^{\infty} F_n^k x^n$  of the

$k$ -th powers of the Fibonacci numbers. Riordan [7] found a general recurrence for  $f_k(x)$ . Carlitz [1], Horadam [4] and Mansour [6] presented some generalizations of Riordan's results and found similar recurrences for the generating functions of powers of any second-order recurrence sequences.

Horadam gave some closed forms for the numerator and the denominator of this generating function. From his results follows, for example

$$f_k(x) = \frac{\sum_{i=0}^k \sum_{j=0}^i (-1)^{\frac{j(j+1)}{2}} \begin{bmatrix} k+1 \\ j \end{bmatrix} F_{i-j}^k x^i}{\sum_{i=0}^{k+1} (-1)^{\frac{i(i+1)}{2}} \begin{bmatrix} k+1 \\ i \end{bmatrix} x^i}, \quad (1)$$

where  $\begin{bmatrix} n \\ k \end{bmatrix}$  are the Fibonomial coefficients defined for any nonnegative integers  $n$  and  $k$  by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \prod_{i=0}^{k-1} \frac{F_{n-i}}{F_{i+1}} = \frac{F_n F_{n-1} \cdots F_{n-k+1}}{F_1 F_2 \cdots F_k},$$

with  $\begin{bmatrix} n \\ 0 \end{bmatrix} = 1$  and  $\begin{bmatrix} n \\ k \end{bmatrix} = 0$  for  $n < k$ .

Using Carlitz' method, Shannon [11] obtained some special results for the numerator and the denominator in the expression of the generating function  $f_k(x)$ . For example, he used the  $q$ -analog of the terminating binomial theorem (firstly published by Rothe [9], but from Gauss's posthumous papers it is known that he had found it around 1808, see [5]) and obtained the relation

$$\prod_{i=0}^k (1 - q^i x) = \sum_{i=0}^{k+1} (-1)^i q^{\frac{i}{2}(i-1)} \begin{Bmatrix} k+1 \\ i \end{Bmatrix} x^i.$$

$Q$ -binomial coefficients are defined  $\begin{Bmatrix} k+1 \\ i \end{Bmatrix} = \frac{(q^{k+1}-1)(q^k-1)\cdots(q^{k-i+2}-1)}{(q-1)(q^2-1)\cdots(q^i-1)}$  for  $i \geq 1$  and any complex numbers  $q, x$  and any positive integer  $k$ , where  $\begin{Bmatrix} k+1 \\ 0 \end{Bmatrix} = 1$ . Replacing  $q$  by  $\beta/\alpha$  and  $x$  by  $\alpha^k x$  he got

$$\prod_{i=0}^k (1 - \alpha^{k-i} \beta^i x) = \sum_{i=0}^{k+1} (-1)^{\frac{i}{2}(i+1)} \begin{Bmatrix} k+1 \\ i \end{Bmatrix} x^i. \quad (2)$$

We paid attention [10] to a generalization of a type of the well-known formulas for the Fibonacci and Lucas numbers, see [12, pp. 179–183], for example

$$\sum_{i=0}^n (-1)^i L_{n-2i} = 2F_{n+1}.$$

In this paper we concentrate on the sums

$$\sum_{i_n=0}^{\lfloor \frac{k-1}{2} \rfloor} \sum_{i_{n-1}=i_n+1}^{\lfloor \frac{k-1}{2} \rfloor} \cdots \sum_{i_{n-2}=i_{n-1}+1}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^{i_1+i_2+\cdots+i_n} \prod_{j=1}^n L_{k-2i_j}, \quad (3)$$

where  $k$  is an arbitrary positive integer. The special case of (3) for an odd  $k$  was solved up in [10]. Here we use analogous method to find formulas for an even integer  $k$ .

Throughout the paper we adopt the conventions that the sum and the product over an empty set is 0 and 1, respectively,  $\lfloor x \rfloor$  represents the greatest integer less than or equal to  $x$ , the relation  $f(x) \sim g(x)$  means that  $f(x)$  is asymptotic to  $g(x)$  and Iverson's notation (see, e. g., [3]) that

$$[P(k)] = \begin{cases} 1, & \text{if statement } P(k) \text{ is true;} \\ 0, & \text{if statement } P(k) \text{ is false.} \end{cases}$$

## 2 The main results

**Definition 1.** Let  $k$  be any positive integer. We define the sequence  $\{S_n(k)\}_{n=0}^{\infty}$  in the following way

$$S_0(k) = 1, \quad S_1(k) = \sum_{i_1=0}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^{i_1} L_{k-2i_1}$$

and

$$S_n(k) = \sum_{i_n=0}^{\lfloor \frac{k-1}{2} \rfloor} \sum_{i_{n-1}=i_n+1}^{\lfloor \frac{k-1}{2} \rfloor} \cdots \sum_{i_1=i_2+1}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^{i_1+i_2+\cdots+i_n} \prod_{j=1}^n L_{k-2i_j}, \quad (4)$$

for any integer  $n > 1$ .

Let us denote

$$\Theta(i, k, n) = \binom{\lfloor \frac{k+1}{2} \rfloor - n + i}{i} + \binom{\lfloor \frac{k+1}{2} \rfloor - n + i - 1}{i - 1}$$

for any positive integers  $i, k$  and any nonnegative integer  $n$ .

**Theorem 2.** Let  $n$  be any nonnegative integer and let  $k$  be any positive integer. Then

$$S_n(k) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{\lfloor \frac{n}{2} \rfloor - i} \Theta(i, k, n) \begin{bmatrix} k+1 \\ n-2i \end{bmatrix} \quad (5)$$

if  $k$  is odd and

$$S_n(k) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{n-2i} (-1)^{i+n(\frac{k}{2}+1)+\frac{j}{2}(j+k+1)} \Theta(i, k, n) \begin{bmatrix} k+1 \\ j \end{bmatrix} \quad (6)$$

if  $k$  is even.

**Corollary 3.** Let  $n$  be any nonnegative integer and let  $k$  be any positive integer. Then the asymptotic formula

$$S_n(k) \sim \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{\lfloor \frac{n}{2} \rfloor + ik} \Theta(i, k, n) \begin{bmatrix} k+1 \\ n-2i \end{bmatrix} \quad (7)$$

holds as  $k \rightarrow \infty$ .

**Theorem 4.** Let  $m$  be any integer and let  $k$  be any even positive integer. Then

$$\sum_{j=0}^m (-1)^{\frac{j}{2}(j+k+1)} \begin{bmatrix} k+1 \\ j \end{bmatrix} = (-1)^{\frac{m}{2}(m+k+1)} \frac{1}{F_{\frac{k}{2}+1}} \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^i \begin{bmatrix} k+2 \\ m-2i \end{bmatrix} F_{\frac{k+2}{2}-m+2i} .$$

**Corollary 5.** Let  $n$  be any nonnegative integer and let  $k$  be any even positive integer. Then

$$S_n(k) = \frac{(-1)^{\lfloor \frac{n}{2} \rfloor}}{F_{\frac{k}{2}+1}} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=i}^{\lfloor \frac{n}{2} \rfloor} (-1)^{i+j} \Theta(i, k, n) \begin{bmatrix} k+2 \\ n-j \end{bmatrix} F_{\frac{k+2}{2}-n+2j} . \quad (8)$$

**Theorem 6.** Let  $m$  be any integer. Then

$$\begin{aligned} \sum_{j=0}^m (-1)^{\frac{j}{2}(j+k+1)} \begin{bmatrix} k+1 \\ j \end{bmatrix} &= \frac{(-1)^{\frac{m}{2}(m+k+1)}}{F_{\frac{k}{2}+1} F_{k+3} F_{k+4}} \sum_{i=0}^{\lfloor \frac{m}{4} \rfloor} \begin{bmatrix} k+4 \\ m-4i \end{bmatrix} \times \\ &\times \left( F_{\frac{k}{2}+1-(m-4i)} L_{\frac{k}{2}+2-(m-4i)} F_{k+3} - F_{m-4i} F_{m-4i-1} \right) . \end{aligned}$$

**Corollary 7.** Let  $n$  be any nonnegative integer. Then

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \left( \binom{\frac{k+1}{2} - n + i}{i} + \binom{\frac{k-1}{2} - n + i}{i-1} \right) \begin{bmatrix} k+1 \\ n-2i \end{bmatrix} = 0 \quad (9)$$

if  $k$  is an odd positive integer,  $k < 2n - 1$ , and

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{n-2i} (-1)^{i+\frac{j}{2}(j+k+1)} \left( \binom{\frac{k}{2} - n + i}{i} + \binom{\frac{k-2}{2} - n + i}{i-1} \right) \begin{bmatrix} k+1 \\ j \end{bmatrix} = 0 \quad (10)$$

if  $k$  is an even integer,  $k < 2n$ .

**Corollary 8.** Let  $k$  be any even positive integer. Then

$$\begin{aligned} \sum_{i=0}^{\frac{k-2}{2}} (-1)^i L_{k-2i} &= F_{k+1} - (-1)^{\frac{k}{2}} , \\ \sum_{i_2=0}^{\frac{k-2}{2}} \sum_{i_1=i_2+1}^{\frac{k-2}{2}} (-1)^{i_1+i_2+1} L_{k-2i_1} L_{k-2i_2} &= \frac{k-2}{2} + (-1)^{\frac{k}{2}} F_{k+1} + F_k F_{k+1} \end{aligned}$$

and

$$\begin{aligned} \sum_{i_3=0}^{\frac{k-2}{2}} \sum_{i_2=i_3+1}^{\frac{k-2}{2}} \sum_{i_1=i_2+1}^{\frac{k-2}{2}} (-1)^{i_1+i_2+i_3} L_{k-2i_1} L_{k-2i_2} L_{k-2i_3} \\ = \frac{k-4}{2} \left( (-1)^{\frac{k}{2}} - F_{k+1} \right) + F_k F_{k+1} \left( (-1)^{\frac{k}{2}} - \frac{1}{2} F_{k-1} \right) . \end{aligned}$$

### 3 The preliminary results

**Lemma 9.** *Let  $k$  be any positive integer. Then  $S_n(k) = 0$  for each positive integer  $n > \lfloor \frac{k+1}{2} \rfloor$ .*

*Proof.* After rewriting relation (4) from Definition 1 into the form

$$S_n(k) = \sum_{\substack{i_1, i_2, \dots, i_n \\ 0 \leq i_n < i_{n-1} < \dots < i_1 \leq \lfloor \frac{k-1}{2} \rfloor}} (-1)^{i_1+i_2+\dots+i_n} \prod_{j=1}^n L_{k-2i_j}$$

the assertion easily follows from the condition

$$0 \leq i_n < i_{n-1} < \dots < i_1 \leq \left\lfloor \frac{k-1}{2} \right\rfloor$$

which does not hold for any values  $i_1, i_2, \dots, i_n$  if  $\lfloor \frac{k-1}{2} \rfloor < n-1$ .  $\square$

**Lemma 10.** *Let  $k$  be any even positive integer and let  $n$  be any positive integer. Then*

$$(i) \quad \sum_{i=0}^n \binom{\frac{k}{2} - 2i}{n-i} S_{2i}(k) = 0 \quad \text{for } n \geq \frac{k}{2} + 1$$

$$(ii) \quad \sum_{i=0}^n \binom{\frac{k}{2} - (2i+1)}{n-i} S_{2i+1}(k) = 0 \quad \text{for } n \geq \frac{k}{2}.$$

*Proof.* We show the proof of (i). Case (ii) can be proved analogously. Each positive integer  $n \geq \frac{k}{2} + 1$  can be written in the form  $n = \frac{k}{2} + l$ , where  $l$  is any positive integer. We will show that just one of factors in the product  $\binom{\frac{k}{2} - 2i}{n-i} S_{2i}(k)$  is equal to zero. Concretely, the first one equals zero for  $i \leq \lfloor \frac{k}{4} \rfloor$  and the second one equals zero for  $i > \lfloor \frac{k}{4} \rfloor$ . For the sum in (i) the following holds:

$$\sum_{i=0}^{\frac{k}{2}+l} \binom{\frac{k}{2} - 2i}{\frac{k}{2} + l - i} S_{2i}(k) = Q_1(k, l) + Q_2(k, l),$$

where

$$Q_1(k, l) = \sum_{i=0}^{\lfloor \frac{k}{4} \rfloor} \binom{\frac{k}{2} - 2i}{\frac{k}{2} + l - i} S_{2i}(k)$$

and

$$Q_2(k, l) = \sum_{i=\lfloor \frac{k}{4} \rfloor + 1}^{\frac{k}{2}+l} \binom{\frac{k}{2} - 2i}{\frac{k}{2} + l - i} S_{2i}(k) = \sum_{p=1}^{\frac{k}{2} - \lfloor \frac{k}{4} \rfloor + l} \binom{\frac{k}{2} - 2\lfloor \frac{k}{4} \rfloor - 2p}{\frac{k}{2} - \lfloor \frac{k}{4} \rfloor + l - p} S_{2\lfloor \frac{k}{4} \rfloor + 2p}(k).$$

It is obvious that  $\binom{\frac{k}{2} - 2i}{\frac{k}{2} + l - i} = 0$  if  $i \leq \lfloor \frac{k}{4} \rfloor$  and therefore  $Q_1(k, l) = 0$  for any  $k$  and  $l$ . Since the equality  $S_{2\lfloor \frac{k}{4} \rfloor + 2p}(k) = 0$  is implied by Lemma 9 for any nonnegative integer  $p$ , it follows that  $Q_2(k, l) = 0$ .  $\square$

**Lemma 11.** *Let  $n$  be any positive integer and let  $q$  be any integer. Then the following inverse formula holds:*

$$a_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^n \binom{q-n+2i}{i} b_{n-2i}$$

if and only if

$$b_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{n+i} \left( \binom{q-n+i}{i} + \binom{q-n+i-1}{i-1} \right) a_{n-2i}. \quad (11)$$

*Proof.* Riordan [8, p. 243] gave the following inverse formula:

$$a_n = \sum_{i=0}^n \binom{q-2i}{n-i} b_i$$

if and only if

$$b_n = \sum_{i=0}^n (-1)^{n+i} \left( \binom{q-n-i}{n-i} + \binom{q-n-i-1}{n-i-1} \right) a_i.$$

To get Lemma 11 from this formula first we substitute  $\{a_n\}$  by  $\{a_{2n}\}$ ,  $\{b_i\}$  by  $\{b_{2i}\}$ ,  $n$  by  $\frac{n}{2}$  and  $i$  by  $\frac{n}{2} - i$  and then  $\{a_n\}$  by  $\{a_{2n+1}\}$ ,  $\{b_i\}$  by  $\{-b_{2i+1}\}$ ,  $n$  by  $\frac{n-1}{2}$ ,  $i$  by  $\frac{n-1}{2} - i$  and  $q$  by  $q-1$ . This leads to the proved formula.  $\square$

**Lemma 12.** *Let  $n, k, l$  be any positive integers,  $l < n < k$ . Let  $c_i, i = 1, 2, \dots, n$ , be any real numbers,  $c_n \neq 0$ . Then*

$$(i) \quad \lim_{k \rightarrow \infty} \begin{bmatrix} k \\ l \end{bmatrix} \begin{bmatrix} k \\ n \end{bmatrix}^{-1} = 0, \quad (ii) \quad \sum_{i=l}^n c_i \begin{bmatrix} k \\ i \end{bmatrix} \sim c_n \begin{bmatrix} k \\ n \end{bmatrix} \quad \text{as } k \rightarrow \infty.$$

*Proof.* Relation (i) follows from the definition of the Fibonomial coefficients and the obvious fact that  $\lim_{k \rightarrow \infty} F_k = \infty$ . Thus,

$$\begin{aligned} \lim_{k \rightarrow \infty} \begin{bmatrix} k \\ l \end{bmatrix} \begin{bmatrix} k \\ n \end{bmatrix}^{-1} &= \lim_{k \rightarrow \infty} \frac{F_k F_{k-1} \cdots F_{k-l+1}}{F_1 F_2 \cdots F_l} \cdot \frac{F_1 F_2 \cdots F_n}{F_k F_{k-1} \cdots F_{k-n+1}} = \\ &= \frac{F_1 F_2 \cdots F_n}{F_1 F_2 \cdots F_l} \lim_{k \rightarrow \infty} \frac{F_k F_{k-1} \cdots F_{k-l+1}}{F_k F_{k-1} \cdots F_{k-n+1}} = \\ &= \prod_{i=l+1}^n F_i \cdot \lim_{k \rightarrow \infty} \frac{1}{F_{k-l} \cdots F_{k-n+1}} = 0. \end{aligned}$$

Asymptotic formula (ii) is implied by (i).  $\square$

**Lemma 13.** *Let  $\{a_n\}, \{b_n\}$  be any sequences of real numbers, with  $b_{-1} = 0$ , and let  $h$  be any integer. Then for an arbitrary positive integer  $n$*

$$a_n = b_n - (-1)^h b_{n-1} \quad (12)$$

if and only if

$$b_n = \sum_{i=0}^n (-1)^{h(n+i)} a_i. \quad (13)$$

*Proof.* Let us show that identity (12) implies identity (13). We have

$$\begin{aligned}
\sum_{i=0}^n (-1)^{h(n+i)} a_i &= \sum_{i=0}^n (-1)^{h(n+i)} (b_i - (-1)^h b_{i-1}) \\
&= \sum_{i=0}^n (-1)^{h(n+i)} b_i - \sum_{i=1}^n (-1)^{h(n-1+i)} b_{i-1} - (-1)^{h(n-1)} b_{-1} \\
&= \sum_{i=0}^n (-1)^{h(n+i)} b_i - \sum_{j=0}^{n-1} (-1)^{h(n+j)} b_j = b_n .
\end{aligned}$$

Thus, this part of the assertion is true and similarly we can prove the reversed implication.  $\square$

**Lemma 14.** *Let  $k$  be any even positive integer and let  $a$  be any positive integer. Then*

$$\begin{bmatrix} k+1 \\ a \end{bmatrix} + (-1)^{\frac{k}{2}+a} \begin{bmatrix} k+1 \\ a-1 \end{bmatrix} = \frac{F_{\frac{k}{2}+1-a}}{F_{\frac{k}{2}+1}} \begin{bmatrix} k+2 \\ a \end{bmatrix}.$$

*Proof.* Using the definition of the Fibonomial coefficients we get the relation

$$F_{\frac{k}{2}-a+1} F_{k+2} = F_{\frac{k}{2}+1} \left( F_{k-a+2} + (-1)^{\frac{k}{2}+a} F_a \right) ,$$

which can be written in the form

$$F_{\frac{k}{2}-a+1} L_{\frac{k}{2}+1} = F_{k-a+2} + (-1)^{\frac{k}{2}+a} F_a$$

as  $F_{2n} = F_n L_n$  ([12, p. 176]). We get the previous relation by setting  $l = \frac{k}{2} - a + 1$  and  $n = \frac{k}{2} + 1$  into the identity ([12, p. 177])

$$F_{l+n} = F_l L_n + (-1)^{n+1} F_{l-n} , \tag{14}$$

which holds for any integers  $l, n$ . The assertion follows at once.  $\square$

The following form of  $\Theta(i, k, n)$  is more effective for the computation of the sums  $S_n(k)$ :

**Lemma 15.** *Let  $i, n$  be any integers and let  $k$  be any even positive integer. Then*

$$\Theta(i, k, n) = \begin{cases} 0, & i < 0 ; \\ 1, & i = 0 ; \\ \frac{k-2(n-2i)}{2i} \prod_{j=1}^{i-1} \frac{k-2(n+j-i)}{2(i-j)} , & i > 0 . \end{cases}$$

*Proof.* The cases for  $i \leq 0$  are clear. For  $i > 0$  we can write:

$$\begin{aligned}
\Theta(i, k, n) &= \binom{\frac{k}{2} - n + i}{i} + \binom{\frac{k}{2} - n + i - 1}{i-1} \\
&= \frac{k-2(n-2i)}{2i} \binom{\frac{k}{2} - n + i - 1}{i-1} = \frac{k-2(n-2i)}{2i} \prod_{j=1}^{i-1} \frac{\frac{k}{2} - n + i - j}{i-j}
\end{aligned}$$

and the proof is over.  $\square$

## 4 Additional properties of the inner sum

Now we will investigate properties of the inner sum involved in (6). Let us denote

$$\sigma_k(m) = \sigma(m) := \sum_{j=0}^{k-m} (-1)^{\frac{j}{2}(j+k+1)} \begin{bmatrix} k+1 \\ j \end{bmatrix}, \quad (15)$$

where  $k$  is any even positive integer and  $m$  is any integer.

**Lemma 16.** *Let  $k$  be any even positive integer and let  $m$  be any integer. Then*

(i)

$$\sigma(m) = 0, \text{ for } m \leq -1 \text{ or } m \geq k+1,$$

(ii)

$$\sigma(k-m) = \sigma(m),$$

(iii)

$$\begin{aligned} \sigma(0) &= 1, & \sigma(1) &= 1 + (-1)^{\frac{k-2}{2}} F_{k+1}, \\ \sigma(2) &= 1 - L_{\frac{k+2}{2}} F_{k+1} F_{\frac{k-2}{2}}, & \sigma(3) &= 1 - \frac{1}{2} (-1)^{\frac{k}{2}} F_{k+1} \left( 2 - F_k F_{\frac{k-4}{2}} L_{\frac{k+2}{2}} \right). \end{aligned}$$

*Proof.* (i) First we prove the case for  $m = -1$ :

$$\begin{aligned} \sigma(-1) &= \sum_{j=0}^{k+1} (-1)^{\frac{j}{2}(j+k+1)} \begin{bmatrix} k+1 \\ j \end{bmatrix} \\ &= \sum_{j=0}^{\frac{k}{2}} (-1)^{\frac{j}{2}(j+k+1)} \begin{bmatrix} k+1 \\ j \end{bmatrix} + \sum_{j=\frac{k}{2}+1}^{k+1} (-1)^{\frac{j}{2}(j+k+1)} \begin{bmatrix} k+1 \\ j \end{bmatrix} \\ &= \sum_{j=0}^{\frac{k}{2}} (-1)^{\frac{j}{2}(j+k+1)} \begin{bmatrix} k+1 \\ j \end{bmatrix} + \sum_{i=0}^{\frac{k}{2}} (-1)^{\frac{k+1-i}{2}(2k+2-i)} \begin{bmatrix} k+1 \\ k+1-i \end{bmatrix} \\ &= \sum_{j=0}^{\frac{k}{2}} (-1)^{\frac{j}{2}(j+k+1)} \begin{bmatrix} k+1 \\ j \end{bmatrix} + \sum_{i=0}^{\frac{k}{2}} (-1)^{-1} (-1)^{\frac{i}{2}(i+k+1)} \begin{bmatrix} k+1 \\ i \end{bmatrix} = 0. \end{aligned}$$

For  $m \geq k+1$  the assertion is obvious, according to defining formula (15). The case for  $m < -1$  follows from  $\sigma(-1) = 0$  and  $\begin{bmatrix} k+1 \\ i \end{bmatrix} = 0$ , for  $i > k+1$ , with respect to the definition of the Fibonomial coefficients.



(ii) We can write successively

$$\begin{aligned}
\sigma(k-m) &= \sum_{j=0}^m (-1)^{\frac{j}{2}(j+k+1)} \begin{bmatrix} k+1 \\ j \end{bmatrix} = \sum_{i=k-m+1}^{k+1} (-1)^{\frac{k+1-i}{2}(2k+2-i)} \begin{bmatrix} k+1 \\ k+1-i \end{bmatrix} \\
&= \sum_{i=k-m+1}^{k+1} (-1)^1 (-1)^{\frac{i}{2}(i+k+1)} \begin{bmatrix} k+1 \\ i \end{bmatrix} \\
&= \sum_{i=0}^{k+1} (-1)^1 (-1)^{\frac{i}{2}(i+k+1)} \begin{bmatrix} k+1 \\ i \end{bmatrix} - \sum_{i=0}^{k-m} (-1)^1 (-1)^{\frac{i}{2}(i+k+1)} \begin{bmatrix} k+1 \\ i \end{bmatrix} \\
&= -\sigma(-1) + \sum_{i=0}^{k-m} (-1)^{\frac{i}{2}(i+k+1)} \begin{bmatrix} k+1 \\ i \end{bmatrix} = \sigma(m) .
\end{aligned}$$

(iii) Identities for  $\sigma(0)$  and  $\sigma(1)$  are directly implied by  $\sigma(-1) = 0$ . Using case (ii) and identity (14) we have

$$\begin{aligned}
\sigma(2) &= \sum_{j=0}^{k-2} (-1)^{\frac{j}{2}(j+k+1)} \begin{bmatrix} k+1 \\ j \end{bmatrix} = 1 + (-1)^{\frac{k-2}{2}} F_{k+1} - F_{k+1} F_k \\
&= 1 - F_{k+1} \left( F_k + (-1)^{\frac{k}{2}} \right) = 1 - F_{k+1} L_{\frac{k}{2}+1} F_{\frac{k}{2}-1} ,
\end{aligned}$$

$$\begin{aligned}
\sigma(3) &= \sigma(2) - \frac{1}{2} (-1)^{\frac{k-2}{2}} F_{k+1} F_k F_{k-1} \\
&= 1 - F_{k+1} F_k - (-1)^{\frac{k}{2}} F_{k+1} + \frac{1}{2} (-1)^{\frac{k}{2}} F_{k+1} F_k F_{k-1} \\
&= 1 - \frac{1}{2} (-1)^{\frac{k}{2}} F_{k+1} \left( 2 - F_k \left( F_{k-1} - 2(-1)^{\frac{k}{2}} \right) \right) \\
&= 1 - \frac{1}{2} (-1)^{\frac{k}{2}} F_{k+1} \left( 2 - F_k F_{\frac{k}{2}-2} L_{\frac{k}{2}+1} \right) .
\end{aligned}$$

This finishes the proof. □

The sum  $\sigma(m)$  can be simplified by the following lemma.

**Lemma 17.** *Let  $k$  be any even positive integer and let  $m$  be any integer. Then*

$$\sigma(m) - \sigma(m-2) = (-1)^{\frac{m}{2}(m+k+1)} \begin{bmatrix} k+2 \\ m \end{bmatrix} \frac{F_{\frac{k}{2}+1-m}}{F_{\frac{k}{2}+1}} .$$

*Proof.* For  $m < 2$  the assertion follows from the definition of the Fibonomial coefficients

and Lemma 16. For  $m \geq 2$  we have, with respect to Lemma 16,

$$\begin{aligned}
\sigma(m) - \sigma(m-2) &= \sigma(k-m) - \sigma(k-m+2) \\
&= \sum_{j=0}^m (-1)^{\frac{j}{2}(j+k+1)} \begin{bmatrix} k+1 \\ j \end{bmatrix} - \sum_{j=0}^{m-2} (-1)^{\frac{j}{2}(j+k+1)} \begin{bmatrix} k+1 \\ j \end{bmatrix} \\
&= (-1)^{\frac{m}{2}(m+k+1)} \begin{bmatrix} k+1 \\ m \end{bmatrix} + (-1)^{\frac{m-1}{2}((m-1)+k+1)} \begin{bmatrix} k+1 \\ m-1 \end{bmatrix} \\
&= (-1)^{\frac{m}{2}(m+k+1)} \left( \begin{bmatrix} k+1 \\ m \end{bmatrix} + (-1)^{\frac{k}{2}+m} \begin{bmatrix} k+1 \\ m-1 \end{bmatrix} \right),
\end{aligned}$$

which, by Lemma 14, implies the assertion.  $\square$

**Lemma 18.** *Let  $k$  be any even positive integer and let  $m$  be any integer. Then*

$$\sigma(m) - \sigma(m-4) = (-1)^{\frac{m}{2}(m+k+1)} \begin{bmatrix} k+4 \\ m \end{bmatrix} \frac{F_{\frac{k}{2}+2-m}}{F_{\frac{k}{2}+1} F_{k+3} F_{k+4}} \omega(m, k), \quad (16)$$

where

$$\omega(m, k) = F_{\frac{k}{2}+1-m} L_{\frac{k}{2}+2-m} F_{k+3} - F_m F_{m-1}.$$

*Proof.* With respect to Lemma 17 we have for any integer  $m$

$$\begin{aligned}
\sigma(m) - \sigma(m-4) &= (\sigma(m) - \sigma(m-2)) + (\sigma(m-2) - \sigma(m-4)) = \\
&= (-1)^{\frac{m}{2}(m+k+1)} \frac{1}{F_{\frac{k}{2}+1}} \left( F_{\frac{k}{2}+1-m} \begin{bmatrix} k+2 \\ m \end{bmatrix} - F_{\frac{k}{2}+3-m} \begin{bmatrix} k+2 \\ m-2 \end{bmatrix} \right).
\end{aligned}$$

The bracket term can be rewritten as

$$\begin{aligned}
&F_{\frac{k}{2}+1-m} \begin{bmatrix} k+2 \\ m \end{bmatrix} - F_{\frac{k}{2}+3-m} \begin{bmatrix} k+2 \\ m-2 \end{bmatrix} = \\
&= \begin{bmatrix} k+4 \\ m \end{bmatrix} \frac{1}{F_{k+3} F_{k+4}} \left( F_{\frac{k}{2}+1-m} F_{k+3-m} F_{k+4-m} - F_{\frac{k}{2}+3-m} F_m F_{m-1} \right).
\end{aligned}$$

The identity

$$F_{k+3-m} F_{k+4-m} = F_{k+4-2m} F_{k+3} + F_m F_{m-1}$$

follows from the identity ([12, p. 177])

$$F_{n+h} F_{n+l} - F_n F_{n+h+l} = (-1)^n F_h F_l,$$

with any integers  $h, n, l$ . Hence, we obtain

$$\begin{aligned}
&F_{\frac{k}{2}+1-m} \begin{bmatrix} k+2 \\ m \end{bmatrix} - F_{\frac{k}{2}+3-m} \begin{bmatrix} k+2 \\ m-2 \end{bmatrix} \\
&= \begin{bmatrix} k+4 \\ m \end{bmatrix} \frac{1}{F_{k+3} F_{k+4}} \left( F_{\frac{k}{2}+2-m} \left( F_{k+4-2m} F_{k+3} - F_m F_{m-1} \right) - F_{\frac{k+6}{2}-m} F_m F_{m-1} \right) \\
&= \begin{bmatrix} k+4 \\ m \end{bmatrix} \frac{1}{F_{k+3} F_{k+4}} \left( F_{\frac{k}{2}+1-m} F_{k+4-2m} F_{k+3} - \left( F_{\frac{k}{2}+3-m} - F_{\frac{k}{2}+1-m} \right) F_m F_{m-1} \right) \\
&= \begin{bmatrix} k+4 \\ m \end{bmatrix} \frac{F_{\frac{k}{2}+2-m}}{F_{k+3} F_{k+4}} \left( F_{\frac{k}{2}+1-m} L_{\frac{k}{2}+2-m} F_{k+3} - F_m F_{m-1} \right)
\end{aligned}$$

and the assertion follows.  $\square$

**Lemma 19.** *Let  $m \geq 5$  be any integer and let  $k$  be any positive even integer in one of the following forms*

$$(i) \ k = m - 4 + [2 \nmid m] , \quad (ii) \ k = 2(m - 3) , \quad (iii) \ k = 2(m - 1) .$$

*Then  $\omega(m, k)$  can be factored into a product of the Fibonacci or Lucas numbers.*

*Proof.* Condition (i), with respect to the identities ([12, pp. 176–177])  $F_{-n} = (-1)^{n+1}F_n$ ,  $L_{-n} = (-1)^n L_n$  and  $F_{2n} = F_n L_n$ , leads to the relation

$$\begin{aligned} \omega(m, m - 3) &= F_{-\frac{m+1}{2}} F_m L_{-\frac{m-1}{2}} - F_m F_{m-1} = F_m L_{\frac{m-1}{2}} (F_{\frac{m+1}{2}} - F_{\frac{m-1}{2}}) \\ &= F_m F_{\frac{m-3}{2}} L_{\frac{m-1}{2}} \end{aligned}$$

if  $m$  is odd and to the relation

$$\begin{aligned} \omega(m, m - 4) &= F_{\frac{m+2}{2}} F_{m-1} L_{\frac{m}{2}} - F_m F_{m-1} = F_{m-1} L_{\frac{m}{2}} (F_{\frac{m+2}{2}} - F_{\frac{m}{2}}) \\ &= F_{m-1} F_{\frac{m-2}{2}} L_{\frac{m}{2}} \end{aligned}$$

if  $m$  is even.

Using the identity  $F_{n+1}^2 + F_n^2 = F_{2n+1}$  ([12, p. 177]), we have from condition (ii)

$$\begin{aligned} \omega(m, 2(m - 3)) &= F_{2m-3} - F_m F_{m-1} = F_{m-2}^2 + F_{m-1}^2 - F_m F_{m-1} \\ &= F_{m-2}^2 - F_{m-1} (F_m - F_{m-1}) = F_{m-2}^2 - F_{m-1} F_{m-2} \\ &= F_{m-2} (F_{m-2} - F_{m-1}) = -F_{m-2} F_{m-3} . \end{aligned}$$

Condition (iii) gives  $\omega(m, 2(m - 1)) = -F_m F_{m-1}$ .  $\square$

**Remark 20.** The right-hand side of (16) can not be factored in a product of the Fibonacci or Lucas numbers for arbitrary values of  $k$  and  $m$ . The trivial factorization can be done for  $m = 0$  and  $m = 1$ . Table 1 lists the values of  $m$  and  $k$ ,  $2 \leq m \leq 10$ ,  $2 \leq k \leq 170$ , for which  $\omega(m, k)$  can be factored into a product of the Fibonacci or Lucas numbers. These values were found by computer. The computer search for  $10 \leq m \leq 100$  showed that  $\omega(m, k)$  can be factored into a product of the Fibonacci or Lucas numbers only at values of  $m$ ,  $k$  satisfying conditions from Lemma 20.

Table 1. The values for which  $\omega(m, k)$  is factorizable.

$m$	$k$				
2	2	6			
3	2	4	6		
4	2	4	6	8	
5	2	4	8	10	
6	2	6	10		
7	4	6	8	12	
8	4	6	8	10	14
9	2	6	10	12	16
10	2	6	14	18	

## 5 The proofs of the main results

**Proof of Theorem 2.** First we prove identity (5). We showed [10] that for any positive odd integer  $k$  and any positive integer  $n$

$$S_{2n-1}(k) = \sum_{i=1}^n (-1)^{i+1} \left( \binom{\frac{k+3}{2} - n - i}{n - i} + \binom{\frac{k+1}{2} - n - i}{n - i - 1} \right) \begin{bmatrix} k + 1 \\ 2i - 1 \end{bmatrix} \quad (17)$$

and

$$S_{2(n-1)}(k) = \sum_{i=1}^n (-1)^{i+1} \left( \binom{\frac{k+5}{2} - n - i}{n - i} + \binom{\frac{k+3}{2} - n - i}{n - i - 1} \right) \begin{bmatrix} k + 1 \\ 2(i - 1) \end{bmatrix}. \quad (18)$$

Relation (5) can be obtained from (17) and (18). Replacing  $n$  by  $n + 1$  and  $i$  by  $n + 1 - i$  we have for any nonnegative integer  $n$

$$S_{2n+1}(k) = \sum_{i=0}^n (-1)^{n-i} \left( \binom{\frac{k+1}{2} - (2n+1) + i}{i} + \binom{\frac{k-1}{2} - (2n+1) + i}{i - 1} \right) \begin{bmatrix} k + 1 \\ 2n+1-2i \end{bmatrix}$$

and

$$S_{2n}(k) = \sum_{i=0}^n (-1)^{n-i} \left( \binom{\frac{k+1}{2} - 2n + i}{i} + \binom{\frac{k-1}{2} - 2n + i}{i - 1} \right) \begin{bmatrix} k + 1 \\ 2n - 2i \end{bmatrix},$$

which can be joined into the proved identity.

We begin the proof of relation (6) by defining the polynomial

$$P_k(x) = \sum_{i=0}^k p_i(k) x^i = \prod_{j=0}^{\frac{k}{2}-1} (1 - (-1)^j L_{k-2j} x + x^2) \quad (19)$$

for an even nonnegative integer  $k$ . By direct multiplication of the factors in (19) we get the identities

$$p_{2i+1}(k) = - \sum_{j=0}^i \binom{\frac{k}{2} - (2j+1)}{i-j} S_{2j+1}(k), \quad (20)$$

for  $i = 0, 1, 2, \dots, \frac{k-2}{2}$ , and

$$p_{2i}(k) = \sum_{j=0}^i \binom{\frac{k}{2} - 2j}{i-j} S_{2j}(k), \quad (21)$$

for  $i = 0, 1, 2, \dots, \frac{k}{2}$ . By shifting indexes of summation it is possible to join (20) and (21) into the relation

$$p_n(k) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^n \binom{\frac{k}{2} - n + 2i}{i} S_{n-2i}(k), \quad (22)$$

for  $n = 0, 1, 2, \dots, k$ . This identity can be extended to any positive integer  $n$  with respect to Lemma 9, as  $p_n(k) = 0$  for  $n < 0$  or  $n > k$ .

If  $k$  is an even positive integer, the denominator in (1) is a polynomial of an odd degree  $k+1$ :

$$D_{k+1}(x) = \sum_{i=0}^{k+1} d_{k+1,i} x^i,$$

where integers  $d_{k+1,i} = (-1)^{\frac{i(i+1)}{2}} \begin{bmatrix} k+1 \\ i \end{bmatrix}$  are terms of sequence [A055870](#), called the ‘‘signed Fibonomial triangle’’ in Sloane’s *On-Line Encyclopedia of Integer Sequences* [13]. Identity (2) implies

$$\begin{aligned} D_{k+1}(x) &= \prod_{j=0}^k (1 - \alpha^{k-j} \beta^j x) = (1 - (\alpha\beta)^{\frac{k}{2}} x) \prod_{\substack{j=0 \\ j \neq \frac{k}{2}}}^k (1 - \alpha^{k-j} \beta^j x) \\ &= (1 - (-1)^{\frac{k}{2}} x) \prod_{j=0}^{\frac{k}{2}-1} (1 - (-1)^j \alpha^{k-2j} x) (1 - (-1)^j \beta^{k-2j} x) \\ &= (1 - (-1)^{\frac{k}{2}} x) \prod_{j=0}^{\frac{k}{2}-1} (1 - (-1)^j (\alpha^{k-2j} + \beta^{k-2j}) x + (\alpha\beta)^{k-2j} x^2) \\ &= (1 - (-1)^{\frac{k}{2}} x) \prod_{j=0}^{\frac{k}{2}-1} (1 - (-1)^j L_{k-2j} x + x^2), \end{aligned}$$

according to the relation  $\alpha\beta = -1$  and the formula  $L_{k-2j} = \alpha^{k-2j} + \beta^{k-2j}$ . Thus, with respect to (19),  $D_{k+1}(x) = (1 - (-1)^{\frac{k}{2}} x) P_k(x)$ . By multiplying on the right-hand side and comparing coefficients of  $x^i$  we have the following relations between coefficients  $d_{k+1,i}$  of  $D_{k+1}(x)$  and

coefficients  $p_i(k)$  of  $P_k(x)$

$$\begin{aligned} d_{k+1,0} &= p_0(k) = 1, \\ d_{k+1,i} &= p_i(k) + (-1)^{\frac{k}{2}+1} p_{i-1}(k), \quad i = 1, 2, \dots, k, \\ d_{k+1,k+1} &= (-1)^{\frac{k}{2}+1} p_k(k) = (-1)^{\frac{k}{2}+1}. \end{aligned}$$

As  $p_n(k) = 0$  for  $n < 0$  or  $n > k$  we can rewrite the previous relations in the recurrence

$$p_n(k) + (-1)^{\frac{k}{2}+1} p_{n-1}(k) = d_{k+1,n},$$

which holds for any integer  $n$ . Using Lemma 13 we have

$$p_n(k) = \sum_{j=0}^n (-1)^{\frac{k}{2}(n+j)} d_{k+1,j} \quad (23)$$

for any nonnegative integer  $n$ .

To complete the proof of (6) we have to invert identity (22). Setting  $a_n = p_{2n}(k)$ ,  $b_n = S_{2n}(k)$  and  $q = \frac{k}{2}$  in inverse formula (11) we obtain

$$S_n(k) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{n+i} \left( \binom{\frac{k}{2} - n + i}{i} + \binom{\frac{k}{2} - n + i - 1}{i-1} \right) p_{n-2i}(k). \quad (24)$$

From (23) and (24) we deduce that

$$S_n(k) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{n-2i} (-1)^{n+i} (-1)^{\frac{k}{2}(n+j)} \left( \binom{\frac{k}{2} - n + i}{i} + \binom{\frac{k}{2} - n + i - 1}{i-1} \right) d_{k+1,j}.$$

Putting  $d_{k+1,j} = (-1)^{\frac{j}{2}(j+1)} \begin{bmatrix} k+1 \\ j \end{bmatrix}$  we obtain (6) after simplification.  $\square$

**Proof of Corollary 3.** The assertion is obviously true with respect to (5) if  $k$  is any odd integer. For even values of  $k$  identity (6) can be written using (15) as

$$S_n(k) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{n+i+n\frac{k}{2}} \sigma(n-2i) \Theta(i, k, n).$$

With respect to Lemma 12 for  $k \rightarrow \infty$

$$\sigma(n-2i) \sim (-1)^{\frac{n-2i}{2}(n-2i+k+1)} \begin{bmatrix} k+1 \\ n-2i \end{bmatrix} = (-1)^i (-1)^{\frac{n}{2}(n+k+1)} \begin{bmatrix} k+1 \\ n-2i \end{bmatrix}.$$

Hence, we obtain

$$\begin{aligned} S_n(k) &\sim \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{n+i+n\frac{k}{2}} (-1)^i (-1)^{\frac{n}{2}(n+k+1)} \Theta(i, k, n) \begin{bmatrix} k+1 \\ n-2i \end{bmatrix} \\ &= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{\frac{n}{2}(n-1)} \Theta(i, k, n) \begin{bmatrix} k+1 \\ n-2i \end{bmatrix} \end{aligned}$$

and the assertion follows from the congruence  $\frac{n}{2}(n-1) \equiv \lfloor \frac{n}{2} \rfloor \pmod{2}$ .  $\square$

**Proof of Theorem 4.** For any even  $m$  we have

$$\sum_{i=0}^{\frac{m}{2}} (\sigma(m-2i) - \sigma(m-2(i+1))) = \sigma(m) - \sigma(-2)$$

and analogously for any odd  $m$

$$\sum_{i=0}^{\frac{m-1}{2}} (\sigma(m-2i) - \sigma(m-2(i+1))) = \sigma(m) - \sigma(-1) .$$

Thus, using Lemma 16 we obtain for any integer  $m$

$$\sigma(m) = \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} (\sigma(m-2i) - \sigma(m-2(i+1)))$$

and with respect to Lemma 17

$$\begin{aligned} \sigma(m) &= \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^{\frac{m-2i}{2}(m-2i+k+1)} \frac{1}{F_{\frac{k}{2}+1}} \begin{bmatrix} k+2 \\ m-2i \end{bmatrix} F_{\frac{k}{2}+1-(m-2i)} \\ &= (-1)^{\frac{m}{2}(m+k+1)} \frac{1}{F_{\frac{k}{2}+1}} \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^i \begin{bmatrix} k+2 \\ m-2i \end{bmatrix} F_{\frac{k+2}{2}-m+2i} . \end{aligned}$$

□

**Proof of Corollary 5.** Applying Theorem 2 and Theorem 4, consecutively, we get

$$\begin{aligned} S_n(k) &= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{n+i+n\frac{k}{2}} \sigma(n-2i) \Theta(i, k, n) \\ &= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{n+i+n\frac{k}{2}} \Theta(i, k, n) \frac{(-1)^{\frac{n}{2}(n+k+1)}}{F_{\frac{k}{2}+1}} \sum_{j=i}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \begin{bmatrix} k+2 \\ n-2j \end{bmatrix} F_{\frac{k+2}{2}-n+2j} \\ &= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{\frac{n}{2}(n-1)+i} \frac{1}{F_{\frac{k}{2}+1}} \Theta(i, k, n) \sum_{j=i}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \begin{bmatrix} k+2 \\ n-2j \end{bmatrix} F_{\frac{k+2}{2}-n+2j} \\ &= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{\lfloor \frac{n}{2} \rfloor + i} \Theta(i, k, n) \frac{1}{F_{\frac{k}{2}+1}} \sum_{j=i}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \begin{bmatrix} k+2 \\ n-2j \end{bmatrix} F_{\frac{k+2}{2}-n+2j} \\ &= \frac{(-1)^{\lfloor \frac{n}{2} \rfloor}}{F_{\frac{k}{2}+1}} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=i}^{\lfloor \frac{n}{2} \rfloor} (-1)^{i+j} \Theta(i, k, n) \begin{bmatrix} k+2 \\ n-2j \end{bmatrix} F_{\frac{k+2}{2}-n+2j} . \end{aligned}$$

□

**Proof of Theorem 6.** Similarly as in the proof of Theorem 4 we obtain for any integer  $m$  the relation

$$\sum_{i=0}^{\lfloor \frac{m}{4} \rfloor} (\sigma(m-4i) - \sigma(m-4(i+1))) = \sigma(m) - \sigma\left(m-4\left(\left\lfloor \frac{m}{4} \right\rfloor + 1\right)\right).$$

Thus, using Lemma 16 we obtain

$$\sigma(m) = \sum_{i=0}^{\lfloor \frac{m}{4} \rfloor} (\sigma(m-4i) - \sigma(m-4(i+1))).$$

With respect to Lemma 18 we have

$$\begin{aligned} \sigma(m) &= \sum_{i=0}^{\lfloor \frac{m}{4} \rfloor} (-1)^{\frac{m-4i}{2}(m-4i+k+1)} \begin{bmatrix} k+4 \\ m-4i \end{bmatrix} \frac{F_{\frac{k}{2}+2-(m-4i)}}{F_{\frac{k}{2}+1}F_{k+3}F_{k+4}} \\ &\cdot \left( F_{\frac{k}{2}+1-(m-4i)} L_{\frac{k}{2}+2-(m-4i)} F_{k+3} - F_{m-4i} F_{m-4i-1} \right) \\ &= \frac{(-1)^{\frac{m}{2}(m+k+1)}}{F_{\frac{k}{2}+1}F_{k+3}F_{k+4}} \sum_{i=0}^{\lfloor \frac{m}{4} \rfloor} \begin{bmatrix} k+4 \\ m-4i \end{bmatrix} F_{\frac{k}{2}+2-(m-4i)} \\ &\cdot \left( F_{\frac{k}{2}+1-(m-4i)} L_{\frac{k}{2}+2-(m-4i)} F_{k+3} - F_{m-4i} F_{m-4i-1} \right). \end{aligned}$$

□

**Proof of Corollary 7.** Identities (9) and (10) can be obtained from identities (5) and (6) with respect to  $S_n(k) = 0$  for positive integers  $k$ ,  $n > \lfloor \frac{k+1}{2} \rfloor$  (see Lemma 9). □

**Proof of Corollary 8.** Each of these three sums follows from identity (6) after some tedious simplification. □

## 6 Concluding remark

It is interesting to compare the effectiveness of formulas (6) and (8) in contrast to defining formula (4) for computation of  $S_n(k)$ . Therefore, we found the CPU time (in seconds) required for computation of sums  $S_3(k)$  for some values of  $k$  using the system Mathematica on a standard PC. There is the measured time in Table 2.

Table 2. CPU time for  $S_3(k)$

	$k$							
	100	200	300	400	500	600	700	800
(4)	0.297	2.438	8.547	21.296	43.172	77.078	130.125	203.594
(6)	0	0	0.047	0.094	0.172	0.297	0.484	0.719
(8)	0	0	0.015	0.046	0.078	0.156	0.25	0.359



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(Concerned with sequence [A055870](#).)

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