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An Inequality for Macaulay Functions

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Abstract

Given integers $k \geq 1$ and $n \geq 0$, there is a unique way of writing n as $n = \binom{n_k}{k} + \binom{n_{k-1}}{k-1} + \cdots + \binom{n_1}{1}$ so that $0 \leq n_1 < \cdots < n_{k-1} < n_k$. Using this representation, the k^{th} *Macaulay function of* n is defined as $\partial^k(n) = \binom{n_k-1}{k-1} + \binom{n_{k-1}-1}{k-2} + \cdots + \binom{n_1-1}{0}$. We show that if $a \geq 0$ and $a < \partial^{k+1}(n)$, then $\partial^k(a) + \partial^{k+1}(n-a) \geq \partial^{k+1}(n)$. As a corollary, we obtain a short proof of Macaulay's theorem. Other previously known results are obtained as direct consequences.

1 Introduction

Given integers $k \geq 1$ and $n \geq 0$, there is a unique way of writing n as

$$n = \binom{n_k}{k} + \binom{n_{k-1}}{k-1} + \cdots + \binom{n_2}{2} + \binom{n_1}{1} \tag{1}$$

so that $0 \leq n_1 < n_2 < \cdots < n_{k-1} < n_k$. Using this representation, called the *k-binomial representation of n*, the k^{th} *Macaulay function of n* is defined as

$$\partial^k(n) = \binom{n_k - 1}{k-1} + \binom{n_{k-1} - 1}{k-2} + \cdots + \binom{n_2 - 1}{1} + \binom{n_1 - 1}{0}.$$

(See [2, 6, 12, 14] for details.) The main goal of this paper is to prove the following inequality for Macaulay functions and show some of its consequences.

Theorem 1. *Let k, a, and n be integers such that $k \geq 1$ and $n \geq a \geq 0$. If $a < \partial^{k+1}(n)$, then*

$$\partial^k(a) + \partial^{k+1}(n-a) \geq \partial^{k+1}(n). \tag{2}$$

Moreover, if $n = \binom{N}{k+1}$ for some $N \geq k+1$, then the equality occurs only when $a = 0$.

Macaulay functions and the related Kruskal–Katona functions (defined below) are relevant for their applications to the study of antichains in multisets (see for example [12, 2]), posets, rings and polyhedral combinatorics (see [5] and the survey [3]). In particular, they play an important role in proving results, extensions and generalizations of classical problems concerning the Kruskal–Katona [13, 11, 16], Macaulay [14], and Erdős-Ko-Rado [9] theorems. More recently, the authors [1] applied Theorem 1 to the problem of finding the maximum number of translated copies of a pattern that can occur among n points in a d-dimensional space, a typical problem related to the study of repeated patterns in Combinatorial Geometry. For every $P \subseteq \mathbb{R}^d$, a fixed finite point set (called a *pattern*), we say that P is a *rational simplex* if all the points of P are rationally affinely independent. We proved [1] that the maximum number of translated copies of a rational simplex P with $|P| = k+1$ determined by a set of n points of $\mathbb{R}^d$ is equal to $n - \partial^k(n)$.

We now introduce some terminology needed to state the Kruskal–Katona and Macaulay theorems. Let M_k and S_k denote the set of nonincreasing, respectively decreasing, sequences of natural integers of length k, i.e.,

$$M_k = \left\{(x_1, x_2, \ldots, x_k) \in \mathbb{N}^k : x_1 \geq x_2 \geq \cdots \geq x_k \geq 1\right\}$$
$$S_k = \left\{(x_1, x_2, \ldots, x_k) \in \mathbb{N}^k : x_1 > x_2 > \cdots > x_k \geq 1\right\}.$$

If $A \subseteq M_k$ (or S_k), then the *shadow* of A, denoted by ∂A, consists of all nonincreasing (decreasing) subsequences of length $k-1$ of elements of A ($\partial(\emptyset) = \emptyset$). That is,

$$\partial A = \{x : x \text{ is a subsequence of } y \text{ of length } k-1, \text{ for some } y \in A\}.$$

By analogy, one can think of M_k (or S_k) as multisets (or sets) of size k, with positive integers as elements. In this context ∂A consists of the subsets of multisets (or sets) in A of cardinality $k-1$.

The Kruskal–Katona function ∂_k (defined below) is the analogue of the Macaulay function defined before. For n as in Identity (1),

$$\partial_k(n) = \binom{n_k}{k-1} + \binom{n_{k-1}}{k-2} + \cdots + \binom{n_2}{1} + \binom{n_1}{0}.$$

The sets of sequences M_k and S_k are *lexicographically ordered*. That is, for x and y in M_k (or S_k), $x \prec y$ if for some index i, $x_i < y_i$, and $x_j = y_j$ whenever $j < i$. There is an important relationship between shadows of multisets and sets and the functions ∂^k and ∂_k. Namely, if we denote by $FM_k(n)$ and $FS_k(n)$ the first n members, in lexicographic order, of M_k and S_k, respectively; then

$$|\partial FM_k(n)| = \partial^k(n) \text{ and } |\partial FS_k(n)| = \partial_k(n) \,.. \tag{3}$$

The Kruskal–Katona and Macaulay theorems show that in fact $\partial_k(n)$ and $\partial^k(n)$ are the best lower bounds for the shadow of a set with n elements.

Theorem K. (Kruskal [13]–Katona [11]) Let $k \geq 0$; for every $A \subseteq S_{k+1}$,

$$|\partial A| \geq |\partial FS_{k+1}(|A|)| = \partial_{k+1}(|A|) \,.$$

Theorem M. (Macaulay [14]) Let $k \geq 0$; for every $A \subseteq M_{k+1}$,

$$|\partial A| \geq |\partial FM_{k+1}(|A|)| = \partial^{k+1}(|A|) \,.$$

The theorems just stated correspond only to the necessity of their polyhedral versions, where f-vectors of complexes or multi-complexes are characterized by these inequalities.

We present, in Section 2, a short and simple proof of Theorem M obtained as a corollary of Theorem 1. For a short textbook proof of Theorem K, see Frankl's proof [10]. For an approach similar to ours, we point out that Eckhoff and Wegner [8], (see also Daykin [7]) obtained a proof of Theorem K as a consequence of an inequality similar to Inequality (2). Namely, for $n \geq a \geq 0$,

$$\max(\partial_k(a), n-a) + \partial_{k+1}(n-a) \geq \partial_{k+1}(n) \,. \tag{4}$$

The equivalent inequality to Inequality (4) for the functions ∂^k and ∂^{k+1} is true, and it was in fact generalized by Björner and Vrećica [5] to a larger number of terms (see Corollary 3). The proof of their result depends on Macaulay's theorem. However, we are not aware of, nor could we find, a proof of Theorem M obtained as a consequence of this result. We show, in Section 2, how Björner and Vrećica's inequalities follow easily from Theorem 1.

Our proof of Theorem 1, presented in Section 3, is elementary as it only relies on properties of binomial coefficients. Some of the ideas are similar to those used in [8] for the proof of Inequality (4).

The condition $a < \partial^{k+1}(n)$ in Theorem 1 cannot be strengthened. For instance, whenever $k \geq 2$, $a = \partial^{k+1}(n)$, and the last three coefficients of the $(k+1)$-binomial representation of n are $n_1 = 1$, $n_2 = 2$, and $n_3 = 4$; it follows that

$$\partial^k(a) + \partial^{k+1}(n-a) = \partial^{k+1}(n) - 1 < \partial^{k+1}(n) \,.$$

Finally, it is an interesting open problem to determine the pairs (n, a) with $a < \partial^{k+1}(n)$ that achieve equality in Inequality (2). So far we were able to classify the pairs when n is of the form $\binom{N}{k+1}$. The solution to this problem would be the first step to classify all patterns P for which the maximum number of translates of P, among n points in $\mathbb{R}^d$; is equal to $n - \partial^k(n)$.

2 Consequences of the theorem

We first prove Macaulay's theorem as a corollary of Theorem 1.

Proof of Theorem M. Let $A \subseteq M_{k+1}$. We proceed by induction on $k + |A|$. If $k = 0$ or $A = \emptyset$, the result is trivially true. Suppose $k \geq 1$ and $A \neq \emptyset$. Set $A_{11} = \{x \in M_k : x_k = 1$ and $x * 1 \in A\}$, $A_{12} = \{x \in M_k : x_k \geq 2$ and $x * 1 \in A\}$, and $A_2 = \{x \in A : x_{k+1} \geq 2\}$. Here $x * 1$ denotes the concatenation of x and 1, that is $x * 1$ is the k-tuple x with an entry 1 appended in the $(k+1)^{\text{th}}$ position. Clearly, $A = (A_{11} * 1) \cup (A_{12} * 1) \cup A_2$ and the terms in the union are pairwise disjoint. Moreover, we can assume that $A_{11} \cup A_{12} \neq \emptyset$. Otherwise, since all entries of members of A are ≥ 2, we can work with the set A' obtained by subtracting 1 to every entry in the sequences of A ($|A'| = |A|$ and $|\partial A'| = |\partial A|$.) Let $a = |A_{11}| + |A_{12}|$ and $b = |A_2|$. Note that $|A| = a + b$ and $a \geq 1$.

If $x = (x_1, x_2, \ldots, x_k) \in A_{11}$, then $(x_1, x_2, \ldots, x_{k-1}) \in \partial A_{11}$ and $(x_1, x_2, \ldots, x_{k-1}, 1) = x \in \partial A_{11} * 1$. That is, $A_{11} \subseteq \partial A_{11} * 1$. We now calculate ∂A in terms of A_{11}, A_{12}, and A_2. We use the property that $\partial(A \cup B) = \partial A \cup \partial B$.

$$\begin{aligned} \partial A &= \partial A_2 \cup A_{12} \cup A_{11} \cup (\partial A_{11} * 1) \cup (\partial A_{12} * 1) \\ &= \partial A_2 \cup A_{12} \cup (\partial A_{11} * 1) \cup (\partial A_{12} * 1) \\ &= (\partial A_2 \cup A_{12}) \cup (\partial (A_{11} \cup A_{12}) * 1). \end{aligned}$$

If $x \in (\partial A_2 \cup A_{12})$, then $x_k \geq 2$. Thus

$$(\partial A_2 \cup A_{12}) \cap (\partial (A_{11} \cup A_{12}) * 1) = \emptyset,$$

and consequently

$$|\partial A| = |\partial A_2 \cup A_{12}| + |\partial (A_{11} \cup A_{12})|. \tag{5}$$

We consider two cases. If $a \geq \partial^{k+1}(|A|)$, then

$$|\partial A| = |\partial A_2 \cup A_{12}| + |\partial (A_{11} \cup A_{12})| \geq |A_{12}| + |A_{11}| = a \geq \partial^{k+1}(|A|).$$

Assume $a < \partial^{k+1}(|A|)$. Since $a \geq 1$ then $b < |A|$ and thus, by induction and Identity (3),

$$|\partial A_2 \cup A_{12}| \geq |\partial A_2| \geq |\partial F_{k+1}(b)| = \partial^{k+1}(b) \text{ and}$$

$$|\partial (A_{11} \cup A_{12})| \geq |\partial F_k(a)| = \partial^k(a).$$

Therefore, by Identity (5), Theorem 1, and Identity (3); it follows that

$$|\partial A| \geq \partial^{k+1}(b) + \partial^k(a) \geq \partial^{k+1}(|A|) = |\partial F_{k+1}(|A|)|.$$

□

In terms of shadows of sets, and using our previous corollary, Theorem 1 can be generalized as follows.

Corollary 2. *Given sets $A \subseteq M_k$ and $B \subseteq M_{k+1}$ with $|A| < |\partial F_{k+1}(|A| + |B|)|$ we have*

$$|\partial A| + |\partial B| \geq |\partial F_{k+1}(|A| + |B|)|.$$

Proof. By the previous corollary and Identity (3), $|\partial A| + |\partial B| \geq \partial^k(|A|)+ \partial^{k+1}(|B|)$ and $|A| < \partial^{k+1}(|A| + |B|)$. Thus, by Theorem 1, $\partial^k(|A|)+ \partial^{k+1}(|B|) \geq |\partial F_{k+1}(|A| + |B|)|$. □

The following inequality, proved by Björner and Vrećica, follows directly from our Theorem. We recall that their proof makes use of Macaulay's theorem. Note that $r = 1$, $n_0 = a$, and $n_1 = n - a$ give the equivalent inequality to Inequality (4) for the function ∂^k.

Corollary 3. (Lemma 3.2 [4], also Lemma 2.1 [15]). *For $k > 0$, the function ∂^k satisfies that*

$$\partial^k \left(\sum_{i=0}^{r} n_i \right) \leq \sum_{i=0}^{r} \max \left\{ n_{i+1}, \partial^{k-i} \left(n_i \right) \right\},$$

$$\partial^k \left(1 + \sum_{i=0}^{k} n_i \right) \leq 1 + \sum_{i=0}^{k-1} \max \left\{ n_{i+1}, \partial^{k-i} \left(n_i \right) \right\}.$$

for all nonnegative integers n_i and $r < k$.

Proof. By induction on k. If $k = 1$ the inequalities are trivially true.

Let $r < k + 1$, $a = \sum_{i=1}^{r} n_i$, and $n = \sum_{i=0}^{r} n_i$. If $a \geq \partial^{k+1}(n)$, then

$$\begin{aligned}\partial^{k+1} \left(\sum_{i=0}^{r} n_i \right) &= \partial^{k+1} \left(n \right) \leq a = \sum_{i=1}^{r} n_i \leq \sum_{i=0}^{r-1} \max \left\{ n_{i+1}, \partial^{k+1-i} \left(n_i \right) \right\} \\ &\leq \sum_{i=0}^{r} \max \left\{ n_{i+1}, \partial^{k+1-i} \left(n_i \right) \right\}.\end{aligned}$$

If on the other hand, $a < \partial^{k+1}(n)$, then by Theorem 1,

$$\partial^{k+1} \left(\sum_{i=0}^{r} n_i \right) = \partial^{k+1} \left(n \right) \leq \partial^{k+1} \left(n - a \right) + \partial^{k} \left(a \right) = \partial^{k+1} \left(n_0 \right) + \partial^{k} \left(\sum_{i=0}^{r-1} n_{i+1} \right);$$

then by induction,

$$\begin{aligned}\partial^{k+1} \left(\sum_{i=0}^{r} n_i \right) &\leq \partial^{k+1} \left(n_0 \right) + \sum_{i=0}^{r-1} \max \left\{ n_{i+2}, \partial^{k-i} \left(n_{i+1} \right) \right\} \\ &\leq \sum_{i=0}^{r} \max \left\{ n_{i+1}, \partial^{k+1-i} \left(n_i \right) \right\}.\end{aligned}$$

This proves the first inequality. The second inequality is proved exactly the same way by letting $a = 1 + \sum_{i=1}^{k+1} n_i$ and $n = 1 + \sum_{i=0}^{k+1} n_i$. □

3 Proof of the theorem

We first present a simple observation. If $n > k \geq 0$ then by Pascal's identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-2}{k-1} + \cdots + \binom{n-k}{1} + \binom{n-k-1}{0}. \tag{6}$$

Let $a = \sum_{i=1}^{k} \binom{a_i}{i}$ be the k-binomial representation of a. We say that a is *k-long* if $a_1 \geq 1$, and *k-short* if $a_1 = 0$.

Lemma 4. *Let $a \geq 0$ be an integer. If a is k-short, then $\partial^k(a+1) = \partial^k(a) + 1$, otherwise $\partial^k(a+1) = \partial^k(a)$.*

Proof. The result is clear for $a = 0$. If $a \geq 1$ is k-short, then $a = \sum_{i=v}^{k} \binom{a_i}{i}$ for some $v \geq 2$ and $a_v \geq v$. Thus $a + 1 = \sum_{i=v}^{k} \binom{a_i}{i} + \binom{v-1}{v-1}$ is the k-binomial representation of $a+1$ where all the zero terms have been omitted. Then $\partial^k(a+1) = \partial^k(a) + \binom{v-2}{v-2} = \partial^k(a) + 1$.

Now suppose a is k-long. There is $v \geq 2$ such that $a_j = a_1 + j - 1$ for $j < v$, and either $v = k+1$ or $v \leq k$ and $a_v > a_1 + v - 1$. Then

$$a + 1 = \binom{a_k}{k} + \cdots + \binom{a_v}{v} + \binom{a_1+v-2}{v-1} + \cdots + \binom{a_1+1}{2} + \binom{a_1}{1} + \binom{a_1-1}{0}$$

and by Identity (6) the k-binomial representation of $a+1$ is

$$a + 1 = \binom{a_k}{k} + \cdots + \binom{a_v}{v} + \binom{a_1+v-1}{v-1}.$$

Then, again by Identity (6),

$$\partial^k(a+1) - \partial^k(a) = \binom{a_1+v-2}{v-2} - \left(\binom{a_1+v-3}{v-2} + \cdots + \binom{a_1}{1} + \binom{a_1-1}{0}\right) = 0.$$

□

To prove the Theorem, we need to consider the *extended k-binomial representation* of a positive integer a, by requiring an a_0 coefficient. That is, we write

$$a = \binom{a'_k}{k} + \binom{a'_{k-1}}{k-1} + \cdots + \binom{a'_2}{2} + \binom{a'_1}{1} + \binom{a'_0}{0},$$

with $0 \leq a'_0 = a'_1 - 1 < a'_1 < \cdots < a'_k$. The condition $a'_0 = a'_1 - 1$ is necessary to make this representation unique when it exists. Clearly $a = 0$ does not have an extended representation. In general the following is true.

Lemma 5. *Let $a = \sum_{i=v}^{k} \binom{a_i}{i} \geq 1$ be the k-binomial representation of a, where the terms equal to zero have been omitted. The extended k-binomial representation of a exists (and it is unique), if and only if $a_v \geq v + 1$.*

Proof. If $a_v \geq v+1$, then, by Identity (6),

$$\binom{a_v}{v} = \binom{a_v - 1}{v} + \binom{a_v - 2}{v-1} + \cdots + \binom{a_v - v - 1}{0}.$$

Thus

$$a = \sum_{i=v+1}^{k} \binom{a_i}{i} + \sum_{i=0}^{v} \binom{a_v - v - 1 + i}{i}$$

is an extended k-representation of a. Reciprocally, if $a = \sum_{i=0}^{k} \binom{a'_i}{i}$ is an extended k-representation, then $\binom{a'_0}{0} = \binom{a'_1 - 1}{0}$, and there is $v \geq 1$ such that $a'_j = a'_1 + j - 1$ for $0 \leq j \leq v$ with either $v = k$ or $a'_{v+1} > a'_1 + v$. Then, by Identity (6),

$$a = \sum_{i=v+1}^{k} \binom{a'_j}{j} + \sum_{j=0}^{v} \binom{a'_1 + j - 1}{j} = \sum_{i=v+1}^{k} \binom{a'_j}{j} + \binom{a'_1 + v}{v},$$

is the k-representation of a. Thus $a_v = a'_1 + v \geq v + 1$. □

We can define $\partial_e^k(a) = \sum_{i=1}^{k} \binom{a'_i - 1}{i-1}$ for the extended k-representation of a (if it exists). It turns out that both definitions agree, i.e., $\partial^k(a) = \partial_e^k(a)$. Indeed, if $a = \sum_{i=v}^{k} \binom{a_i}{i}$ with $a_v \geq v+1$, then by Identity (6) and the last proof,

$$\partial^k(a) - \partial_e^k(a) = \tbinom{a_v - 1}{v-1} - \sum_{i=0}^{v} \binom{a_v - v - 2 + i}{i - 1} = 0.$$

Let $n = \sum_{i=1}^{k+1} \binom{n_i}{i}$, $a = \sum_{i=1}^{k} \binom{a_i}{i}$, and $n - a = b = \sum_{i=1}^{k+1} \binom{b_i}{i}$ be binomial representations.

Lemma 6. *If $0 \leq a < \partial^{k+1}(n)$, then $a_k < n_{k+1} \leq b_{k+1} + 1$.*

Proof. We prove the contrapositives. If $a_k \geq n_{k+1}$, then

$$a \geq \binom{a_k}{k} \geq \binom{n_{k+1}}{k} = \partial^{k+1}\left(\binom{n_{k+1}+1}{k+1}\right) \geq \partial^{k+1}(n),$$

since $\binom{n_{k+1}+1}{k+1} \geq n$ and ∂^{k+1} is a non-decreasing function by Lemma 4. Now, if $b_{k+1} + 1 \leq n_{k+1} - 1$, then $b < \binom{b_{k+1}+1}{k+1} \leq \binom{n_{k+1}-1}{k+1}$. Thus

$$a = n - b > n - \binom{n_{k+1} - 1}{k+1} = \binom{n_{k+1} - 1}{k} + \binom{n_k}{k} + \binom{n_{k-1}}{k-1} + \cdots + \binom{n_1}{1},$$

but

$$\partial^{k+1}(n) = \binom{n_{k+1} - 1}{k} + \binom{n_k - 1}{k-1} + \binom{n_{k-1} - 1}{k-2} + \cdots + \binom{n_1 - 1}{0},$$

and clearly $\binom{n_i}{i} \geq \binom{n_i - 1}{i-1}$. Thus $a \geq \partial^{k+1}(n)$. □

Proof of Theorem 1. Recall that $b = n - a$. Clearly Inequality (2) holds if $a = 0$, and the case $a = 1$ is a consequence of Lemma 4. We consider two cases.

Case 1. $a_k < b_{k+1}$.

Let $a = \sum_{i=v}^{k} \binom{a_i}{i} \geq 2$ be the k-binomial representation of a without the zero terms. Assume that the pair (a, b) minimizes $\partial^k(a)+ \partial^{k+1}(b)$ with a as small as possible.

(i) Suppose first that $a_v \geq v + 1$. Then, by Lemma 5, a has an extended representation, say $a = \sum_{i=0}^{k} \binom{a'_i}{i}$. Let

$$\alpha = \sum_{i=1}^{k} \binom{\min(a'_i, b_i)}{i} \text{ and } \beta = \binom{b_{k+1}}{k+1} + \sum_{i=1}^{k} \binom{\max(a'_i, b_i)}{i} + \binom{a'_0}{0}.$$

Note that $a + b = \alpha + \beta$ and $\alpha < a$. Also

$$0 \leq \min(a'_1, b_1) < \min(a'_2, b_2) < \cdots < \min(a'_k, b_k) \text{ and}$$

$$0 \leq a'_0 < \max(a'_1, b_1) < \cdots < \max(a'_k, b_k) < b_{k+1}$$

(since $a'_k \leq a_k < b_{k+1}$ by assumption). Therefore the definitions we gave for α and β are k-binomial representations (extended for β). This means that

$$\partial^k(\alpha) + \partial^{k+1}(\beta) = \partial^k(\alpha) + \partial_e^{k+1}(\beta) = \partial^k(a) + \partial^{k+1}(b),$$

a contradiction to the minimality of a.

(ii) Assume now that $a_v = v$. This means that $a - 1 = a - \binom{a_v}{v} = \sum_{i=v+1}^{k} \binom{a_i}{i} \geq 1$ is the k-representation of $a - 1$, and thus $a - 1$ is short. Then by Lemma 4,

$$\partial^k(a-1) + \partial^{k+1}(b+1) = \partial^k(a) - 1 + \partial^{k+1}(b+1) \leq \partial^k(a) + \partial^{k+1}(b),$$

again a contradiction to the minimality of a.

Case 1 is settled.

Case 2. $b_{k+1} \leq a_k$.

Since $a < \partial^{k+1}(n)$ then, by Lemma 6, $a_k < n_{k+1} \leq b_{k+1}+1$. That is, $a_k = b_{k+1} = n_{k+1}-1$. We proceed by induction on k. If $k = 1$, then $a_1 = b_2 = n_2 - 1$. Thus

$$\binom{n_2}{2} + \binom{n_1}{1} = n = a + b = \binom{n_2 - 1}{1} + \binom{n_2 - 1}{2} + \binom{b_1}{1},$$

i.e., $b_1 = n_1$. Hence,

$$\partial^1(a) + \partial^2(b) = \binom{n_2 - 2}{0} + \binom{n_2 - 2}{1} + \binom{n_1 - 1}{0} = \partial^2(n).$$

Assume $k \geq 2$ and that the result holds for $k-1$. Let $n' = n-\binom{n_{k+1}}{k+1}$, $b' = b-\binom{n_{k+1}-1}{k+1}$, and $a' = a - \binom{n_{k+1}-1}{k}$. Because $a \geq \binom{a_k}{k} = \binom{n_{k+1}-1}{k}$, it follows that $a' \geq 0$. Clearly, $a' + b' = n'$, and $a' < \partial^k(n')$ since $a < \partial^{k+1}(n) = \binom{n_{k+1}-1}{k} + \partial^k(n')$. By induction on k the result holds for a', b', n', and thus

$$\begin{aligned}\partial^{k+1}(b) + \partial^k(a) - \partial^{k+1}(n) &= \binom{n_{k+1}-2}{k} + \partial^k(b') + \binom{n_{k+1}-2}{k-1} + \partial^{k-1}(a') \\ &\quad - \binom{n_{k+1}-1}{k} - \partial^k(n') \\ &= \partial^k(b') + \partial^{k-1}(a') - \partial^k(n') \geq 0.\end{aligned}$$

Case 2 is now proved.

It is only left to be shown that if $n = \binom{N}{k+1}$ for some $N \geq k+1$, then there is equality in Inequality (2), i.e.,

$$\partial^k(a) + \partial^{k+1}\left(\binom{N}{k+1} - a\right) = \partial^{k+1}\left(\binom{N}{k+1}\right), \tag{7}$$

occurs only when $a = 0$.

If $N = k+1$ (and thus $a = 0$) or $a = 0$, the equality trivially holds. Suppose that $N \geq k+2$, $a \geq 1$, and Identity (7) holds. Let $b = \binom{N}{k+1} - a$; as before, we consider two cases. First suppose that $a_k < b_{k+1}$. Assume that a and b are the smallest integers such that Identity (7) is satisfied with $a \geq 1$. If $a = 1$, then by Identity (6),

$$\binom{N}{k+1} - 1 = \binom{N-1}{k+1} + \binom{N-2}{k} + \cdots + \binom{N-k-1}{1}$$

is $(k+1)$-long. Thus, by Lemma 4, $\partial^{k+1}(\binom{N}{k+1}-1) = \partial^{k+1}(\binom{N}{k+1}) < 1 + \partial^{k+1}(\binom{N}{k+1}-1)$, which is a contradiction. If $a \geq 2$, then we proceed as in Case 1 to get a contradiction. Now assume that $b_{k+1} \leq a_k$. In this case, $a_k < N \leq b_{k+1} + 1$ and following the proof of Case 2, we have that $a_k = b_{k+1} = N-1$, a contradiction since $\binom{N-1}{k} = \binom{a_k}{k} \leq a < \partial^{k+1}(\binom{N}{k+1}) = \binom{N-1}{k}$. $\square$

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